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Completely mixed bimatrix games

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Abstract. Kaplansky (*Ann. Math.* **46(3)** (1945) 474–479) introduced the notion of completely mixed games. Fifty years later in 1995, he wrote another beautiful paper where he gave a set of necessary and sufficient conditions for a skew symmetric matrix game to be completely mixed. In this work, we attempt to answer when bimatrix games will be completely mixed. In particular, we give a set of necessary and sufficient condition for a bimatrix game (A, B) to be completely mixed when A and B are skew symmetric matrices of order 3. We give several examples to show the sharpness of our result. We also formulate a conjecture when A and B are skew symmetric matrices of order 5.

Keywords. Zero sum matrix games; bimatrix games; skew symmetric matrices; completely mixed bimatrix games.

Mathematics Subject Classification. 91A05, 91A10.

1. Introduction

In this section, we briefly discuss zero-sum matrix games and non-zero sum bimatrix games. We will also state many results without proof.

Let G_A denote a matrix game corresponding to a given matrix A of order $m \times n$. In this game, player 1 chooses one of the rows, say i, $i \in \{1, 2, \dots, m\}$ while player 2 chooses one of the columns, say j, $j \in \{1, 2, \dots, n\}$. Both of them choose simultaneously. As a result, player 1 receives an amount a_{ij}. If $a_{ij} > 0$, he or she gets this amount from player 2 and if $a_{ij} < 0$, he or she gives this amount to player 2. If $a_{ij} = 0$, no one gets anything. Player 1 wants to maximize his/her pay-off while player 2 tries to minimize the first player's pay-off.

Strategy for player 1 is a vector $x = (x_1, x_2, \dots, x_m)$ such that $x_i \geq 0$ and $\sum x_i = 1$. Such a vector is called a probability vector. Player 1 choosing x means, he chooses the i-th row with probability x_i. Analogously a strategy $y = (y_1, y_2, \dots, y_n)$ is defined for player 2. A strategy x is called a pure strategy if $x_i = 1$ for some i and $x_j = 0$ for $j \neq i$. Otherwise, it is a called a mixed strategy. A strategy x is called completely mixed if $x_i > 0$ for all i.

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If (x, y) is chosen by the two players, expected pay-off to player 1 is

$$A(x, y) = x^t Ay = \sum_i \sum_j a_{ij} x_i y_j$$

which may be positive, negative or zero.

DEFINITION 1

A pair of strategies (x^o, y^o) is called an *optimal pair* for the game G_A if

$$A(x^o, y^o) \geq A(x, y^o) \quad \text{and} \quad A(x^o, y^o) \leq A(x^o, y)$$

for all strategies x of player 1 and for all strategies y for player 2. The strategies x^o and y^o are optimal strategies for player 1 and player 2 respectively.

We are ready to state the celebrated theorem, called the minimax theorem due to John Von Neumann theorem (minimax theorem).

Theorem 1 (Minimax theorem [8]). *Let G_A be a matrix game represented by a matrix A of order $m \times n$. Then there exists a unique real number v and (at least) a pair of strategy (x^o, y^o) such that*

$$v = \max_x \min_y A(x, y) = \min_y \max_x A(x, y) = A(x^o, y^o). \tag{1}$$

Then v is called the value of the game G_A and (x^o, y^o) is an optimal strategy for G_A.

Note that v is unique but optimal pair (x^o, y^o) need not be unique, in general. Matrix A is called the pay-off matrix associated with player 1. Easy to check $-A$ is the payoff matrix associated with player 2. Payoff for player 2 is $x^t(-A)y = -x^t Ay$. Since the sum of payoffs to player 1 and player 2 is 0, G_A is called zero-sum matrix game associated with A.

DEFINITION 2

A game G_A or simply matrix game A is said to be completely mixed if all its optimal pair of strategies (x^o, y^o) are completely mixed, i.e. x^o and y^o are completely mixed strategies.

Kaplansky proves the following remarkable theorem, which characterises completely mixed games.

Theorem 2 [1]. *A game G_A represented by $m \times n$ matrix A with value 0 is completely mixed iff*

(1) *A is a square matrix $(m = n)$.*
(2) $\text{rank}(A) = n - 1$.
(3) *All the cofactors A_{ij} are different from zero and all are of the same sign.*

Here $A_{ij} = (-1)^{i+j}\det B$, where B is the submatrix of A omitting i-th row and j-th column from A. A_{ii} is called the i-th principal cofactor of A.

Kaplansky wrote two papers [1,2] on completely mixed zero-sum games, one in 1945 and another after five decades in 1995. The main result of this paper crucially depends on a beautiful result obtained by Kaplansky [2], where he gives a set of necessary and sufficient condition for G_A to be completely mixed when A is a skew symmetric matrix.

If $A = -A^t$, $\det A = 0$ if the order of the matrix is odd. If the order is even, $\det A =$ square of a polynomial and hence non-negative. Also, when $A = -A^t$, one can show using minimax theorem that the value $v = 0$. Also, if x^o is optimal for one player, then it is also optimal for the other player. It follows from Theorem 2 that if $A = -A^t$ and order n is even, then G_A is never completely mixed since all of its principal cofactors are zero. Kaplansky [2] gave a nice characterization for G_A to be completely mixed when n is odd.

In 1849, Arthur Cayley proved that determinant of a skew symmetric even order matrix is the square of polynomial of its elements with integer coefficients [3]. He called the polynomial pfaffian, named after Johann Friedrich Pffaf. Kaplansky [2] made use of this to give a description of completely mixed games G_A, when $A = -A^t$. We start with some definition.

DEFINITION 3 [9]

Let A be a $2n \times 2n$ skew symmetric matrix. We partition set $\{1, 2, \ldots, 2n\}$ into pairs $(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)$ such that $i_1 < i_2 < \cdots < i_n$ and $i_k < j_k$ for all $1 \leq k \leq n$. Let

$$\pi_\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2n-1 & 2n \\ i_1 & j_1 & i_2 & j_2 & \ldots & i_n & j_n \end{pmatrix}$$

be the permutation corresponding to partition

$$\alpha = \{(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)\}.$$

Then

$$\mathrm{pf}(A) = \sum_{\alpha \in \Pi} sgn(\pi_\alpha) a_{i_1 j_1} a_{i_2 j_2} \ldots a_{i_n j_n},$$

where Π is the set of all partitions of $\{1, 2, \ldots, 2n\}$ into pairs without regard to order.

For example, for $n = 1$, $\Pi = \{(1, 2)\}$. Therefore, for

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \qquad \mathrm{pf}(A) = a_{12} = a.$$

For $n = 2$, $\Pi = \{(1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$. Then for

$$A = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix},$$

$$\mathrm{pf}(A) = (-1)^0 a_{12}a_{34} + (-1)^1 a_{13}a_{24} + (-1)^2 a_{14}a_{23} = af - be + cd.$$

Pfaffian of any odd order skew symmetric matrix is 0. Cayley (1849) has shown that if A is a skew symmetric matrix of even order, then det $A = (\mathrm{pf}(A))^2$, where pf(A) is a polynomial of elements of A with integer coefficients.

Let A be a skew symmetric matrix of odd order. Let p_i denote the pfaffian of B, where B is a matrix (of even order) obtained from A by omitting the i-th row and i-th column from A. These p_i's are called principal pfaffians of A. We are now ready to state the main result from [2].

Theorem 3 [2]. *Let A be an $n \times n$ skew symmetric matrix where n is odd, then A is completely mixed iff principal pfaffians are all different from zero and alternate in sign. Also, optimal strategy for both the players is proportional to the vector* $(p_1, -p_2, p_3, \ldots, p_n)$.

We now introduce the concept of equilibrium point (for non-zero sum bimatrix games) due to [4]. Bimatrix games (A, B) are two person non-zero sum games where each player has a finite number of pure strategies. If player 1 chooses i and player 2 chooses j, then player 1 receives a_{ij} and player 2 receives b_{ij}. Here $a_{ij} + b_{ij}$ need not be equal to zero. Both want to maximize their income.

DEFINITION 4

A strategy pair (x_0, y_0) is called an equilibrium pair, extreme point or a *Nash equilibrium* for bimatrix game $G = (A, B)$ if

$$A(x_0, y_0) = x_0^t A y_0 \ \geq \ x^t A y_0 \ = \ A(x, y_0),$$
$$B(x_0, y_0) = x_0^t B y_0 \ \geq \ x_0^t B y \ = \ B(x_0, y)$$

for all strategies x of player 1 and y of player 2.

Nash [4] proved the following remarkable theorem.

Theorem 4 [4]. *Every bimatrix game has at least one equilibrium pair.*

John Nash [4] proved a more general theorem in 1951. Raghavan [6] proved one part of Kaplansky"s theorem ([1]) for bimatrix games.

Theorem 5 [7]. *If the bimatrix game (A, B) is completely mixed, that is, every equilibrium point is completely mixed, then A and B are square matrices and the equilibrium point is unique and it is completely mixed.*

Theorem 6 *Bimatrix game $(A, -A)$ is completely mixed iff A is a completely mixed zero sum matrix game.*

The aim of this article is to give a set of necessary and sufficient conditions such that bimatrix game (A, B) is completely mixed. We show in Section 2 that if A and B are skew symmetric matrices of order 3, then the bimatrix game (A, B) is completely mixed if and only if matrix games A and B are completely mixed and $p_i q_i < 0$ for all $i \in \{1, 2, 3\}$, where p_i and q_i are the principle pfaffians of A and B respectively. We give several counterexamples to show that this result is not valid even when $n = 5$. We formulate a conjecture when $n = 5$.

2. Completely mixed bimatrix games

In this section, we prove the following

Theorem 7 (Main result). *Let A and B be completely mixed skew symmetric matrices of order 3×3. Then (A, B) is completely mixed as a bimatrix game if and only if zero sum matrix games G_A and G_B are completely mixed and $p_i q_i < 0$ for all $i, j \in \{1, 2, 3\}$, where $p_i = pf(A_{ii})$ and $q_j = pf(B_{jj})$.*

Proof. Suppose G_A and G_B are completely mixed and $p_i q_i < 0$ for all $i \in \{1, 2, 3\}$. We will show the bimatrix game (A, B) is completely mixed. In other words, we will show that every equilibrium point is completely mixed. From our hypothesis, we may and do assume that

$$A = \begin{pmatrix} 0 & + & - \\ - & 0 & + \\ + & - & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & - & + \\ + & 0 & - \\ - & + & 0 \end{pmatrix}$$

Observe that B need not be equal to $-A$. Note that if $B = -A$, the result follows from Theorem 6.

Clearly, from the above sign structure, it follows that there is no pure equilibrium for the bimatrix game (A, B). We now analyse different cases:

(1) Suppose $x = (1, 0, 0)$ and $y = (y_1, y_2, 0)$ is an equilibrium pair.
Then $(1 \;\; 0 \;\; 0)B = (0 \;\; - \;\; +)$ for which player 2 would get maximum payoff by playing strategy $(0 \;\; 0 \;\; 1)$, thus contradicting that (x, y) is an equilibrium pair.
We will reach similar contradictions whenever either of x or y are pure strategies.

(2) Suppose $x = (x_1, x_2, 0)$ and $y = (y_1, y_2, 0)$ is an equilibrium pair. Then

$$Ay^t = \begin{pmatrix} 0 & + & - \\ - & 0 & + \\ + & - & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix} = \begin{pmatrix} + \\ - \\ * \end{pmatrix}.$$

But since $x_1 > 0$, $x_2 > 0$, Ay^t should be of the form $(\delta_1 \;\; \delta_2 \;\; *)$, where δ_1 and δ_2 are some positive numbers. This gives us a contradiction.
Similar contradictions are obtained in case for any $i \in \{1, 2, 3\}$, $x_i = 0$ or $y_i = 0$.

Thus, we have shown that if (x, y) is an equilibrium point then every coordinate in x and as well as in y must be positive. In other words, every equilibrium point (x, y) must be completely mixed. This proves that bimatrix game (A, B) is completely mixed.
Conversely, if the bimatrix game (A, B) is completely mixed and if A and B are skew symmetric matrices, it follows that A and B as matrix games are completely mixed. In other words, every optimal in G_A (as well as in G_B) is completely mixed. From Theorem 3 of [2], it follows that p_i's and q_i's for all $i \in \{1, 2, 3\}$ are non-zero and alternate in sign. To complete the proof, we need to show $p_i q_i < 0$ for all $i \in \{1, 2, 3\}$. Suppose $p_1 q_1 > 0$. Then $p_1 > 0, q_1 > 0$ or $p_1 < 0, q_1 < 0$. This will imply that sign pattern of A and sign pattern of B are the same which further implies that the bimatrix game (A, B) has a pure

equilibrium contradicting our assumption that every equilibrium of (A, B) is completely mixed. This terminates the proof of the converse. □

We can also prove the following (weaker version) of the main theorem proved above.

Theorem 8 *Let A and B be both symmetric. Then the bimatrix game (A, B) has a unique completely mixed equilibrium if and only if matrix games A and B are completely mixed.*

Proof. Suppose bimatrix game (A, B) has a unique completely mixed equilibrium say (x^o, y^o). If matrix game A is not completely mixed, then there exists an optimal y for A which is not completely mixed (that is, one coordinate of y is zero). Now we have $Ay^o = v_1 e_n$ ($\because$ (x^o, y^o) is the unique completely mixed equilibrium). Since $A = A^t$, it follows that y^o is optimal for both players and v_1 is the value of the matrix game. Thus $Ay = v_1 e_n$. Now it is easy to observe that $(x^o, (y^0 + y)/2)$ is another completely mixed equilibrium contradicting our assumption. Hence both A and B are completely mixed.

Conversely, if $A = A^t$ and $B = B^t$ are completely mixed then it is not hard to see the bimatrix (A, B) has a unique completely mixed equilibrium since (A, B) having another completely mixed equilibrium would imply that A and B are not completely mixed. □

COROLLARY 8.1

Let A and B be both skew-symmetric matrices of odd order. Then (A, B) has a unique completely mixed equilibrium if and only if A and B are completely mixed.

COROLLARY 8.2

If G_A and G_B are completely mixed then the bimatrix game (A, B) has a unique completely mixed equilibrium.

We omit the proofs of these corollaries here (see [7] for proofs).

We have also seen that if A and B are symmetric (or skew symmetric) and if (A, B) is completely mixed then matrix games A and B are completely mixed. The converse is not true as the following simple example shows. Take

$$A = B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly A is completely mixed (so is B) but the bimatrix game is not completely mixed. Is it possible to extend the main theorem when A and B are skew symmetric matrices of odd order $n \geq 5$? In the next section, we give several examples to illustrate the sharpness of our results. We will also show, in general, that if the bimatrix game (A, B) is completely mixed, then the zero-sum matrix games A and B are not necessarily completely mixed.

3. Counterexamples in higher dimension

In this section, we give several counterexamples to illustrate the sharpness of the main theorem proved in the previous section.

Example 1. Consider the following two matrices A and B:

$$A = \begin{pmatrix} 0 & 1 & -5 & 4 & 4 \\ -1 & 0 & -1 & 4 & -3 \\ 5 & 1 & 0 & 0 & -4 \\ -4 & -4 & 0 & 0 & 5 \\ -4 & 3 & 4 & -5 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & 0 & -4 \\ -1 & -2 & 0 & 2 & 2 \\ 2 & 0 & -2 & 0 & -3 \\ 0 & 4 & -2 & 3 & 0 \end{pmatrix}.$$

Here

$$\begin{array}{lllll} p_1 = 11, & p_2 = -9, & p_3 = 33, & p_4 = -23, & p_5 = 16, \\ q_1 = -14, & q_2 = 1, & q_3 = -8, & q_4 = 4, & q_5 = -4. \end{array}$$

Since p_i's and q_i's alternate in sign, it follows from Theorem 3 that matrix games A and B are completely mixed. Also, $p_i q_i < 0$ for all $i \in \{1, 2, 3, 4, 5\}$. But there exists a pure equilibrium pair which is $(0, 0, 0, 0, 1), (0, 1, 0, 0, 0)$, since a_{52} and b_{52} are column and row maximas respectively.

In the 3×3 case, if A and B (as matrix games) are completely mixed and $p_i q_i < 0$ for all $i \in \{1, 2, 3\}$, it follows that the off-diagonal entries in A and B are non-zero. Also, one can observe $a_{ij} b_{ij} < 0$ for $i \neq j$. Unfortunately, even if we impose these additional restrictions, bimatrix games need not be completely mixed as the following example shows.

Example 2. Consider the following two matrices

$$A = \begin{pmatrix} 0 & -\frac{6}{40} & 2 & -3 & \frac{1}{8} \\ \frac{6}{40} & 0 & -1 & 2 & -\frac{1}{8} \\ -2 & 1 & 0 & -\frac{1}{6} & \frac{3}{7} \\ 3 & -2 & \frac{1}{6} & 0 & -\frac{5}{12} \\ -\frac{1}{8} & \frac{1}{8} & -\frac{3}{7} & -\frac{5}{12} & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & \frac{1}{6} & -1 & 2 & -\frac{5}{6} \\ -\frac{1}{6} & 0 & 1 & -1 & \frac{1}{6} \\ 1 & -1 & 0 & \frac{7}{5} & -1 \\ -2 & 1 & -\frac{7}{5} & 0 & 1 \\ \frac{5}{6} & -\frac{1}{6} & 1 & -1 & 0 \end{pmatrix}.$$

One can check $p_1 < 0, p_2 > 0, p_3 < 0, p_4 > 0$ and $p_5 < 0$ and $q_1 > 0, q_2 < 0, q_3 > 0, q_4 < 0$ and $q_5 > 0$.

In other words, matrix games are completely mixed (from Theorem 3) and $p_i q_i < 0$ for all $i \in \{1, 2, 3, 4, 5\}$. Also $a_{ij} b_{ij} < 0$ for $i \neq j$. Clearly the bimatrix game has no pure equilibrium because $a_{ij} b_{ij} < 0$ for $i \neq j$.

However, one can show that $\{(2/5, 3/5, 0, 0, 0), (0, 0, 5/8, 3/8, 0)\}$ is an equilibrium point and it is not completely mixed.

Observe in the examples given above that there is a unique completely mixed equilibrium. In view of the above examples, we formulate the following conjecture when $n = 5$.

Conjecture 1. Let $A = -A^t$, $B = -B^t$. Assume that order of A and B be 5. Then the bimatrix game (A, B) is completely mixed if

(1) Matrix games A and B are completely mixed.
(2) Off-diagonal entries of A and B are nonzero and $a_{ij} b_{ij} < 0, \forall i \neq j$.

(3) If (x, y) is an equilibrium point of (A, B), then (support of x + support of y) is at least 5, where support of z = # of coordinates that are positive in z.

Note that in Example 1, condition (1) is violated and the bimatrix game (A, B) has a pure equilibrium (see that condition (1) is also violated). In Example 2, condition (1) is violated.

4. Concluding remarks

We start with an example to illustrate that if bimatrix game (A, B) is completely mixed, zero-sum matrix games G_A and G_B may or may not be completely mixed.

Example 3.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

In this example, bimatrix game (A, B) is completely mixed, but the matrix game A is not completely mixed.

We also give an example to show that if G_A and G_B are completely mixed, we have a unique completely mixed equilibrium pair for bimatrix game (A, B) but that does not imply that bimatrix game (A, B) is completely mixed.

Example 4.

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = B.$$

Here G_A and G_B are completely mixed and (A, B) has a unique completely mixed equilibrium $((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}))$. But ((1, 0, 0), (0, 1, 0)) is also an equilibrium pair for (A, B) which is a pure equilibrium.

We have given a set of necessary and sufficient conditions for a bimatrix game (A, B) to be completely mixed when A and B are skew symmetric matrices of order 3. We formulated conjecture when A and B are skew symmetric matrices of order 5. We believe this result is true but we do not have a complete proof yet.

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