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A submultiplicative property of the Carathéodory metric on planar domains

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Abstract. Given a pair of smoothly bounded domains $D_1, D_2 \subset \mathbb{C}$, the purpose of this paper is to obtain an inequality that relates the Carathéodory metrics on D_1, D_2, $D_1 \cap D_2$ and $D_1 \cup D_2$.

Keywords. Carathéodory metric; curvature; Ahlfors map and localization.

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1. Introduction

Let $\lambda_D(z)|\mathrm{d}z|$ denote the Poincaré metric on a hyperbolic domain $D \subset \mathbb{C}$. To quickly recall the construction of this metric, note that there exists a holomorphic covering from the unit disc $\mathbb{D}$ to D,

$$\pi : \mathbb{D} \to D$$

whose deck transformations form a Fuchsian group G that acts on $\mathbb{D}$. The Poincaré metric on $\mathbb{D}$,

$$\lambda_{\mathbb{D}}(z)|\mathrm{d}z| = \frac{|\mathrm{d}z|}{1 - |z|^2}$$

is invariant under $\mathrm{Aut}(\mathbb{D})$ (hence, in particular, G) and therefore, for $z \in D$, the prescription

$$\lambda_D(\pi(z))|\pi'(z)| = \lambda_{\mathbb{D}}(z) \tag{1}$$

defines the Poincaré metric λ_D on D in an unambiguous manner. For an arbitrary $z \in D$, we may choose the covering projection so that $\pi(0) = z$ and hence (1) implies that

$$\lambda_D(z) = |\pi'(0)|^{-1}.$$

Solynin [8,9] proved the following remarkable relation between the Poincaré metrics on a pair of hyperbolic domains and those on their union and intersection:

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Let $D_1, D_2 \subset \mathbb{C}$ be domains such that $D_1 \cap D_2 \neq \emptyset$. Suppose that $D_1 \cup D_2$ is hyperbolic. Then

$$\lambda_{D_1\cap D_2}(z)\lambda_{D_1\cup D_2}(z) \leq \lambda_{D_1}(z)\lambda_{D_2}(z)$$

for all $z \in D_1 \cap D_2$. If equality holds at one point $z_0 \in D_1 \cap D_2$, then $D_1 \subset D_2$ or $D_2 \subset D_1$ and in this case, equality holds for all points $z \in D_1 \cap D_2$.

A direct proof of this was given by Kraus and Roth [3] that relied on a computation reminiscent of the classical Ahlfors lemma and the fact that

$$\Delta \log \lambda_D(z) = 4\lambda_D^2(z), \tag{2}$$

which precisely means $\lambda_D(z)|\mathrm{d}z|$ has constant curvature -4 on D. Solynin's result follows from a comparison result for solutions to non-linear elliptic PDE's of the form

$$\Delta u - \mu(u) - f = 0, \tag{3}$$

where $\mu : \mathbb{R} \to \mathbb{R}$ and $f : D \to \mathbb{R}$ are suitable continuous non-negative functions that satisfy an additional convexity condition. Clearly, this reduces to the curvature equation (3) by writing $u = \log \lambda_D$ and letting $\mu(x) = 4\mathrm{e}^{2x}$ and $f \equiv 0$.

The purpose of this paper is to prove an analogue of Solynin's theorem for the Carathéodory metric on planar domains. Let $D \subset \mathbb{C}$ be a domain that admits at least one non-constant bounded holomorphic function. Recall that for $z \in D$, the Carathéodory metric $c_D(z)|\mathrm{d}z|$ is defined by

$$c_D(z)|\mathrm{d}z| = \sup\{|f'(z)| : f : D \to \mathbb{D} \text{ holomorphic and } f(z) = 0\}.$$

This is a distance-decreasing (and hence conformal) metric in the sense that if $h : U \to V$ is a holomorphic map between a pair of planar domains U, V, then

$$c_V(h(z))|h'(z)| \leq c_U(z), \quad z \in U.$$

If $U \subset V$, applying this to the inclusion $i : U \to V$ shows that the Carathéodory metric is monotonic as a function of the domain, i.e., $c_V(z) \leq c_U(z)$ for $z \in U$. Furthermore, for each $\zeta \in D$, there is a unique holomorphic map $f_\zeta : D \to \mathbb{D}$ that realizes the supremum in the definition of $c_D(z)$ – this is the Ahlfors map.

2. Statement and proof of the main result

Theorem 2.1. *Let $D_1, D_2 \subset \mathbb{C}$ be smoothly bounded domains with $D_1 \cap D_2 \neq \emptyset$. Then there exists a constant $C = C(D_1, D_2) > 0$ such that*

$$c_{D_1\cap D_2}(z) \cdot c_{D_1\cup D_2}(z) \leq C\, c_{D_1}(z) \cdot c_{D_2}(z)$$

for all $z \in D_1 \cap D_2$.

Note that $D_1 \cap D_2$ may possibly have several components. The notation $c_{D_1\cap D_2}(z)$ refers to the Carathéodory metric on that component of $D_1 \cap D_2$ which contains a given $z \in D_1 \cap D_2$. Furthermore, the constant $C > 0$ is independent of which component of $D_1 \cap D_2$ is being considered and only depends on D_1 and D_2. The proof, which is inspired by [3], uses the following known supplementary properties of the Carathéodory metric:

First, $c_D(z)$ is continuous and $\log c_D(z)$ is subharmonic on D (see [4] for instance). The possibility that $\log c_D(z) \equiv -\infty$ can be ruled out for bounded domain since for every

$\zeta \in D$, the affine map $f(z) = (z - \zeta)/a$ vanishes at ζ and maps D into the unit disc $\mathbb{D}$ for every positive a that is bigger than the diameter of D. The fact that $f'(z) \equiv 1/a > 0$ shows that $c_D(\zeta) > 0$ and hence $\log c_D(\zeta) > -\infty$.

Second, Suita [6] showed that $c_D(z)$ is real analytic in fact and hence we may speak of its curvature

$$\kappa_D(z) = -c_D^{-2}(z)\Delta \log c_D(z)$$

in the usual sense. The subharmonicity of $\log c_D(z)$ already implies that $\kappa_D \le 0$ everywhere on D, but by using the method of supporting metrics, Suita [6] was also able to prove a much stronger inequality namely, $\kappa_D \le -4$ on D. Following this line of inquiry further, Suita [7] (see [1] as well) showed that if the boundary of D consists of finitely many Jordan curves, the assumption that $\kappa_D(\zeta) = -4$ for some point $\zeta \in D$, implies that D is conformally equivalent to $\mathbb{D}$.

Finally, it is known that this metric admits a localization near C^∞-smooth boundary points – see for example [2] which contains a proof for the case of strongly pseudoconvex points that works verbatim in the planar case too. That is, if $p \in \partial D$ is a C^∞-smooth boundary point of a bounded domain $D \subset C$, then for a small enough neighbourhood U of p in $\mathbb{C}$,

$$\lim_{z \to p} \frac{c_{U \cap D}(z)}{c_D(z)} = 1.$$

Using this, it was shown in [5] that the curvature $\kappa_D(z)$ of the Carathéodory metric approaches -4 near each C^∞-smooth boundary point of a bounded domain $D \subset \mathbb{C}$. The point here being that, as one moves nearer to such a point, the metric begins to look more and more like the Carathéodory metric on $\mathbb{D}$ – the use of the scaling principle makes all this precise. To recall this in brief, let $p \in \partial D$ be a C^∞-smooth boundary point and ψ a local defining function. Then for a sequence $p_j \to p$ ($p_j \in D$), the family

$$T_j(z) = \frac{z - p_j}{-\psi(p_j)}$$

expands each neighbourhood of p in the sense that every compact $K \subset \mathbb{C}$ is eventually contained in $T_j(U)$, where U is a given neighbourhood of p.

The family of domains $D_j = T_j(U \cap D)$ is then defined by $-\psi(p_j)^{-1}\psi \circ T_j^{-1}$ which is seen to converge to

$$\psi_\infty(z) = -1 + 2\mathrm{Re}\left(\frac{\partial \psi}{\partial z}(p)z\right).$$

Thus D_j converges to the half-space $\mathcal{H} = \{\psi_\infty < 0\}$ in the Hausdorff sense. It has been shown in [5] that the Carathéodory metrics C_{D_j} converge to $C_{\mathcal{H}}$ uniformly on each compact subset of $\mathcal{H}$ and hence the curvature κ_{D_j} of the Carathéodory metric on D_j converges to -4, which is the curvature of the Carathéodory metric of $\mathcal{H}$, uniformly on compact subsets of $\mathcal{H}$.

It follows on any smoothly bounded planar domain D, $\kappa_D \approx -4$ for points close to the boundary and for those that are at a fixed positive distance away from it, there is a large negative lower bound for κ_D as a result of its continuity. Hence for every such $D \subset \mathbb{C}$, there is a constant $C = C(D) > 0$ such that

$$-C \le \kappa_D(z) \le -4$$

for all $z \in D$. Another consequence of the localization principle is that $c_D(z) \to +\infty$ as z approaches a smooth boundary point. Indeed, near such a point, $c_D(z) \approx c_{U \cap D}(z)$ and the latter is the same as the Carathéodory metric on the disc $\mathbb{D}$ (since $U \cap D$ can be taken to be simply connected) which blows up near every point on $\partial\mathbb{D}$.

To prove the theorem, let κ_1 and κ_2 be the curvatures of the Carathéodory metric $c_{D_1}(z)|\mathrm{d}z|$ and $c_{D_2}(z)|\mathrm{d}z|$ on D_1 and D_2 respectively.

Consider the metric

$$c(z)|\mathrm{d}z| = \frac{c_{D_1}(z) \cdot c_{D_2}(z)}{c_{D_1 \cup D_2}(z)}|\mathrm{d}z|$$

on the possibly disconnected open set $D_1 \cap D_2$. What follows applies to each component of $D_1 \cap D_2$ without any regard to its analytic or topological properties and hence we will continue to write $c(z)|\mathrm{d}z|$ to denote this metric on any given component. Its curvature is

$$\begin{aligned} \kappa(z) &= -c^{-2}(z)\Delta \log c(z) \\ &= -c^{-2}(z) \left(\Delta \log c_{D_1}(z) + \Delta \log c_{D_2}(z) - \Delta \log c_{D_1 \cup D_2}(z)\right) \\ &= I_1 + I_2 + I_{12}, \end{aligned} \tag{4}$$

where

$$I_1 = -c^{-2}(z)\Delta \log c_{D_1}(z),$$
$$I_2 = -c^{-2}(z)\Delta \log c_{D_2}(z)$$

and

$$I_{12} = c^{-2}(z)\Delta \log c_{D_1 \cup D_2}(z).$$

Note that $I_{12} \geq 0$ since $\log c_{D_1 \cup D_2}(z)$ is subharmonic and hence

$$\kappa(z) \geq I_1 + I_2.$$

To analyze each of these terms, note that $c_{D_1} \geq c_{D_1 \cup D_2}$ and $c_{D_2} \geq c_{D_1 \cup D_2}$. Combining this with the fact that the curvature of the Carathéodory metric is negative everywhere, it follows that

$$I_1 \geq \kappa_1(z)$$

and

$$I_2 \geq \kappa_2(z).$$

Hence there is a constant $C = C(D_1, D_2) > 0$ such that

$$\kappa(z) \geq \kappa_1(z) + \kappa_2(z) > -C$$

for all $z \in D_1 \cap D_2$.

Let U be a component of $D_1 \cap D_2$. Since D_1, D_2 have smooth boundaries, U has finite connectivity, say $m \geq 1$ and is non-degenerate in the sense that the interior of its closure coincides with itself. In particular, its boundary cannot contain isolated points. Let U_ϵ be an ϵ-thickening of U. For all sufficiently small $\epsilon > 0$, U_ϵ also has connectivity m and is non-degenerate. Furthermore, $U_\epsilon \to U$ in the sense of Carathéodory as $\epsilon \to 0$.

For a fixed $\epsilon > 0$, let $c_\epsilon(z)|\mathrm{d}z|$ be the Carathéodory metric on U_ϵ. Consider the function

$$u(z) = \log\left(\frac{c_\epsilon(z)}{\sqrt{C/4}\, c(z)}\right)$$

on U. Since U is compactly contained in U_ϵ, c_ϵ is bounded on U and if $\xi \in \partial U$, then as $z \to \xi$ within U,

$$\lim_{z\to\xi} \frac{c_{D_1}(z) \cdot c_{D_2}(z)}{c_{D_1\cup D_2}(z)} \geq \lim_{z\to\xi} c_{D_2}(z) = +\infty,$$

where the inequality follows from the monotonocity of the Carathéodory metric, i.e., $c_{D_1}(z) \geq c_{D_1\cup D_2}(z)$. The fact that ξ is a smooth boundary point of D_2 implies that c_{D_2} blows up near it and this means that $u(z) \to -\infty$ at ∂U. Therefore, u attains a maximum at some point $z_0 \in U$. As a result,

$$\begin{aligned} 0 \geq \Delta u(z_0) &= \Delta \log c_\epsilon(z_0) - \Delta \log c(z_0) \\ &\geq -\kappa_\epsilon(z_0) c_\epsilon^2(z_0) - C c^2(z_0) \\ &\geq 4c_\epsilon^2(z_0) - C c^2(z_0), \end{aligned} \tag{5}$$

where κ_ϵ is the curvature of $c_\epsilon(z)|\mathrm{d}z|$. It follows that

$$u(z_0) = \log\left(\frac{c_\epsilon(z_0)}{\sqrt{C/4}\, c(z_0)}\right) \leq 0.$$

For an arbitrary $z \in U$, $u(z) \leq u(z_0) \leq 0$ and this gives

$$\log\left(\frac{c_\epsilon(z)}{\sqrt{C/4}\, c(z)}\right) \leq 0$$

or

$$c_\epsilon(z) \leq \sqrt{C/4}\, c(z)$$

which is same as

$$c_\epsilon(z) \cdot c_{D_1\cup D_2}(z) \leq \sqrt{C/4}\, c_{D_1}(z) \cdot c_{D_2}(z)$$

and this holds for all $z \in U$. It remains to show that $c_\epsilon \to c_U$ pointwise on U for then we can pass to the limit as $\epsilon \to 0$, keeping in mind that C is independent of ϵ, to get

$$c_U(z) \cdot c_{D_1\cup D_2}(z) \leq \sqrt{C/4}\, c_{D_1}(z) \cdot c_{D_2}(z)$$

as claimed. Fix $p \in U$. To show that $c_\epsilon(p) \to c_U(p)$, it suffices to prove that $|f_\epsilon'(p)| \to |f'(p)|$ as $\epsilon \to 0$, where $f_\epsilon : U_\epsilon \to \mathbb{D}$ and $f : U \to \mathbb{D}$ are the respective Ahlfors maps for the domains U_ϵ and U at p. What this is really asking for is the continuous dependence of the Ahflors maps on the domains. But this is addressed in [10] – indeed, let us recall Theorem 3.2 therein:

Let D_k be a sequence of nondegenerate n-connected domains containing ∞ in the Riemann sphere. Suppose D_k converges to a domain D, which is also nondegenerate n-connected, in the Carathéodory kernel sense. Let f_k and f be the Ahlfors maps of D_k and D respectively. Then f_k converges to f uniformly on compact subsets of D.

This can be applied since U_ϵ and U have the same connectivity by construction and the U_ϵ's decrease to U as $\epsilon \to 0$. The nuance, in this theorem, about the base point being the point at infinity can be arranged by sending $p \mapsto \infty$ by the map $T(z) = 1/(z-p)$ and working with the domains $T(U_\epsilon)$ and $T(U)$. This completes the proof.

Question. Let $\Omega \subset \mathbb{C}$ be a domain on which the Carathéodory metric $c_\Omega(z)|\mathrm{d}z|$ is not degenerate. Does its curvature κ_Ω admit a global lower bound?

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