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Repdigits as products of two Fibonacci or Lucas numbers

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Abstract. In this study, we show that if $2 \leq m \leq n$ and $F_m F_n$ represents a repdigit, then (m, n) belongs to the set

$$\{(2, 2), (2, 3), (3, 3), (2, 4), (3, 4), (4, 4), (2, 5), (2, 6), (2, 10)\}.$$

Also, we show that if $0 \leq m \leq n$ and $L_m L_n$ represents a repdigit, then (m, n) belongs to the set

$$\left\{ \begin{array}{c} (0, 0), (0, 1), (1, 1), (0, 2), (1, 2), (2, 2), (0, 3), \\ (1, 3), (1, 4), (1, 5), (2, 5), (3, 5), (4, 5) \end{array} \right\}.$$

Keywords. Fibonacci number; Lucas number; repdigit; Diophantine equations; linear forms in logarithms.

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1. Introduction

Let (F_n) and (L_n) be the sequences of Fibonacci and Lucas numbers given by $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, respectively. Binet formulas for Fibonacci and Lucas numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$, which are the roots of the characteristic equations $x^2 - x - 1 = 0$. It can be seen that $1 < \alpha < 2$, $-1 < \beta < 0$ and $\alpha\beta = -1$. For more about Fibonacci and Lucas sequences with their applications, see [7]. The relation between n-th Fibonacci number F_n and α is given by

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \tag{1}$$

for $n \geq 1$. Also, the relation between n-th Lucas number L_n and α is given by

$$\alpha^{n-1} \leq L_n \leq 2\alpha^n. \tag{2}$$

The inequality (1) and (2) can be proved by induction. A repdigit is a non-negative integer whose digits are all equal. Investigation of the repdigits in the second-order linear

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recurrence sequences has been of interest to mathematicians. In [8], Luca has found all Fibonacci and Lucas numbers which are repdigits. The largest repdigits in Fibonacci and Lucas sequences are $F_5 = 55$ and $L_5 = 11$. Later, in [1], it was shown that the largest Fibonacci number which is a sum of two repdigits is $F_{20} = 6765 = 6666 + 99$. Furthermore, in [9], Luca found all repdigits which are sums of three Fibonacci numbers. In this paper, we study the Diophantine equations

$$F_m F_n = \frac{d \cdot (10^k - 1)}{9} \tag{3}$$

and

$$L_m L_n = \frac{d \cdot (10^k - 1)}{9}, \tag{4}$$

where d and k are non-negative integers with $1 \leq d \leq 9$. Assuming that $2 \leq m \leq n$, it is shown that if $F_m F_n$ is a repdigit, then $m \leq 4$ and $n \leq 10$. Also, it is shown that if $L_m L_n$ is a repdigit, then $m \leq 4$ and $n \leq 5$ for $0 \leq m \leq n$. In Section 2, we introduce necessary lemmas and theorems. Then we prove our main theorems in Section 3.

2. Auxiliary results

As in [5], we use Baker's theory of linear forms to solve equations (3) and (4) in m and n for non-negative integers d and k. Since such bounds are of crucial importance in effectively solving the Diophantine equations of similar form, we start by recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with minimal polynomial

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[x],$$

where the a_i's are relatively prime integers with $a_0 > 0$ and $\eta^{(i)}$'s are conjugates of η. Then

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log(\max\{|\eta^{(i)}|, 1\})\right) \tag{5}$$

is called the logarithmic height of η. In particular, if $\eta = a/b$ is a rational number with $\gcd(a, b) = 1$ and $b > 1$, then $h(\eta) = \log(\max\{|a|, b\})$.

The following properties of logarithmic height can be found in the book [4]:

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \tag{6}$$

$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma), \tag{7}$$

$$h(\eta^m) = |m| h(\eta). \tag{8}$$

Now we give a lemma which is deduced from Corollary 2.3 of Matveev [10] and provide a large upper bound for the subscript n in equations (3) and (4) (also see, Theorem 9.4 in [3]).

Lemma 1. *Assume that $\gamma_1, \gamma_2, \ldots, \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, b_2, \ldots, b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \ldots \gamma_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1+\log D)(1+\log B)A_1 A_2 \dots A_t),$$

where

$$B \geq \max\{|b_1|, \dots, |b_t|\},$$

and $A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ *for all* $i = 1, \dots, t$.

The following lemma was proved by Dujella and Pethő [6] and is a variation of a lemma of Baker and Davenport [2]. This lemma will be used to reduce the upper bound for the subscript n in equations (3) and (4). Let the function $||\cdot||$ denote the distance from x to the nearest integer, i.e., $||x|| = \min\{|x-n| : n \in \mathbb{Z}\}$ for any real number x. Then we have as follows.

Lemma 2. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational number γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := ||\mu q|| - M||\gamma q||$. If $\epsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

in positive integers u, v and w with

$$u \leq M \quad \textit{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

The following lemmas are given in [8].

Lemma 3. *The only repdigits in the Lucas sequence are* 1, 2, 3, 7, 11.

Lemma 4. *The only repdigits in the Fibonacci sequence are* 0, 1, 2, 3, 5, 8, 55.

The identity

$$L_{2mn+k} \equiv (-1)^{mn} L_k (\mathrm{mod}\ F_m) \tag{9}$$

is well known, which will be needed in the proof of Theorem 6.

3. Main theorems

Theorem 5. *Let $2 \leq m \leq n$. A product of the form $F_m F_n$ represents a repdigit if and only if (m, n) belongs to the set*

$$\{(2,2), (2,3), (3,3), (2,4), (3,4), (4,4), (2,5), (2,6), (2,10)\}.$$

Proof. Assume that $F_m F_n$ represents a repdigit for $2 \leq m \leq n$. If $m = 2$, then we have $F_n \in \{1, 2, 3, 5, 8, 55\}$, by Lemma 4. This implies that $n = 2, 3, 4, 5, 6, 10$. Assume that

$3 \le m \le n \le 99$. Thus, a simple computation shows that $k \le 40$. Then, with the help of the *Mathematica* command,

$$\begin{aligned}
&\text{For}[n = 3, n \le 99, n++, \text{For}[m = 3, m \le n,\\
&\qquad m++, x = \text{Fibonacci}[m] * \text{Fibonacci}[n];\\
&\text{For}[d = 1, d \le 9, d++, \text{For}[k = 1, k \le 40,\\
&\qquad k++, y = (d * (10^\wedge k - 1))/(9);\\
&\ \text{If}[x == y, \text{Print}[\{m, n, k, d, x\}]];];];];],
\end{aligned}$$

we obtain only the solutions $F_m F_n = 4, 6, 9$. This gives us that $(m, n) = (3, 3), (3, 4), (4, 4)$. From now on, assume that $n \ge 100$. Combining the right side of (1) with (3), we obtain

$$10^{k-1} \le \frac{d \cdot (10^k - 1)}{9} = F_m F_n \le \alpha^{n+m-2} < \alpha^{n+m} \le \alpha^{2n}.$$

From this, we get $k \le n$. On the other hand, we rewrite equation (3) as

$$\frac{\alpha^m - \beta^m}{\sqrt{5}} \cdot \frac{\alpha^n - \beta^n}{\sqrt{5}} = \frac{d \cdot (10^k - 1)}{9}$$

to obtain

$$\frac{\alpha^{m+n}}{5} - \frac{d \cdot 10^k}{9} = -\frac{d}{9} + \frac{\alpha^m \beta^n + \beta^m \alpha^n - \beta^{m+n}}{5}. \tag{10}$$

Taking absolute values on both sides of (10), we get

$$\left|\frac{\alpha^{m+n}}{5} - \frac{d \cdot 10^k}{9}\right| \le \frac{d}{9} + \frac{\alpha^m |\beta|^n}{5} + \frac{|\beta|^m \alpha^n}{5} + \frac{|\beta|^{m+n}}{5}. \tag{11}$$

Dividing both sides of (11) by $\dfrac{\alpha^{n+m}}{5}$, we obtain

$$\begin{aligned}
\left|1 - \frac{5 \cdot d \cdot 10^k}{9 \cdot \alpha^{n+m}}\right| &\le \frac{5d}{9 \cdot \alpha^{n+m}} + \frac{|\beta|^n}{\alpha^n} + \frac{|\beta|^m}{\alpha^m} + \frac{|\beta|^{m+n}}{\alpha^{n+m}}\\
&< \frac{5}{\alpha^{2m}} + \frac{1}{\alpha^{2m}} + \frac{1}{\alpha^{2m}} + \frac{1}{\alpha^{4m}} < 8 \cdot \alpha^{-2m}.
\end{aligned}$$

From this, it follows that

$$\left|1 - \frac{5 \cdot d \cdot 10^k}{9 \cdot \alpha^{n+m}}\right| < 8 \cdot \alpha^{-2m}. \tag{12}$$

Now, let us apply Lemma 1 with $\gamma_1 := \alpha$, $\gamma_2 := 10$, $\gamma_3 := 5d/9$ and $b_1 := -(n + m)$, $b_2 := k$, $b_3 := 1$. Note that the numbers γ_1, γ_2 and γ_3 are positive real numbers and elements of the field $\mathbb{K} = \mathbb{Q}(\sqrt{5})$. It is obvious that the degree of the field $\mathbb{K}$ is 2. So $D = 2$. Now, we show that

$$\Lambda_1 := 1 - \frac{5 \cdot d \cdot 10^k}{9 \cdot \alpha^{n+m}}$$

is nonzero. For, if $\Lambda_1 = 0$, then $\alpha^{n+m} = 5 \cdot d \cdot 10^k/9$, which is impossible since α^{n+m} is irrational. Moreover, since

$$h(\gamma_1) = h(\alpha) = \frac{\log \alpha}{2} \approx \frac{0.4812\ldots}{2}, \quad h(\gamma_2) = h(10) = \log 10$$

and

$$h(\gamma_3) = h\left(\frac{5 \cdot d}{9}\right) \leq h(5) + h(d) + h(9) < 6.01,$$

by (7), we can take $A_1 := 0.5$, $A_2 := 4.7$ and $A_3 = 12.1$. Also, since $k \leq n$, $m \leq n$ and $B \geq \max\{|-(n+m)|, |k|, |1|\}$. We can take $B := 2n$. Thus, taking into account the inequality (12) and using Lemma 1, we obtain

$$8 \cdot \alpha^{-2m} > |\Lambda_1| > \exp(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2)(1 + \log 2n)(0.5)(4.7)(12.1)).$$

By a simple computation, it follows that

$$2m \log \alpha < 2.8 \cdot 10^{13} \cdot (1 + \log 2n) + 2.08. \tag{13}$$

Rearranging equation (3) as

$$\frac{\alpha^n}{\sqrt{5}} - \frac{d \cdot 10^k}{9F_m} = \frac{\beta^n}{\sqrt{5}} - \frac{d}{9F_m} \tag{14}$$

and taking absolute values on both sides of (14), we get

$$\left|\frac{\alpha^n}{\sqrt{5}} - \frac{d \cdot 10^k}{9F_m}\right| \leq \frac{|\beta|^n}{\sqrt{5}} + \frac{d}{9F_m}. \tag{15}$$

Dividing both sides of (15) by $\frac{\alpha^n}{\sqrt{5}}$, we obtain

$$\left|1 - \frac{\sqrt{5} \cdot d \cdot 10^k}{9 \cdot \alpha^n \cdot F_m}\right| \leq \frac{|\beta|^n}{\alpha^n} + \frac{d\sqrt{5}}{9 \cdot \alpha^n \cdot F_m} < \frac{1}{\alpha^{2n}} + \frac{\sqrt{5}}{\alpha^n} < 4 \cdot \alpha^{-n}.$$

From this, it follows that

$$\left|1 - \frac{\sqrt{5} \cdot d \cdot 10^k}{9 \cdot \alpha^n \cdot F_m}\right| < 4 \cdot \alpha^{-n}. \tag{16}$$

Taking $\gamma_1 := \alpha$, $\gamma_2 := 10$, $\gamma_3 := 9F_m/\sqrt{5}d$, and $b_1 := -n$, $b_2 := k$, $b_3 := -1$, we can apply Theorem 1. The numbers γ_1, γ_2 and γ_3 are positive real numbers and elements of the field $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ and so $D = 2$. Now, we show that

$$\Lambda_2 := 1 - \frac{\sqrt{5} \cdot d \cdot 10^k}{9 \cdot \alpha^n \cdot F_m}$$

is nonzero. For, if $\Lambda_2 = 0$, then $\alpha^n = \sqrt{5} \cdot d \cdot 10^k/9F_m$. Conjugating in $\mathbb{Q}(\sqrt{5})$ gives us $\beta^n = -\sqrt{5} \cdot d \cdot 10^k/9F_m$ and so, $L_n = \alpha^n + \beta^n = 0$. This is a contradiction. By using (5) and the properties of the logarithmic height (see (6), (7) and (8)), we get $h(\gamma_1) = \dfrac{\log \alpha}{2} \approx \dfrac{0.4812\ldots}{2}$, $h(\gamma_2) = \log 10$, and

$$\begin{aligned} h(\gamma_3) &\le \frac{1}{2}\left(\log 5d^2 + 2\log\left(\frac{9F_m}{\sqrt{5}d}\right)\right) = \log(\sqrt{5}d) + \log(9F_m) - \log(\sqrt{5}d) \\ &\le 1.8 + m\log\alpha. \end{aligned}$$

So we can take $A_1 := 0.5$, $A_2 := 4.7$ and $A_3 := 4 + 2m\log\alpha$. Since $k \le n$ and $B \ge \max\{|-n|, |k|, |-1|\}$, we can take $B := n$. Thus, taking into account the inequality (16) and using Lemma 1, we obtain

$$4 \cdot \alpha^{-n} > |\Lambda_2| > \exp(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 (1+\log 2)(1+\log n)(0.5)(4.7)(4+2m\log\alpha)).$$

or

$$n\log\alpha < 2.3 \cdot 10^{12} \cdot (1+\log n)(4+2m\log\alpha) + \log 4. \tag{17}$$

Inserting the equality (13) into the inequality (17), we get $n < 5.4 \cdot 10^{29}$. This inequality can be quickly verified using the *Mathematica* command

$$\begin{aligned} &\text{N[Reduce[}n * \text{Log[GoldenRatio]–Log[4]} < 2.3 * 10\hat{}12 * (1 + \text{Log}[n]) \\ &\quad *(2.3 * 10\hat{}13 * (1 + \text{Log}[2n]) + 6.08), n]]. \end{aligned}$$

Now, let us try to reduce the upper bound on n by applying Lemma 2 . Let

$$z_1 := k\log 10 - (n+m)\log\alpha + \log(5d/9).$$

If $z_1 > 0$, then by (12), we have the inequality

$$|z_1| = z_1 < e^{z_1} - 1 = |1 - e^{z_1}| < 8 \cdot \alpha^{-2m}$$

since $x < e^x - 1$ for $x > 0$. If $z_1 < 0$, then

$$1 - e^{z_1} = |1 - e^{z_1}| < 8 \cdot \alpha^{-2m} < \frac{1}{2}.$$

From this, we get $e^{z_1} > \frac{1}{2}$ and therefore

$$e^{|z_1|} = e^{-z_1} < 2.$$

Consequently, we get

$$|z_1| < e^{|z_1|} - 1 = e^{|z_1|}|1 - e^{z_1}| < 16 \cdot \alpha^{-2m}.$$

In both cases, the inequality

$$0 < |z_1| < 16 \cdot \alpha^{-2m}$$

is true. That is,

$$0 < |k \log 10 - (n+m)\log\alpha + \log(5d/9)| < 16 \cdot \alpha^{-2m}.$$

Dividing this inequality by $\log\alpha$, we get

$$0 < \left| k\left(\frac{\log 10}{\log\alpha}\right) - (n+m) + \left(\frac{\log(5d/9)}{\log\alpha}\right)\right| < 33.3 \cdot \alpha^{-2m}. \tag{18}$$

Now, we show that $\frac{\log 10}{\log\alpha}$ is irrational. On the contrary, assume that

$$\frac{\log 10}{\log\alpha} = \frac{p}{q}$$

for some positive integers p and q. This shows that $10^q = \alpha^p$, which is impossible since α^p is irrational. Take $\gamma = \frac{\log 10}{\log\alpha}$ and $M = 5.4 \cdot 10^{29}$. Then we found that the denominator of the 63rd convergent

$$\frac{p_{63}}{q_{63}} = \frac{1230843912905865341327206555418148}{257231164874247142657591800250993}$$

of γ exceeds $6M$. Now take

$$\mu := \frac{\log(5d/9)}{\log\alpha}.$$

In this case, considering the fact that $1 \le d \le 9$, a quick computation with *Mathematica* gives us the inequality $0 < \epsilon = \epsilon(\mu) := ||\mu q_{63}|| - M||\gamma q_{63}|| \le 0.162533$. To calculate this value of ϵ, we use the following command in the *Mathematica* program:

$$\begin{aligned}
&\text{For}[d = 1, d \le 9, d++, z = \text{Max}[\text{Denominator}[\text{Convergents}[\gamma, k]]];\\
&\quad x = \text{Min}[N[\text{Abs}[\text{FractionalPart}[\mu * z]], 20], 1\\
&\qquad -N[\text{Abs}[\text{FractionalPart}[\mu * z]], 20]]\\
&\qquad -M * \text{Min}[N[\text{Abs}[\text{FractionalPart}[\gamma * z]], 20], 1\\
&\qquad -N[\text{Abs}[\text{FractionalPart}[\gamma * z]], 20]];\\
&\ \ \text{If}[z > 6 * M, \text{Print}[\{d, x\}];];].
\end{aligned} \tag{19}$$

Let $A := 33.3$, $B := \alpha$ and $w := 2m$ in Lemma 2. Thus, we can say that the inequality (18) has no solution for

$$2m = w \ge \frac{\log(Aq_{63}/\epsilon(\mu))}{\log B} \ge \frac{\log(Aq_{63}/0.162533)}{\log B} \ge 162.9.$$

So

$$m \le 81.$$

Here and later on, for evaluating this task, we use the following command in the *Mathematica* program:

$$N\left[\frac{\mathrm{Log}[A * q_k/\epsilon]}{\mathrm{Log}[B]}\right]. \tag{20}$$

Substituting this upper bound for m into (17), we obtain $n < 1.5 \cdot 10^{16}$. Now, let

$$z_2 := k \log 10 - n \log\alpha + \log(\sqrt{5}d/9F_m).$$

If $z_2 > 0$, then by (16), we get the inequality

$$|z_2| = z_2 < e^{z_2} - 1 = |1 - e^{z_2}| < 4 \cdot \alpha^{-n}.$$

If $z_2 < 0$, then $1 - e^{z_2} = |1 - e^{z_2}| < 4 \cdot \alpha^{-n} < \dfrac{1}{4}$ as $n \geq 100$. Therefore, we get $e^{z_2} > \frac{3}{4}$ and so $e^{|z_2|} = e^{-z_2} < \frac{4}{3}$. Thus it follows that

$$|z_2| < e^{|z_2|} - 1 = e^{|z_2|}|1 - e^{z_2}| < 5.4 \cdot \alpha^{-n}.$$

This means that the inequality

$$0 < |z_2| < 5.4 \cdot \alpha^{-n}$$

is always true. That is,

$$0 < |k \log 10 - n \log\alpha + \log(\sqrt{5}d/9F_m)| < 5.4 \cdot \alpha^{-n}.$$

Dividing both sides of the above inequality by $\log\alpha$, we obtain

$$0 < \left|k\left(\frac{\log 10}{\log\alpha}\right) - n + \frac{\log(\sqrt{5}d/9F_m)}{\log\alpha}\right| < 11.3 \cdot \alpha^{-n}. \tag{21}$$

Putting $\gamma := \dfrac{\log 10}{\log\alpha}$ and taking $M := 1.5 \cdot 10^{16}$, we found that the denominator of the 40-th convergent

$$\frac{p_{40}}{q_{40}} = \frac{41924177609269798247}{8761634947982290092}$$

of γ exceeds $6M$. Taking

$$\mu := \frac{\log(\sqrt{5}d/9F_m)}{\log\alpha}$$

and considering the fact that $m \leq 81$ and $1 \leq d \leq 9$, a quick computation with *Mathematica* gives us the inequality $0 < \epsilon = \epsilon(\mu) = ||\mu q_{40}|| - M||\gamma q_{40}|| \leq 0.497798$.

Let $A := 11.3$, $B := \alpha$ and $w := n$ in Lemma 2. Thus, with the help of *Mathematica* command given in (20), we can say that the inequality (21) has no solution for

$$n = w \geq \frac{\log(Aq_{40}/\epsilon)}{\log B} \geq \frac{\log(Aq_{40}/0.497798)}{\log B} \geq 97.1283.$$

Therefore $n \leq 97$. This contradicts our assumption that $n \geq 100$. Thus, the proof is completed. □

Theorem 6. *Let $0 \leq m \leq n$. A product of the form $L_m L_n$ represents a repdigit if and only if (m, n) belongs to the set*

$$\left\{ \begin{array}{c} (0,0), (0,1), (1,1), (0,2), (1,2), (2,2), (0,3), \\ (0,5), (1,3), (1,4), (1,5), (2,5), (3,5), (4,5) \end{array} \right\}.$$

Proof. Assume that $L_m L_n$ represents a repdigit for $0 \leq m \leq n$. If $m = 0$, then $2L_n = \frac{d \cdot (10^k - 1)}{9}$ for some non-negative k and d. Therefore d must be even. That is, there exists $d_1 \in \{1, 2, 3, 4\}$ such that $d = 2d_1$. Thus $L_n = \frac{d_1 \cdot (10^k - 1)}{9}$. By Lemma 3, we get $L_n \in \{1, 2, 3, 4, 11\}$ and this implies that

$$(m, n) = (0, 0), (0, 1), (0, 2), (0, 3), (0, 5).$$

If $m = 1$, then L_n is a repdigit and so by Lemma 3, we get $L_n \in \{1, 3, 4, 7, 11\}$.This shows that

$$(m, n) = (1, 1), (1, 2), (1, 3), (1, 4), (1, 5).$$

Now assume that $2 \leq m \leq n \leq 99$. Thus, a simple computation shows that $k \leq 41$. Then by using the *Mathematica* command given for Fibonacci numbers in the proof of Theorem 5, we obtain the solutions

$$(m, n) = (2, 2), (2, 5), (3, 5), (4, 5).$$

Here, *Mathematica* command for Lucas numbers is "*LucasL*". From now on, assume that $n \geq 100$. Then

$$L_{100} \leq L_m L_n \leq \frac{d \cdot (10^k - 1)}{9} \leq 10^k - 1,$$

which gives us

$$20 \leq \frac{\log(1 + L_{100})}{\log 10} \leq k.$$

That is, $k \geq 20$. Combining the right side of inequality of (2) with (4), we obtain

$$10^{k-1} \leq \frac{d \cdot (10^k - 1)}{9} = L_m L_n \leq 4\alpha^{n+m} \leq 4\alpha^{2n}.$$

From this, we get $k < n$. On the other hand, we can rewrite equation (4) as

$$(\alpha^m + \beta^m) \cdot (\alpha^n + \beta^n) = \frac{d \cdot (10^k - 1)}{9}$$

to obtain

$$\alpha^{m+n} - \frac{d \cdot 10^k}{9} = -\frac{d}{9} - \alpha^m \beta^n - \beta^m \alpha^n - \beta^{m+n}. \tag{22}$$

Taking absolute values on both sides of equation (22), we get

$$\left| \alpha^{m+n} - \frac{d \cdot 10^k}{9} \right| \leq \frac{d}{9} + \alpha^m |\beta|^n + |\beta|^m \alpha^n + |\beta|^{m+n}. \tag{23}$$

Dividing both sides of (23) by α^{n+m}, we obtain

$$\begin{aligned} \left| 1 - \frac{d \cdot 10^k}{9 \cdot \alpha^{n+m}} \right| &\leq \frac{d}{9 \cdot \alpha^{n+m}} + \frac{|\beta|^n}{\alpha^n} + \frac{|\beta|^m}{\alpha^m} + \frac{|\beta|^{m+n}}{\alpha^{n+m}} \\ &< \frac{1}{\alpha^{2m}} + \frac{1}{\alpha^{2m}} + \frac{1}{\alpha^{2m}} + \frac{1}{\alpha^{4m}} < 4 \cdot \alpha^{-2m}. \end{aligned}$$

From this, it follows that

$$\left| 1 - \frac{d \cdot 10^k}{9 \cdot \alpha^{n+m}} \right| < 4 \cdot \alpha^{-2m}. \tag{24}$$

Now, let us apply Lemma 1 with $\gamma_1 := \alpha$, $\gamma_2 := 10$, $\gamma_3 := d/9$ and $b_1 := -(n+m)$, $b_2 := k$, $b_3 := 1$. Note that the numbers γ_1, γ_2 and γ_3 are positive real numbers and elements of the field $K = \mathbb{Q}(\sqrt{5})$. It is obvious that the degree of the field K is 2. So $D = 2$. It can be seen that $\Lambda_3 := 1 - d \cdot 10^k/9 \cdot \alpha^{n+m}$ is nonzero. Moreover, since

$$h(\gamma_1) = h(\alpha) = \frac{\log \alpha}{2} \approx \frac{0.4812\ldots}{2}, h(\gamma_2) = h(10) = \log 10$$

and

$$h(\gamma_3) = h\left(\frac{d}{9}\right) \leq h(d) + h(9) < 4.4,$$

by (7), we can take $A_1 := 0.5$, $A_2 := 4.7$ and $A_3 := 8.8$. On the other hand, as $k < n$, $m \leq n$ and $B \geq \max\{|-(n+m)|, |k|, |1|\}$, we can take $B := 2n$. Thus, taking into account the inequality (24) and using Theorem 1, we obtain

$$4 \cdot \alpha^{-2m} > |\Lambda_3| > \exp(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 (1 + \log 2)(1 + \log 2n)(0.5)(4.7)(8.8)).$$

By a simple computation, it follows that

$$2m \log \alpha < 2.1 \cdot 10^{13} \cdot (1 + \log 2n) + 1.4. \tag{25}$$

Rearranging equation (4) as

$$\alpha^n - \frac{d \cdot 10^k}{9L_m} = -\beta^n - \frac{d}{9L_m} \tag{26}$$

and taking absolute values on both sides of equation (26), we get

$$\left|\alpha^n - \frac{d \cdot 10^k}{9L_m}\right| \le |\beta|^n + \frac{d}{9L_m}. \tag{27}$$

Dividing both sides of (27) by α^n, we obtain

$$\left|1 - \frac{d \cdot 10^k}{9 \cdot \alpha^n \cdot L_m}\right| \le \frac{|\beta|^n}{\alpha^n} + \frac{d}{9 \cdot \alpha^n \cdot L_m} < \frac{1}{\alpha^{2n}} + \frac{1}{\alpha^n} < 2 \cdot \alpha^{-n}.$$

From this, it follows that

$$\left|1 - \frac{d \cdot 10^k}{9 \cdot \alpha^n \cdot L_m}\right| < 2 \cdot \alpha^{-n}. \tag{28}$$

Taking $\gamma_1 := \alpha, \gamma_2 := 10, \gamma_3 := 9L_m/d$ and $b_1 := -n, b_2 := k, b_3 := -1$, we can apply Lemma 1. The numbers γ_1, γ_2 and γ_3 are positive real numbers and elements of the field $K = \mathbb{Q}(\sqrt{5})$ and so, $D = 2$. One can verify that $\Lambda_4 := 1 - d \cdot 10^k/9 \cdot \alpha^n \cdot L_m \neq 0$. By using (5) and the properties of the logarithmic height, we get $h(\gamma_1) = \frac{\log\alpha}{2} \approx \frac{0.4812\ldots}{2}$, $h(\gamma_2) = \log 10$ and

$$h(\gamma_3) \le \log d + \log\left(\frac{9L_m}{d}\right) = \log(9L_m) < 2.9 + m\log\alpha.$$

So, we can take $A_1 := 0.5$, $A_2 := 4.7$ and $A_3 := 6 + 2m\log\alpha$. As $k < n$ and $B \ge \max\{|-n|, |k|, |-1|\}$, we can take $B := n$. Thus, taking into account the inequality (28) and using Lemma 1, we obtain

$$\begin{aligned} 2 \cdot \alpha^{-n} > |\Lambda_4| > \exp(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2)(1 \\ + \log n)(0.5)(4.7)(6 + 2m\log\alpha)) \end{aligned}$$

or

$$n\log\alpha < 2.3 \cdot 10^{12} \cdot (1 + \log n)(6 + 2m\log\alpha) + \log 2. \tag{29}$$

Using the inequalities (25) and (29), a computer search with *Mathematica* command

$$\begin{aligned} &N[\text{Reduce}[n * \text{Log[GoldenRatio]-Log[2]} < 2.3 * 10^\wedge 12 * \\ &(1 + \text{Log}[n]) * (2.1 * 10^\wedge 13 * (1 + \text{Log}[2n]) + 6), n]] \end{aligned}$$

gives us that $n < 4.9 \cdot 10^{29}$. Now, let us try to reduce the upper bound on n by applying Lemma 2. Let

$$z_3 := k\log 10 - (n + m)\log\alpha + \log(d/9).$$

If $z_3 > 0$, then we have the inequality

$$|z_3| = z_3 < e^{z_3} - 1 = |1 - e^{z_3}| < 4 \cdot \alpha^{-2m}.$$

If $z_3 < 0$, then by (24), we get the inequality

$$1 - e^{z_3} = |1 - e^{z_3}| < 4 \cdot \alpha^{-2m} < \frac{3}{5}.$$

From this, we get $e^{z_3} > \frac{2}{5}$ and therefore,

$$e^{|z_3|} = e^{-z_3} < \frac{5}{2}.$$

Consequently, we get

$$|z_3| < e^{|z_3|} - 1 = e^{|z_3|}|1 - e^{z_3}| < 10 \cdot \alpha^{-2m}.$$

In both cases, the inequality

$$0 < |z_3| < 10 \cdot \alpha^{-2m}$$

is true. That is,

$$0 < |k \log 10 - (n + m) \log \alpha + \log(d/9)| < 10 \cdot \alpha^{-2m}.$$

Dividing this inequality by $\log \alpha$, we get

$$0 < \left| k \left(\frac{\log 10}{\log \alpha} \right) - (n + m) + \left(\frac{\log(d/9)}{\log \alpha} \right) \right| < 21 \cdot \alpha^{-2m}. \tag{30}$$

Take $\gamma := \dfrac{\log 10}{\log \alpha} \notin \mathbb{Q}$ and $M := 4.9 \cdot 10^{29}$. Then we found that q_{61}, the denominator of the 61st convergent

$$\frac{p_{61}}{q_{61}} = \frac{1230843912905865341327206554181 48}{257231164874247142657591800250 93}$$

of γ exceeds $6M$. Take

$$\mu := \frac{\log(d/9)}{\log \alpha}.$$

Now assume that $1 \leq d \leq 8$. In this case, a quick computation with *Mathematica* command given in (19) gives us the inequality $0 < \epsilon = \epsilon(\mu) := ||\mu q_{61}|| - M||\gamma q_{61}|| \leq 0.201992$. Let $A := 21$, $B := \alpha$ and $w := 2m$ in Lemma 2. Then with the help of the *Mathematica* command given in (20), we can say that the inequality (30) has no solution for

$$2m = w \geq \frac{\log(Aq_{61}/\epsilon)}{\log B} \geq \frac{\log(Aq_{61}/0.201992)}{\log B} \geq 159.948.$$

So

$$m \leq 79.$$

Substituting this upper bound for m into (29), we obtain $n < 1.5 \cdot 10^{16}$. Now, let

$$z_4 := k \log 10 - n \log \alpha + \log(d/9L_m).$$

If $z_4 > 0$, then by (28), we have the inequality

$$|z_4| = z_4 < e^{z_4} - 1 = |1 - e^{z_4}| < 2 \cdot \alpha^{-n}.$$

If $z_4 < 0$, then $1 - e^{z_4} = |1 - e^{z_4}| < 2 \cdot \alpha^{-n} < \frac{1}{4}$ since $n \geq 100$. Therefore, we get $e^{z_4} > \frac{3}{4}$ and so, $e^{|z_4|} = e^{-z_4} < \frac{4}{3}$. Then it follows that

$$|z_4| < e^{|z_4|} - 1 = e^{|z_4|}|1 - e^{z_4}| < 2.7 \cdot \alpha^{-n}.$$

Therefore, it is always true that

$$0 < |z_4| < 2.7 \cdot \alpha^{-n}.$$

That is,

$$0 < |k \log 10 - n \log \alpha + \log(d/9L_m)| < 2.7 \cdot \alpha^{-n}.$$

Dividing both sides of the above inequality by $\log \alpha$, we get

$$0 < \left| k \left(\frac{\log 10}{\log \alpha} \right) - n + \frac{\log(d/9L_m)}{\log \alpha} \right| < 5.7 \cdot \alpha^{-n}. \tag{31}$$

Putting $\gamma := \frac{\log 10}{\log \alpha}$ and taking $M := 1.5 \cdot 10^{16}$, we found that q_{39}, the denominator of the 39-th convergent

$$\frac{p_{39}}{q_{39}} = \frac{33149137033495047668}{6927759924953310265}$$

of γ exceeds $6M$. Taking

$$\mu := \frac{\log(d/9L_m)}{\log \alpha}$$

and considering the fact that $m \leq 79$ and $1 \leq d \leq 8$, a quick computation with the *Mathematica* command given in (19) gives us the equality $0 < \epsilon = \epsilon(\mu) := ||\mu q_{39}|| - M||\gamma q_{39}|| \leq 0.498401$. Let $A := 5.7,\ B := \alpha$ and $w := n$ in Lemma 2. Then with the help of *Mathematica* command given in (20), we can say that the inequality (31) has no solution for

$$n = w \geq \frac{\log(Aq_{39}/\epsilon)}{\log B} \geq \frac{\log(Aq_{39}/0.498401)}{\log B} \geq 95.2156.$$

Therefore $n \leq 95$. This contradicts our assumption that $n \geq 100$. Now assume that $d = 9$. Then

$$L_n L_m = 10^k - 1$$

and therefore, $9 | L_n L_m$. Let $m = 24k + r, n = 24t + s, 0 \leq r, s \leq 23$. Thus

$$L_m = L_{24k+r} \equiv L_r (\bmod F_{12}) \quad \text{and} \quad L_n = L_{24t+s} \equiv L_s (\bmod F_{12})$$

by (9). Then it follows that

$$L_m L_n \equiv L_r L_s (\bmod F_{12})$$

Since $k \geq 3$, it follows that

$$L_n L_m = 10^k - 1 \equiv -1 \equiv 7 (\bmod 8) \tag{32}$$

Moreover, since $9 | L_n L_m$, we get

$$L_m L_n \equiv 63 (\bmod 72).$$

On the other hand, since $L_m L_n \equiv L_r L_s (\bmod F_{12})$, we get $L_m L_n \equiv L_r L_s (\bmod 72)$ and therefore, $L_r L_s \equiv 63 (\bmod 72)$. A simple computation shows that $L_r L_s \equiv 63 (\bmod 72)$ is impossible for $0 \leq r, s \leq 23$. This completes the proof of the theorem. □

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