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Picard bundle on the moduli space of torsionfree sheaves

USHA N BHOSLE

Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India
Email: usha@math.tifr.res.in

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Abstract. Let Y be an integral nodal projective curve of arithmetic genus $g \geq 2$ with m nodes defined over an algebraically closed field. Let n and d be mutually coprime integers with $n \geq 2$ and $d > n(2g-2)$. Fix a line bundle L of degree d on Y. We prove that the Picard bundle $\mathbf{E}_L$ over the 'fixed determinant moduli space' $U_L(n, d)$ is stable with respect to the polarisation θ_L and its restriction to the moduli space $U'_L(n, d)$, of vector bundles of rank n and determinant L, is stable with respect to any polarisation. There is an embedding of the compactified Jacobian $\bar{J}(Y)$ in the moduli space $U_Y(n, d)$ of rank n and degree d. We show that the restriction of the Picard bundle of rank ng (over $U_Y(n, n(2g-1))$) to $\bar{J}(Y)$ is stable with respect to any theta divisor $\theta_{\bar{J}(Y)}$.

Keywords. Nodal curve; moduli spaces; Picard bundles; stability.

Mathematics Subject Classification. 14H60.

1. Introduction

For a smooth curve X of genus $g \geq 2$, Picard bundles and projective Picard bundles on the moduli spaces of stable vector bundles of rank $n \geq 1$ and degree d (with or without fixed determinant) on X, as well as Poincaré bundles and projective Poincaré bundles have been studied extensively by several authors such as Ein and Lazarsfeld [14] and Kempf [15] for the rank 1 case and Balaji *et al.* [2], Biswas *et al.* [10] and Li [16] for higher ranks. Bhosle and Parameswaran [11] started the study of Picard bundles on nodal curves. They started with rank 1, i.e. Picard bundles on the compactified Jacobian of an integral nodal curve Y and showed that these bundles are stable but, unlike in case X is smooth, the dual Picard bundle is not ample .

In this paper, the case $n \geq 2$ is considered. Let $U_Y(n, d)$ be the moduli variety of stable torsionfree sheaves of rank n, degree d on the nodal curve Y and $U'_Y(n, d)$ its open dense subvariety corresponding to vector bundles. Denote by $U'_L(n, d)$ the subvariety of $U'_Y(n, d)$ corresponding to vector bundles with determinant L, where L is a fixed line bundle of degree d and by $U_L(n, d)$ its closure in $U_Y(n, d)$. Assume that n and d are mutually coprime. Then there is a Poincaré sheaf $\mathcal{U}$ on $U_Y(n, d) \times Y$. For $d > n(2g-2)$, the direct image $\mathbf{E}_{n,d}$ of $\mathcal{U}$ on $U_Y(n, d)$ is a vector bundle on $U_Y(n, d)$ called the Picard bundle. Let $\mathbf{E}_L$ denote the restriction of $\mathbf{E}_{n,d}$ to $U_L(n, d)$. Let θ_L denote the theta line

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bundle on $U_L(n,d)$. One has Pic $U'_L(n,d) \cong \mathbb{Z}$ and the restriction of θ_L to $U'_L(n,d)$ is the ample generator of Pic $U'_L(n,d)$ [5, Theorem I].

The main result of the paper is the following theorem.

Theorem 1.1 (Theorem 4.1). *Let Y be an integral projective curve of arithmetic genus $g \geq 2$ with m nodes (ordinary double points) defined over an algebraically closed field. Let n and d be mutually coprime integers with $n \geq 2$ and $d > n(2g-2)$. Fix a line bundle L of degree d on Y. Then*

- *the Picard bundle $\mathbf{E}_L$ over $U_L(n,d)$ is stable with respect to the polarisation θ_L,*
- *the restriction of the Picard bundle to $U'_L(n,d)$ is stable with respect to any polarisation on $U'_L(n,d)$.*

We denote by $\bar{J}(Y)$ the compactified Jacobian of Y, the space of torsion free sheaves of rank 1 and degree -1 on Y. Let $p : X \to Y$ denote the normalisation map. Fix a general stable vector bundle E_0 on Y of rank n and degree $d+n$, where $-n < d \leq 0$, such that p^*E_0 is stable. There is an embedding [9]

$$\alpha_Y : \bar{J}(Y) \longrightarrow U_Y(n,d)\,; \; \alpha_Y(N) = N \otimes E_0\,. \tag{1.1}$$

Theorem 1.2 (Theorem 5.2). *Let Y be a non-hyperelliptic nodal curve. Let $\mathbf{E}_{n,n(2g-1)}|\bar{J}(Y)$ denote the restriction of the Picard bundle $\mathbf{E}_{n,n(2g-1)}$ on $U_Y(n,n(2g-1))$ to $\bar{J}(Y)$. Then $\mathbf{E}_{n,n(2g-1)}|\bar{J}(Y)$ is stable with respect to any theta divisor $\theta_{\bar{J}(Y)}$.*

A coherent sheaf F on a scheme X is said to have natural cohomology if $H^i(X,F) \neq 0$ for at most one i. We prove the following theorem on natural cohomology and use it for the proof of Theorem 5.2.

Theorem 1.3 (Theorem 5.1: Theorem on natural cohomology). *Let Y be an integral nodal curve of arithmetic genus $g \geq 2$. Then a general vector bundle of rank $n \geq 1$ on Y has natural cohomology. More precisely, one has*

(*a*) *a general stable vector bundle $E \in U_Y(n,d)$ has $H^0(Y,E) = 0$, $H^1(Y,E) = 0$ if $d = n(g-1)$,*
(*b*) *a general stable vector bundle $E \in U_Y(n,d)$ has $H^0(Y,E) = 0$ if $d \leq n(g-1)$,*
(*c*) *a general stable vector bundle $E \in U_Y(n,d)$ has $H^1(Y,E) = 0$ if $d \geq n(g-1)$.*

2. Generalities

2.1 *Notation*

Let Y be an integral projective curve with m nodes (ordinary double points) as singularities defined over an algebraically closed field. Let $g = h^1(Y, \mathcal{O}_Y)$ be the arithmetic genus of Y, we assume that $g \geq 2$. Let

$$p : X \longrightarrow Y$$

be the normalisation map. For a node $y_j \in Y$, let x_j and z_j denote the points of X lying over y_j. For each $j = 1, \ldots, m$, let $D_j = x_j + z_j$ denote the divisor on X.

For a torsion free sheaf F on Y, let $r(F)$ denote the (generic) rank of F and $d(F) = \chi(F) - r(F)\chi(\mathcal{O}_Y)$ denote the degree of F. Let $\mu(F) = d(F)/r(F)$ denote the slope of F.

At a node y_j, let A_j be the local ring and m_j its maximum ideal. The stalk of a torsion free sheaf E at a node y_j is isomorphic to $a_j(E)A_j \oplus b_j(E)m_j$, $a_j(E) + b_j(E) = r(E)$, where $a_j(E)A_j$ denotes the direct sum of $a_j(E)$ copies of A_j (similarly $b_j(E)m_j$) [21]. We call $b_j(E)$ the local type of E at y_j. One has $0 \leq b_j(E) \leq r(E)$.

2.2 *Moduli spaces*

Let $U_Y(n, d)$ denote the moduli space of semistable torsion free sheaves of rank n and degree d on Y. It is irreducible [20] and it is a seminormal projective variety [22, Theorem 4.2]. Let $U'_Y(n, d)$ be its open dense sub variety corresponding to semistable vector bundles on Y. From the remark on p. 167, section 7 of [17], one deduces that $U'_Y(n, d)$ is a normal quasi-projective variety (being the GIT quotient of a nonsingular variety R). For a fixed line bundle L on Y, let $U'_L(n, d)$ denote the sub variety of $U'_Y(n, d)$ consisting of vector bundles with determinant isomorphic to L.

If n and d are coprime, then the points of $U'_Y(n, d)$ correspond to stable vector bundles. Hence it follows from the remark on p. 167, section 7 of [17] that $U'_Y(n, d)$ is a nonsingular quasi-projective variety, so is $U'_L(n, d)$. Let $U_L(n, d)$ denote the closure of $U'_L(n, d)$ in $U_Y(n, d)$. The superscript s will denote sub varieties corresponding to stable sheaves.

The moduli varieties $U'_Y(n, d)$ and $U'_L(n, d)$ are locally factorial [6, Theorem 3A]. One has

$$\text{Pic } U'_L(n, d) \cong \mathbb{Z}$$

(for $g \geq 2$) [5, Theorem I]. There is a canonically defined ample line bundle θ_U on $U_Y(n, d)$, let

$$\theta_L \longrightarrow U_L(n, d)$$

be its restriction. The restriction of θ_L to $U'_L(n, d)$ is the generator of Pic $U'_L(n, d)$ [5, Proposition 3.2].

The subset of $U_Y(n, d)$ corresponding to torsionfree sheaves which are not locally free (i.e., have local type at least 1 at some node) is a closed subset of codimension 1. However, the subset of $U_L(n, d)$ corresponding to torsionfree sheaves which are not locally free is a closed subset of codimension at least 3 as the following proposition shows.

PROPOSITION 2.1

Let Y be an integral nodal curve of genus $g \geq 2$ with m nodes, $m \geq 1$.

(1) *The codimension of $U_L(2, d) - U'_L(2, d)$ in $U_L(2, d)$ is at least* 3.
(2) *The codimension of $U_L(r, d) - U'_L(r, d)$ in $U_L(r, d)$, $r \geq 3$, is at least* 5 *except for* $g(X) = 1$.
(3) *The codimension of $U_L(r, d) - U'_L(r, d)$ in $U_L(r, d)$, $r \geq 3$, is at least* 3 *if* $g(X) = 1$.

Proof.

(1) By arguments in [3, Section 2] and [3, Section 4], we see that the complement of $U'_L(2, d)$ in $U_L(2, d)$ contains no torsion-free sheaves which are of local type 1 at any

node y. Hence this complement is a finite union of sets, each set consisting of semistable torsion-free sheaves which are locally free at $m - m_1$ nodes and are of local type 2 at the remaining $m_1 \geq 1$ nodes, i.e. they are locally isomorphic to $m_y \oplus m_y$ at m_1 nodes. Such sheaves are direct images of semistable vector bundles with fixed determinant on the partial normalisation Z of Y obtained by blowing up the m_1 nodes. It follows that the corresponding set in the boundary $U_L(2,d) - U'_L(2,d)$ has dimension equal to the dimension δ_Z of the moduli space U'_Z of semistable vector bundles with fixed determinant of degree $d - 2m_1$ on the partial normalisation Z.

For $g(Z) = g - m_1 \geq 2$, one has $\delta_Z = 3g(Z) - 3 \geq 3$, hence the corresponding set has codimension $\delta \geq 3$. This implies that for $g(X) \geq 2$, $U_L(2,d) - U'_L(2,d)$ has codimension ≥ 3. For $g(Z) = 1$, d odd or $g(Z) = 0$, either the moduli space U'_Z is empty or has dimension 0, hence the corresponding set in the boundary has codimension $\delta \geq 3$ for $g \geq 2$. It follows that $U_L(2,d) - U'_L(2,d)$ has codimension ≥ 3 for d odd, for $g(X) \leq 1$ as well. For $g(Z) = 1$, d even, U'_Z has dimension 1 (and consists of sheaves which are not stable) so that the corresponding set in the boundary has codimension $\delta \geq 3$ if $g \geq 3$. For $g = 2$ and d even, $U_L(2,d) = U'_L(2,d) \cong \mathbb{P}^3$ [4, Lemmas 3.3, 3.4, Corollary 3.5].

Parts (2) and (3) are proved in [13, Theorem 1.3]. □

2.3 *The* (l_1, l_2)*-stability of torsionfree sheaves*

Generalising [19, Definition 5.1], we make the following definition.

DEFINITION 2.2

Let l_1 and l_2 be integers. A torsionfree sheaf F on Y is (l_1, l_2)-stable if, for every proper subsheaf G of F (with a torsionfree quotient), one has

$$\frac{d(G) + l_1}{r(G)} < \frac{d(F) + l_1 - l_2}{r(F)}.$$

We remark that a torsionfree sheaf F is stable if and only if it is $(0,0)$-stable.

PROPOSITION 2.3

The $(0,1)$*-stable vector bundles* F*, of rank* n *and determinant* L' *of degree* d' *with* n *coprime to* $d' - 1$*, form a non-empty open subset of the moduli space* $U^s_{L'}(n, d')$ *of stable torsionfree sheaves.*

Proof. The proof is on similar lines as that of [19, Propositions 5.3, 5.4] (or [10, Lemma 2]) in the smooth case. We briefly sketch it with modifications necessary in the case of a nodal curve. Let C be the complement in $U'^s_{L'}(n, d')$ of the set of $(0,1)$-stable vector bundles. A bundle $F \in C$ if and only if it has a torsionfree subsheaf G of rank r and degree e satisfying $rd' > ne \geq r(d' - 1)$. This implies that the ranks and degrees of quotients of $F \in C$ are bounded. The closedness of C follows from the properness of quot schemes.

It suffices to show that the dimension of C is strictly less than the dimension of $U'^s_{L'}(n, d')$. Any $F \in C$ comes in an extension of F/G by G with G as above. Such extensions are parametrised by $\mathrm{Ext}^1(F/G, G)$. We may assume that G and F/G are stable (as is done in the smooth case in [19] by using [18, Proposition 2.6]), so that $h^0(\mathrm{Hom}(F/G, G)) = 0$.

By [7, Lemma 2.5(B)],

$$\begin{aligned} h &= \dim \operatorname{Ext}^1(F/G, G) \\ &= h^1(Y, \operatorname{Hom}(F/G, G)) + 2\textstyle\sum_j b_j(F/G)b_j(G) \\ &= rd' - ne + r(n-r)(g-1) + \textstyle\sum_j b_j(F/G)b_j(G)\,. \end{aligned}$$

Hence the corresponding component of C has dimension

$$\begin{aligned} d_0 &\leq (r^2(g-1) + 1 - \textstyle\sum_j b_j(G)^2) + ((n-r)^2(g-1) \\ &\quad +1 - \textstyle\sum_j b_j(F/G)^2) - g + (h-1) \\ &= (n^2 + r^2 - nr - 1)(g-1) + rd' - ne - \textstyle\sum_j (b_j(G) \\ &\quad - b_j(F/G))^2 - \textstyle\sum_j b_j(F/G)b_j(G) \\ &\leq (n^2 + r^2 - nr - 1)(g-1) + rd' - ne \end{aligned}$$

The last expression is strictly less than $\dim U'^{s}_{L'}(n,d) = (n^2-1)(g-1)$ if $rd' - ne < r(n-r)(g-1)$. Since $0 < rd' - ne \leq r$, the last inequality holds except in case $g = 2, r = n-1, ne = r(d'-1)$. Since n and $d'-1$ are mutually coprime, $ne \neq r(d'-1)$ for any r, e proving the proposition. □

3. Picard bundles

Now suppose that n and d are coprime. Then semistability is equivalent to stability, so that $U_Y(n,d) = U^s_Y(n,d), U_L(n,d) = U^s_L(n,d), U'^s_Y(n,d) = U'_Y(n,d), U'^s_L(n,d) = U'_L(n,d)$. Moreover $U'_Y(n,d)$ and $U'_L(n,d)$ are nonsingular. Since n and d are coprime, $U_Y(n,d)$ is a fine moduli space and there is a universal sheaf [17, Theorem 5.12']

$$\mathcal{U} \;\to\; U_Y(n,d) \times Y.$$

Let $\pi_U : U_Y(n,d) \times Y \to U_Y(n,d)$ be the projection.

DEFINITION 3.1

The direct image sheaves

$$\mathbf{E}_{n,d} := \pi_{U*}\mathcal{U} \quad \text{and} \quad \mathbf{F}_{n,d} := R^1\pi_{U*}\mathcal{U}$$

are called the Picard sheaves on $U_Y(n,d)$. For $d > (2g-2)n$ ($d \geq (2g-2)n$ if $n \geq 2$), the Picard sheaf $\mathbf{F}_{n,d} = 0$. Then the Picard sheaf $\mathbf{E}_{n,d}$ is a locally free sheaf of rank $d + n(1-g)$ and is called a Picard bundle.

The restriction of $\mathbf{E}_{n,d}$ to $U_L(n,d)$ is the Picard bundle on $U_L(n,d)$. We denote it by $\mathbf{E}_{L,n,d}$, or by $\mathbf{E}_L$ if n and d are fixed.

3.1 *The morphism* $\psi_{F,x}$

Once for all, we fix a nonsingular point x of Y and a $(0,1)$-stable vector bundle F of rank n and determinant $L(x)$, $d(L) = d$. Let k_x denote the torsion sheaf of length 1 supported at x. Denote by F_x the fibre of F at x and by $\mathbf{P} := \mathbf{P}(F^*_x)$ the projective space of lines in F^*_x.

For every nonzero element $\phi \in \mathbf{P}(F_x^*)$, we have a nonzero homomorphism $\phi : F \to k_x$ giving an exact sequence

$$0 \longrightarrow E \longrightarrow F \xrightarrow{\phi} k_x \longrightarrow 0\,. \tag{3.1}$$

Since x is a nonsingular point of Y and F is locally free, it follows that E is a locally free sheaf of rank n and determinant L. The exact sequence fits in the following commutative diagram D_0:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & F(-x) & = & F(-x) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & k_x & \longrightarrow & 0\,. \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & k_x^{n-1} & \longrightarrow & F_x & \longrightarrow & k_x & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Let p_1 (respectively p_2) be the projection from $Y \times \mathbf{P}$ to Y (respectively to $\mathbf{P}$). The exact sequences (3.1), as ϕ varies over $\mathbf{P}$, combine together to give the following exact sequence on $Y \times \mathbf{P}(F_x^*)$ with $\mathcal{E}$ a vector bundle:

$$0 \longrightarrow \mathcal{E} \longrightarrow p_1^* F \longrightarrow i_* \mathcal{O}_{\mathbf{P}}(1) \longrightarrow 0\,, \tag{3.2}$$

where $i : \mathbf{P} \hookrightarrow Y \times \mathbf{P}$ is the inclusion map defined by $z \mapsto (x, z)$.

By the universal property of $U_L(n, d)$, we have a morphism

$$\psi_{F,x} \colon \mathbf{P}(F_x^*) \longrightarrow U_L(n, d)\,,$$

such that, for some integer j, we have an isomorphism

$$\mathcal{E} \;\cong\; (id \times \psi_{F,x})^* \mathcal{U} \otimes p_2^*(\mathcal{O}_{\mathbf{P}}(-j))\,. \tag{3.3}$$

Lemma 3.2. *$\psi_{F,x}$ is an isomorphism onto its image in $U'_L(n, d)$.*

Proof. This can be proved as [19, Lemma 5.9] ([19, Lemma 5.6] and [10, Lemma 3] for injectivity). As in [19, Lemma 5.5], one can see that (0, 1)-stability of F implies that E is stable. It follows that $\psi_{F,x}$ maps $\mathbf{P}(F_x^*)$ into $U'_L(n, d)$. □

4. Main theorem

Let H be an ample line bundle on a variety Z of dimension r at least 2 and F be a torsionfree sheaf on Z. Then degree of F with respect to H, denoted by $d(F)$, is the degree of the restriction of F to a general complete intersection curve in $|H^{r-1}|$. We remark that if the singular set of Z has codimension at least 2, then the general complete intersection curve can be chosen to lie in the set of nonsingular points of Z.

Our aim in this section is to prove the following theorem.

Theorem 4.1. *Let Y be an integral projective nodal curve of arithmetic genus $g \geq 2$ defined over an algebraically closed field. Let n and d be mutually coprime integers with $n \geq 2$ and $d > n(2g-2)$. Fix a line bundle L of degree d on Y. Then the Picard bundle $\mathbf{E}_L$ over $U_L(n,d)$ is stable with respect to the polarisation θ_L.*

We first prove some results required for the proof of the theorem.

Each point of the projective bundle

$$P_x := \mathbf{P}(\mathcal{U}_{\mathbf{L}}|_{\mathbf{x}\times \mathbf{U_L(n,d)}})$$

corresponds to a pair (E, ℓ), where $E \in U_L(n,d)$ and $\ell \in \mathbf{P}(E_x) \cong \mathbf{P}(\mathrm{Ext}^1(k_x, E))$ and hence determines an exact sequence of type (3.1) and thus a torsionfree sheaf F. Let H_x be the open subset of P_x defined by

$$H_x := \{(E,\ell) \in P_x \mid F \in U_{L(x)}(n, d+1) \text{ is } (0,1)\text{-stable}\}.$$

We have a morphism $q : H_x \to U_{L(x)}(n, d+1)$ defined by $(E,\ell) \mapsto F$ with image the nonempty open subset $V \subset U_{L(x)}(n, d+1)$ of $(0,1)$-stable vector bundles (Lemma 2.3). The fibre of q over $F \in V$ is $\mathbf{P}(F_x^*)$. The restriction of the projection map $p : H_x \to U_L(n,d)$ to the fibre is precisely $\psi_{F,x}$, hence the fibre $P(F_x^*)$ maps isomorphically onto its image $P(F,x) := \psi_{F,x}(P(F_x^*))$ (Lemma 3.2).

Lemma 4.2. *Let $\mathcal{F} \subset \mathbf{E}_L$ be a subsheaf of rank r with $0 < r < r(\mathbf{E}_L)$. Let $x_1, \ldots, x_p \in Y$ be nonsingular points.*

(1) *The singular set S of $\mathcal{F}$ has codimension at least 2 in $U_L(n,d)$.*
(2) *There is a nonempty open set $U \subset U_L(n,d)$ such that for $E \in U$,*
   - *(a) $\mathcal{F}$ is locally free at E,*
   - *(b) the homomorphism of fibres $\mathcal{F}_E \to (\mathbf{E}_L)_E$ is injective,*
   - *(c) for all x_i and for the generic line ℓ in E_{x_i}, the vector bundle F associated to (E,ℓ) is $(0,1)$-stable and $\mathcal{F}$ is locally free at every point of $P(F,x_i)$ outside a subvariety of codimension at least 2.*

Proof.

(1) Recall that the singular set S of $\mathcal{F}$ is the (possibly empty, closed) set of points of $U_L(n,d)$, where $\mathcal{F}$ is not locally free. Since the open subvariety $U'_L(n,d)$ is smooth, $S \cap U'_L(n,d)$ is of codimension at least 2 in $U'_L(n,d)$. The closed subset $U_L(n,d) - U'_L(n,d)$ has codimension at least 3 in $U_L(n,d)$ (Theorem 2.1). It follows that S is of codimension at least 2 in $U_L(n,d)$.
(2) Part (a) follows from Part (1). Part (b) follows similarly (by applying same argument to $\mathbf{E}_L/\mathcal{F}$).

For Part (c), it suffices to show that for a fixed x and general F, either $P(F,x)$ is empty or $\dim(P(F,x) \cap S) \leq n-3$. One may reduce to the case S is irreducible. Let $S' = p^{-1}S \subset H_x$. It is an irreducible variety of dimension $\dim S + n - 1$ (if nonempty) and its fibre over F can be identified with $P(F,x)\cap S$. If $q(S')$ is not dense in V, then for the general F, $P(F,x)\cap S$ is empty. If $q(S')$ is dense in V, then the general fibre $P(F,x)\cap S$ of $q_{|S'}$ has dimension d_S, where

$$\begin{aligned} d_S &= \dim S + n - 1 - \dim V \\ &= \dim S + n - 1 - \dim U_{L(x)}(n, d+1) \\ &= \dim S + n - 1 - \dim U_L(n,d) \\ &\le n-3, \end{aligned}$$

by Part (1). □

Recall that $\mathbf{P} := \mathbf{P}(F_x^*)$. The exact sequence (3.2) on $Y\times\mathbf{P}$ fits in a commutative diagram D_1 of exact sequences

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & p_1^*F(-x) & = & p_1^*F(-x) & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{E} & \longrightarrow & p_1^*F & \longrightarrow & i_*\mathcal{O}_{\mathbf{P}}(1) & \longrightarrow & 0\,. \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & i_*\Omega^1_{\mathbf{P}}(1) & \longrightarrow & F_x\otimes_{\mathbb{C}} i_*\mathcal{O}_{\mathbf{P}} & \longrightarrow & i_*\mathcal{O}_{\mathbf{P}}(1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Since F is (0, 1)-stable, $\mathcal{E}$ is a family of stable vector bundles of rank n and degree $d > n(2g-2)$ over Y parametrised by $\mathbf{P}$. Hence by taking the direct image by p_2 and using the isomorphism (3.3), we get the following exact diagram D_2 over $\mathbf{P}$:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & H^0(Y,F(-x))\otimes\mathcal{O}_{\mathbf{P}} & = & H^0(Y,F(-x))\otimes\mathcal{O}_{\mathbf{P}} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \psi_{F,x}^*\boldsymbol{E}_L(-j) & \longrightarrow & H^0(Y,F)\otimes\mathcal{O}_{\mathbf{P}} & \longrightarrow & \mathcal{O}_{\mathbf{P}}(1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Omega^1_{\mathbf{P}}(1) & \longrightarrow & F_x\otimes_{\mathbb{C}}\mathcal{O}_{\mathbf{P}} & \longrightarrow & \mathcal{O}_{\mathbf{P}}(1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & H^1(Y,F(-x))\otimes\mathcal{O}_{\mathbf{P}} & = & H^1(Y,F(-x))\otimes\mathcal{O}_{\mathbf{P}} & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}.$$

4.1 *Proof of main theorem*

Proof. Let $\mathcal{F}\subset\mathbf{E}_L$ be a subsheaf of rank r with $0<r<r(\mathbf{E}_L)$. We fix nonsingular points $x_1,\ldots,x_p\in Y$ with $p>d/n$. Let $E\in U'_L(n,d)$ such that E satisfies the conditions of Lemma 4.2. The fibre $(\mathbf{E}_L)_E = H^0(Y,E)$ and by Lemma 4.2(2)(b), any nonzero $v\in\mathcal{F}_E$ has a nonzero image $s\in H^0(Y,E)$. If $s(x_i)=0$ for all i, then $s\in H^0(Y,E(-\sum_i x_i))$. Since E is stable and x_i are nonsingular points, $E(-\sum_i x_i)$ is stable. As $p>d/n$, this implies that $H^0(Y,E(-\sum_i x_i))=0$ so that $s=0$. Hence there is some x_i such that $s(x_i)\neq 0$. Write $x:=x_i$, we can choose a general line $\ell\subset E_x$ such that $s(x)\notin\ell$ and the associated bundle F is such that $\mathcal{F}$ is locally free on $P(F, x)$ outside a subvariety of codimension at least 2 (Lemma 4.2(2)(c)).

Let $\mathcal{F}_1$ be the image of $\psi^*_{F,x}\mathcal{F}(-j)$ in $\psi^*_{F,x}\mathbf{E}_L(-j)$ (see diagram D_2). Denote by I the image of the induced homomorphism

$$\mathcal{F}_1 \longrightarrow \Omega^1_{\mathbf{P}}(1)\,. \tag{4.1}$$

Since $s(x) \notin \ell$ and ℓ is the image of $F(-x)_x$ in E_x (see Diagram D_0), one has $s \notin H^0(Y, F(-x))$. Hence s goes to a non-zero element in I, in particular, $I \neq 0$. Since $\Omega^1_{\mathbf{P}}(1)$ is a stable vector bundle of degree -1, one has $d(I) \leq -1$. The kernel of the homomorphism (4.1) is a subsheaf of the trivial sheaf $H^0(Y, F(-x)) \otimes \mathcal{O}_{\mathbf{P}}$ and hence has degree at most 0 so that $d(\mathcal{F}_1) \leq -1$. Since $\mathcal{F}_1$ is isomorphic to $\psi^*_{F,x}\mathcal{F}(-j)$ outside a subvariety of codimension at least 2, we have

$$d\left(\psi^*_{F,x}\mathcal{F}(-j)\right) = d(\mathcal{F}_1) \leq -1\,.$$

The first column of the diagram D_2 shows that $d(\psi^*_{F,x}\mathbf{E}_L(-j)) = -1$. Now

$$\frac{d\left(\psi^*_{F,x}\mathcal{F}(-j)\right)}{r} \leq \frac{-1}{r} < \frac{-1}{r(\mathbf{E}_L)} = \frac{d\left(\psi^*_{F,x}\mathbf{E}_L(-j)\right)}{r(\mathbf{E}_L)}\,.$$

Thus

$$\frac{d(\psi^*_{F,x}\mathcal{F})}{r} < \frac{d(\psi^*_{F,x}\mathbf{E}_L)}{r(\mathbf{E}_L)}\,.$$

Since $\psi^*_{F,x}\theta_L = \mathcal{O}_{\mathbf{P}}(\delta)$ for $\delta > 0$ (by Lemma 3.2), it follows that

$$\frac{d(\mathcal{F})}{r(\mathcal{F})} < \frac{d(\mathbf{E}_L)}{r(\mathbf{E}_L)}\,,$$

showing that $\mathcal{F}$ satisfies the stability condition for $\mathbf{E}_L$. This completes the proof of the theorem. □

COROLLARY 4.3

The restriction of the Picard bundle to $U'_L(n, d)$ is stable with respect to any polarisation on $U'_L(n, d)$.

Proof. The corollary follows from Theorem 4.1, in view of the facts that the singular set $U_L(n, d) - U'_L(n, d)$ is of codimension at least 2 in $U_L(n, d)$ and Pic $U'_L(n, d) \cong \mathbb{Z}$. □

5. Restriction of the Picard bundle to the compactified Jacobian

In [9], we gave an embedding of the compactified Jacobian $\bar{J}(Y)$ of Y in $U_Y(n, d)$ and showed that for $d > n(2g - 1)$, the restriction of the Picard bundle $\mathbf{E}_{n,d}$ on $U_Y(n, d)$ to $\bar{J}(Y)$ is stable. In this section, we extend the result to the case $d = n(2g - 1)$. We first give a complete proof of the theorem on natural cohomology (which was proved in [9] in some cases) and then use the theorem to do this extension.

5.1 *Theorem on natural cohomology*

A coherent sheaf F on a scheme X is said to have natural cohomology if $H^i(X, F) \neq 0$ for at most one i.

Theorem 5.1 (Theorem on natural cohomology). *Let Y be an integral nodal curve of genus $g \geq 2$ with m nodes. Then a general vector bundle of rank $n \geq 1$ on Y has natural cohomology. More precisely, one has*

(*a*) *a general stable vector bundle $E \in U_Y(n, d)$ has $H^0(Y, E) = 0,\ H^1(Y, E) = 0$ if $d = n(g-1)$,*
(*b*) *a general stable vector bundle $E \in U_Y(n, d)$ has $H^0(Y, E) = 0$ if $d \leq n(g-1)$,*
(*c*) *a general stable vector bundle $E \in U_Y(n, d)$ has $H^1(Y, E) = 0$ if $d \geq n(g-1)$.*

Proof. The case $n = 1$ is proved in [9, Lemma 5.2], hence we assume that $n \geq 2$.

(a) The case $d = n(g-1)$ was proved in Theorem [9, Theorem 5.3(c)]. We recall the short proof in [9] for the convenience of the reader. For $d = n(g-1)$ (i.e. $\chi = 0$), there is a canonically defined ample line bundle, the determinant line bundle Det on $U_Y(n, d)$ and a non-trivial section θ of it such that $\theta(u) \neq 0$ if and only if the corresponding vector bundle E_u is cohomologically trivial. Thus the general bundle E is cohomologically trivial.

(b) By [8, Proposition 2.3], if E is a general stable vector bundle of rank n and degree d on Y, then for a general extension

$$0 \to F \to E \to k(y) \to 0\,, \tag{5.1}$$

the torsionfree sheaf F is stable. If $y \in Y$ is a nonsingular closed point, then F is a vector bundle. We assume that y a nonsingular point.

In the exact sequence (5.1), take $E = E_0$ a general stable vector bundle of rank n and degree $n(g-1)$. Then F is a stable vector bundle of rank n of degree $d_1 = n(g-1) - 1$ and $h^0(F) \leq h^0(E) = 0$, by Part (a). By semicontinuity, there is an open subset $U_1 \subset U_Y(n, d_1)$ such that for $E_1 \in U_1$, one has $h^0(E_1) = 0$. Now take $E = E_1 \in U_1$ in the sequence (5.1). Arguing as before, we get a subset $U_2 \subset U_Y(n, d_2)$, $d_2 = n(g-1) - 2$ such that for $E_2 \in U_2$, one has $h^0(E_2) = 0$. Continuing this way (by descending induction on degree), we have $h^0(E) = 0$ for a general stable vector bundle $E \in U_Y(n, d)$ of degree $d \leq n(g-1)$. Note that a stable vector bundle of negative degree has no non-zero sections.

(c) Part (c) follows from Part (b) by Serré duality (as in the proof of [9, Theorem 5.3(b)]). □

5.2 *Restriction of the Picard bundle*

In this section, we denote by $\bar{J}(Y)$ the compactified Jacobian of Y, the space of torsion free sheaves of rank 1 and degree -1 on Y. Fix a general stable vector bundle E_0 on Y of rank n and degree $d+n$, where $-n < d \leq 0$ such that p^*E_0 is stable. We showed that [9, Theorem 1.1] there is an embedding

$$\alpha_Y\colon\ \bar{J}(Y) \longrightarrow U_Y(n, d)\,;\ \ \alpha_Y(N) = N \otimes E_0\,. \tag{5.2}$$

For every line bundle L on Y with $d(L) = 0$, there is a closed embedding ([1, (8.8), p. 108])

$$\phi_L : Y \to \bar{J}(Y)\,, \text{ defined by } \phi_L(y) = L \otimes I_y\,,$$

where I_y is the ideal sheaf of y. We note that ϕ_L maps Y' into $J(Y)$ and nodes into $\bar{J}(Y) - J(Y)$, where Y' is the subset consisting of non-singular points of Y.

Theorem 5.2. *Let Y be a non-hyperelliptic nodal curve. Let $\mathbf{E}_{n,n(2g-1)}|\bar{J}(Y)$ denote the restriction of the Picard bundle $\mathbf{E}_{n,n(2g-1)}$ on $U_Y(n, n(2g-1))$ to $\bar{J}(Y)$. Then $\mathbf{E}_{n,n(2g-1)}|\bar{J}(Y)$ is stable with respect to any theta divisor $\theta_{\bar{J}(Y)}$ on $\bar{J}(Y)$.*

Proof. The proof is a modification of the proof of [9, Theorem 4.4] (which is on similar lines as the proof of [16, Theorem 2.5]). We sketch it briefly with details of the modifications needed. Let E be a vector bundle on $\bar{J}(Y)$ such that $\phi_L^*(E)$ is stable (respectively semistable) for any line bundle L of degree 0 on Y. Then E is stable (respectively semistable) with respect to any theta divisor $\theta_{\bar{J}(Y)}$ on $\bar{J}(Y)$ [9, Lemma 4.3]. Hence it suffices to show that $\mathbf{E}_{n,d}|_{\phi_L(Y)}$ is stable (respectively semistable) for any line bundle L of degree 0 and $d = n(2g-1)$. Let π_1 and π_2 be the first and second projections from $Y \times Y$ to Y. Denote by I_Δ the ideal sheaf of the diagonal in $Y \times Y$. Consider the two families $(Id_Y \times (\alpha_Y \circ \phi_L))^*\mathcal{U}$ and $\pi_1^*(E_0 \otimes L) \otimes I_\Delta$ of torsion free sheaves of rank n and degree d on the first factor Y parametrised by the second factor Y of $Y \times Y$. Both of them give the same morphism $Y \to U_Y(n, d)$ defined by $y \mapsto E_0 \otimes L \otimes I_y$ for $y \in Y$. Hence by the universal property of $U_Y(n, d)$, we have

$$(Id_Y \times (\alpha_Y \circ \phi_L))^*\mathcal{U} \otimes \pi_2^* M \;\cong \pi_1^*(E_0 \otimes L) \otimes I_\Delta\,, \tag{5.3}$$

for some line bundle M of degree 0 on Y.

Using equation (5.3), the fact that $\pi_U \circ (Id_Y \times (\alpha_Y \circ \phi_L)) = \alpha_Y \circ \phi_L \circ \pi_2$ and the sheaf $\mathcal{U}$ on $Y \times U_Y(n, d)$, as in [16, Theorem 2.5], one sees that

$$(\alpha_Y \circ \phi_L)^*\mathbf{E}_{n,d} \;\cong\; \pi_{2*}(\pi_1^*(E_0 \otimes L) \otimes I_\Delta) \otimes M\,. \tag{5.4}$$

Consider the exact sequence

$$0 \to \pi_1^*(E_0 \otimes L) \otimes I_\Delta \to \pi_1^*(E_0 \otimes L) \to \pi_1^*(E_0 \otimes L) \otimes \mathcal{O}_\Delta \to 0\,.$$

Since $h^1(Y, \pi_1^*(E_0 \otimes L) \otimes I_y) = 0$ for $\mu(E_0 \otimes L) \geq 2g - 1$ and for all $y \in Y$ (by [9, Corollary 2.5]), we have $R^1\pi_{2*}(\pi_1^*(E_0 \otimes L) \otimes I_\Delta) = 0$ for $(d/n) \geq 2g - 1$, i.e., $d \geq n(2g-1)$. Hence the direct image of the exact sequence under π_2 gives the exact sequence

$$0 \to \pi_{2*}(\pi_1^*(E_0 \otimes L) \otimes I_\Delta) \to H^0(E_0 \otimes L) \otimes \mathcal{O}_Y \xrightarrow{ev} E_0 \otimes L \to 0\,. \tag{5.5}$$

Thus $\pi_{2*}(\pi_1^*(E_0 \otimes L) \otimes I_\Delta) \cong M(E_0 \otimes L)$, the Syzygy bundle. By [12, Theorem 1.2], the vector bundle $M(E_0 \otimes L)$ is semistable if $\mu(E_0 \otimes L) = 2g$ (i.e. $d = n(2g-1)$). It is stable for $\mu(E_0 \otimes L) = 2g$ except when Y is hyperelliptic or $E_0 \otimes L \supset \omega_Y$, where ω_Y is the dualising sheaf of Y. Hence it is enough to show that for a general stable vector bundle E_0,

$E_0 \otimes L$ does not contain ω_Y, i.e., $H^0(E_0 \otimes L \otimes \omega_Y^{-1}) = 0$. Since Y is non-hyperelliptic, $g \geq 3$. One has $d(E_0 \otimes L \otimes \omega_Y^{-1}) = 2n \leq n(g-1)$ for $g \geq 3$. Therefore, by Theorem 5.1, for a general E_0, one has $H^0(E_0 \otimes L \otimes \omega_Y^{-1}) = 0$. This completes the proof of the theorem. □

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