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On the number of distinct exponents in the prime factorization of an integer

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Abstract. Let $f(n)$ be the number of distinct exponents in the prime factorization of the natural number n. We prove some results about the distribution of $f(n)$. In particular, for any positive integer k, we obtain that

$$\#\{n \leq x : f(n) = k\} \sim A_k x$$

and

$$\#\{n \leq x : f(n) = \omega(n) - k\} \sim \frac{Bx(\log\log x)^k}{k! \log x},$$

as $x \to +\infty$, where $\omega(n)$ is the number of prime factors of n and $A_k, B > 0$ are some explicit constants. The latter asymptotic extends a result of Aktaş and Ram Murty (*Proc. Indian Acad. Sci. (Math. Sci.)* **127(3)** (2017) 423–430) about numbers having mutually distinct exponents in their prime factorization.

Keywords. Prime factorization; squarefree numbers; powerful number.

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1. Introduction

Let $n = p_1^{a_1} \cdots p_s^{a_s}$ be the factorization of the natural number $n > 1$, where $p_1 < \cdots < p_s$ are prime numbers and $a_1, \ldots, a_s$ are positive integers. Several functions of the exponents $a_1, \ldots, a_s$ have been studied, including their product [17], their arithmetic mean [2,4,5,7], and their maximum and minimum [11,13,15,18]. See also [3,8] for a more general function.

Let f be the arithmetic function defined by $f(1) := 0$ and $f(n) := \#\{a_1, \ldots, a_s\}$ for all natural numbers $n > 1$. In other words, $f(n)$ is the number of distinct exponents in the prime factorization of n. The first values of $f(n)$ are listed in sequence A071625 of OEIS [16].

Our first contribution is a quite precise result about the distribution of $f(n)$.

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Theorem 1.1. *There exists a sequence of positive real numbers* $(A_k)_{k\geq 1}$ *such that, given any arithmetic function* ϕ *satisfying* $|\phi(k)| < a^k$ *for some fixed* $a > 1$, *we have that the series*

$$M_\phi := \sum_{k=1}^{\infty} A_k \phi(k) \tag{1}$$

converges and

$$\sum_{n \leq x} \phi(f(n)) = M_\phi x + O_{a,\varepsilon}(x^{1/2+\varepsilon}),$$

for all $x \geq 1$ *and* $\varepsilon > 0$.

From Theorem 1.1, it follows immediately that all the moments of f are finite and that f has a limiting distribution. In particular, we highlight the following corollary.

COROLLARY 1.1

For each positive integer k, *we have*

$$\#\{n \leq x : f(n) = k\} = A_k x + O_\varepsilon(x^{1/2+\varepsilon}),$$

for all $x \geq 1$ *and* $\varepsilon > 0$.

We also provide a formula for A_k. Before stating it, we need to introduce some notations. Let ψ be the Dedekind function defined by

$$\psi(n) := n \prod_{p \mid n} \left(1 + \frac{1}{p}\right)$$

for each positive integer n, and let $(\rho_k)_{k\geq 1}$ be the family of arithmetic functions supported on squarefree numbers and satisfying

$$\rho_1(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \qquad \rho_{k+1}(n) = \begin{cases} 0 & \text{if } n = 1, \\ \frac{1}{n-1} \sum_{\substack{d \mid n \\ d < n}} \rho_k(d) & \text{if } n > 1, \end{cases}$$

for all squarefree numbers n and positive integers k.

Theorem 1.2. *We have*

$$A_k = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{\rho_k(n)}{\psi(n)}$$

for each positive integer k.

Clearly, $f(n) \leq \omega(n)$ for all positive integers n, where $\omega(n)$ denotes the number of prime factors of n. Motivated by a question of Recamán Santos [14], Aktaş and Ram Murty

[1] studied the natural numbers n such that all the exponents in their prime factorization are distinct, that is, $f(n) = \omega(n)$. They called such numbers *special numbers* (sequence A130091 of OEIS [16]) and they proved the following.

Theorem 1.3. *The number of special numbers not exceeding x is*

$$\frac{Bx}{\log x} + O\left(\frac{x}{(\log x)^2}\right),$$

for all $x \geq 2$, where

$$B := \sum_{\ell} \frac{1}{\ell}$$

and the sum of over natural numbers ℓ that are powerful and special.

Let g be the arithmetic function defined by $g(n) := \omega(n) - f(n)$ for all positive integers n. Hence, by the previous observation, g is a nonnegative function and $g(n) = 0$ if and only if n is a special number. We prove the following result about g, which extends Theorem 1.3 and it is somehow dual to Corollary 1.1.

Theorem 1.4. *For each nonnegative integer k, we have*

$$\#\{n \leq x : g(n) = k\} = \frac{Bx(\log\log x)^k}{k!\log x}\left(1 + O_k\left(\frac{1}{\log\log x}\right)\right),$$

for all $x \geq 3$.

Notation. We employ the Landau–Bachmann "Big Oh" notation O, as well as the associated Vinogradov symbol $\ll$, with their usual meaning. Any dependence of the implied constants is explicitly stated. We let ε denote an arbitrary small positive real number, not necessarily the same at each occurrence. We reserve the letter p for prime numbers.

2. Preliminaries

Recall that a natural number n is called *powerful* if $p \mid n$ implies $p^2 \mid n$, for all primes p. For all $x \geq 1$, let $\mathcal{P}(x)$ be the set of powerful numbers not exceeding x.

Lemma 2.1. *We have $\#\mathcal{P}(x) \ll x^{1/2}$ for every $x \geq 1$.*

Proof. See [9]. □

Lemma 2.2. *We have*

$$\sum_{\substack{\ell \in \mathcal{P} \\ \ell > y}} \frac{1}{\ell} \ll \frac{1}{y^{1/2}}, \quad \sum_{\ell \in \mathcal{P}(y)} \frac{1}{\ell^{1/2}} \ll \log y,$$

for all $y \geq 2$.

Proof. By Lemma 2.1 and by partial summation, we have

$$\sum_{\substack{\ell \in \mathcal{P} \\ \ell > y}} \frac{1}{\ell} = \frac{\#\mathcal{P}(t)}{t}\bigg|_{t=y}^{+\infty} + \int_y^{+\infty} \frac{\#\mathcal{P}(t)}{t^2}\,\mathrm{d}t \ll \int_y^{+\infty} \frac{\mathrm{d}t}{t^{1+1/2}} \ll \frac{1}{y^{1/2}}.$$

The proof of the second claim is similar. □

We need the following upper bound for the number of prime factors of a natural number.

Lemma 2.3. *We have*

$$\omega(n) \ll \frac{\log n}{\log\log n}$$

for all integers $n \geq 3$.

Proof. See, for example, [6, Proposition 7.10]. □

For every $x \geq 1$ and every positive integer h, let $Q(x; h)$ denote the number of squarefree numbers not exceeding x and relatively prime with h.

Lemma 2.4. *We have*

$$Q(x; h) = \frac{6}{\pi^2}\frac{h}{\psi(h)}\,x + O(4^{\omega(h)}(x^{1/2} + 1))$$

for all $x \geq 1$ *and all positive integers* h.

Proof. It follows easily from [10, Eq. 8]. □

For every $x \geq 1$ and every positive integers s, h, let $Q_s(x; h)$ denote the number of squarefree numbers not exceeding x, having exactly s prime factors, and relatively prime with h.

Lemma 2.5. *We have*

$$Q_s(x; h) = \frac{x(\log\log x)^{s-1}}{(s-1)!\log x}\left(1 + O_{\delta,s}\left(\frac{\log\log\log(h+15)}{\log\log x}\right)\right)$$

for all $x \geq 3$, $0 < \delta < 1$, *and for all integers* $1 \leq h \leq x^\delta$ *and* $s \geq 1$.

Proof. For $s = 1$, the claim follows from the Prime Number theorem, while for $h = 1$, the claim is a classic result of Landau [12]. Hence, suppose $s, h > 1$. Also, we can assume that $x \geq 3^{1/(1-\delta)}$. If $n \leq x$ is a squarefree number having exactly s prime factors such that $(n, h) > 1$, then $n = pn'$, where p is a prime number dividing h and $n' \leq x/p$ is a squarefree number having exactly $s - 1$ prime factor. Therefore,

$$0 \le Q_s(x;1) - Q_s(x;h) \le \sum_{p\,|\,h} Q_{s-1}\left(\frac{x}{p},1\right) \ll_s \sum_{p\,|\,h} \frac{x}{p}\frac{(\log\log(x/p))^{s-2}}{\log(x/p)}$$
$$\ll_\delta \frac{x(\log\log x)^{s-2}}{\log x}\sum_{p\,|\,h}\frac{1}{p} \ll \frac{x(\log\log x)^{s-1}}{\log x}\frac{\log\log\log(h+15)}{\log\log x},$$

where we used the fact that $p \le x^\delta$ and the upper bound

$$\sum_{p\,|\,h}\frac{1}{p} \le \sum_{p \le \omega(h)}\frac{1}{p} \ll \log\log(\omega(h)+2) \ll \log\log\log(h+15),$$

which in turn follows from Mertens' second theorem [6, Theorem 4.5] and the simple bound $\omega(h) \ll \log h$. Consequently,

$$Q_s(x;h) = Q_s(x;1) + O_{\delta,s}\left(\frac{x(\log\log x)^{s-1}}{\log x}\frac{\log\log\log(h+15)}{\log\log x}\right)$$
$$= \frac{x(\log\log x)^{s-1}}{(s-1)!\log x} + O_{\delta,s}\left(\frac{x(\log\log x)^{s-1}}{\log x}\frac{\log\log\log(h+15)}{\log\log x}\right),$$

as claimed. □

Finally, we need a lemma about certain sums of powers.

Lemma 2.6. *Let a_0 be an integer. For all $x_1, \ldots, x_k > 1$, we have*

$$\sum_{a_0 < a_1 < \cdots < a_k} \frac{1}{x_1^{a_1}\cdots x_k^{a_k}} = \frac{1}{(x_1\cdots x_k)^{a_0}}\prod_{j=1}^{k}\frac{1}{x_j\cdots x_k - 1},$$

where the sum is over all integers $a_1, \ldots, a_k$ satisfying $a_0 < a_1 < \cdots < a_k$.

Proof. We proceed by induction on k. For $k = 1$, we have

$$\sum_{a_0 < a_1}\frac{1}{x_1^{a_1}} = \frac{1}{x_1^{a_0+1}}\sum_{d=0}^{\infty}\frac{1}{x_1^d} = \frac{1}{x_1^{a_0}}\frac{1}{x_1 - 1}, \tag{2}$$

as claimed. Suppose that the claim is true for k, we shall prove it for $k+1$. We have

$$\sum_{a_0 < \cdots < a_{k+1}} \frac{1}{x_1^{a_1}\cdots x_{k+1}^{a_{k+1}}} = \sum_{a_0 < \cdots < a_k}\frac{1}{x_1^{a_1}\cdots x_k^{a_k}}\sum_{a_k < a_{k+1}}\frac{1}{x_{k+1}^{a_{k+1}}}$$
$$= \sum_{a_0 < \cdots < a_{k+1}}\frac{1}{x_1^{a_1}\cdots x_{k-1}^{a_{k-1}}(x_k x_{k+1})^{a_k}}\frac{1}{x_{k+1}-1}$$

$$= \frac{1}{(x_1 \cdots x_{k+1})^{a_0}} \prod_{j=1}^{k} \frac{1}{x_j \cdots x_{k+1} - 1} \frac{1}{x_{k+1} - 1}$$

$$= \frac{1}{(x_1 \cdots x_{k+1})^{a_0}} \prod_{j=1}^{k+1} \frac{1}{x_j \cdots x_{k+1} - 1},$$

where we used (2), with a_0 and x_1 replaced respectively by a_k and x_{k+1}, and the induction hypothesis. □

3. Proof of Theorem 1.1

We begin by proving that for each positive integer k, there exists $A_k > 0$ such that

$$N_k(x) := \#\{n \le x : f(n) = k\} = A_k x + O_\varepsilon(x^{1/2+\varepsilon}), \tag{3}$$

for all $x \ge 1$ and $\varepsilon > 0$. Clearly, every natural number n can be written in a unique way as $n = m\ell$, where m is a squarefree number, ℓ is a powerful number, and $(m, \ell) = 1$. If $m = 1$, then $n = \ell$ is powerful and, by Lemma 2.1, belongs to a set of cardinality $O(x^{1/2})$. If $m > 1$, then $f(n) = k$ is equivalent to $f(\ell) = k - 1$. Also, for each ℓ, there are exactly $Q(x/\ell; \ell) - 1$ choices for $m > 1$. Therefore, we have

$$N_k(x) = \sum_{\substack{\ell \in \mathcal{P}(x) \\ f(\ell) = k-1}} \left(Q\left(\frac{x}{\ell}; \ell\right) - 1\right) + O(x^{1/2}), \tag{4}$$

for all $x \ge 1$. For each positive integer $\ell \le x$, Lemma 2.3 gives $4^{\omega(\ell)} \ll_\varepsilon x^\varepsilon$. Consequently, by Lemma 2.4, we obtain

$$Q\left(\frac{x}{\ell}; \ell\right) = \frac{6}{\pi^2} \frac{x}{\psi(\ell)} + O_\varepsilon\left(\frac{x^{1/2+\varepsilon}}{\ell^{1/2}}\right), \tag{5}$$

for all positive integers $\ell \le x$. By Lemma 2.2, we have

$$\sum_{\substack{\ell \in \mathcal{P} \\ \ell > x}} \frac{1}{\psi(\ell)} < \sum_{\substack{\ell \in \mathcal{P} \\ \ell > x}} \frac{1}{\ell} \ll \frac{1}{x^{1/2}}, \tag{6}$$

for all $x \ge 1$. In particular, the series

$$A_k := \frac{6}{\pi^2} \sum_{\substack{\ell \in \mathcal{P} \\ f(\ell) = k-1}} \frac{1}{\psi(\ell)} \tag{7}$$

converges. Also, again by Lemma 2.2, we have

$$\sum_{\ell \in \mathcal{P}(x)} \frac{1}{\ell^{1/2}} \ll \log x \ll_\varepsilon x^\varepsilon. \tag{8}$$

At this point, putting together (4) and (5), and using (6) and (8), we obtain

$$\begin{aligned} N_k(x) &= \sum_{\substack{\ell \in \mathcal{P}(x) \\ f(\ell)=k-1}} \left(\frac{6}{\pi^2} \frac{x}{\psi(\ell)} + O_\varepsilon\left(\frac{x^{1/2+\varepsilon}}{\ell^{1/2}} \right) \right) + O(x^{1/2}) \\ &= A_k x + O\left(\sum_{\substack{\ell \in \mathcal{P} \\ \ell > x}} \frac{x}{\psi(\ell)} \right) + O_\varepsilon\left(\sum_{\ell \in \mathcal{P}(x)} \frac{x^{1/2+\varepsilon}}{\ell^{1/2}} \right) + O(x^{1/2}) \\ &= A_k x + O_\varepsilon(x^{1/2+\varepsilon}), \end{aligned}$$

as desired. Thus (3) is proved.

Now we shall show that

$$A_k \leq \frac{6}{\pi^2} \frac{1}{(k-1)!} \tag{9}$$

for all positive integers k. For $k = 1$, the claim is obvious since $A_1 = 6/\pi^2$. Hence, assume $k \geq 2$. If ℓ is a powerful number such that $f(\ell) = k-1$, then $\ell = m_1^{a_1} \cdots m_{k-1}^{a_{k-1}}$ for some integers $m_1, \ldots, m_{k-1} \geq 2$ and $2 \leq a_1 < \cdots < a_{k-1}$. Consequently,

$$\begin{aligned} \frac{\pi^2}{6} A_k &= \sum_{\substack{\ell \in \mathcal{P} \\ f(\ell)=k-1}} \frac{1}{\psi(\ell)} < \sum_{\substack{\ell \in \mathcal{P} \\ f(\ell)=k-1}} \frac{1}{\ell} < \prod_{j=1}^{k-1} \sum_{m=2}^{\infty} \sum_{a=j+1}^{\infty} \frac{1}{m^a} \\ &= \prod_{j=1}^{k-1} \sum_{m=2}^{\infty} \frac{1}{m^j(m-1)} \leq \prod_{j=1}^{k-1} \frac{1}{j} = \frac{1}{(k-1)!}, \end{aligned}$$

where we used the facts that

$$\sum_{m=2}^{\infty} \frac{1}{m(m-1)} = \sum_{m=2}^{\infty} \left(\frac{1}{m-1} - \frac{1}{m} \right) = 1$$

and

$$\begin{aligned} \sum_{m=2}^{\infty} \frac{1}{m^j(m-1)} &< \frac{1}{2^j} + \frac{1}{3^j \cdot 2} + \sum_{n=3}^{\infty} \frac{1}{n^{j+1}} \\ &< \frac{1}{2^j} + \frac{1}{3^j \cdot 2} + \int_2^{+\infty} \frac{\mathrm{d}t}{t^{j+1}} = \frac{1}{2^j} + \frac{1}{3^j \cdot 2} + \frac{1}{j 2^j} < \frac{1}{j}, \end{aligned}$$

for all integers $j \geq 2$. Thus (9) is proved.

Now let ϕ be an arithmetic function satisfying $|\phi(k)| < a^k$ for all positive integers k, where $a > 1$ is some constant. From (9) it follows that series (1) converges. Define

$$y := 2a + \lfloor C \log x / \log\log(x+2) \rfloor,$$

where $C > 0$ is some absolute constant. Since $f(n) \leq \omega(n)$ for all positive integers n, by Lemma 2.3, we can choose C sufficiently large so that $f(n) \leq y$ for all natural numbers $n \leq x$. Moreover, from (9) and $y \geq 2a$, we get that

$$\sum_{k>y} A_k \phi(k) \ll \sum_{k>y} \frac{a^k}{(k-1)!} < \frac{a^{y+1}}{y!} \sum_{j=0}^{\infty} \left(\frac{a}{y} \right)^j \ll_a \frac{a^y}{y!} \ll_a \frac{1}{x^{1/2}} \tag{10}$$

and

$$a^y y \ll_{a,\varepsilon} x^{\varepsilon}, \tag{11}$$

for all $x \geq 1$. Therefore, putting together (3), (10) and (11), we have

$$\begin{aligned}\sum_{n \leq x} \phi(f(n)) = \sum_{k \leq y} N_k(x)\phi(k) &= \sum_{k \leq y} (A_k \phi(k)x + O_\varepsilon(\phi(k)x^{1/2+\varepsilon})) \\ &= M_\phi x + O\left(\sum_{k > y} A_k \phi(k)x\right) + O_\varepsilon(a^y y x^{1/2+\varepsilon}) \\ &= M_\phi x + O_{a,\varepsilon}(x^{1/2+\varepsilon}),\end{aligned}$$

for all $x \geq 1$ and $\varepsilon > 0$. The proof is complete.

4. Proof of Theorem 1.2

Recall that A_k is defined by (7). For $k = 1$, the claim is obvious, since $f(\ell) = 0$ if and only if $\ell = 1$. Hence, assume $k \geq 2$. If ℓ is a powerful number such that $f(\ell) = k - 1$, then ℓ can be written in a unique way as $\ell = m_1^{a_1} \cdots m_{k-1}^{a_{k-1}}$, where $1 < a_1 < \cdots < a_{k-1}$ are integers and $m_1, \ldots, m_{k-1} > 1$ are pairwise coprime squarefree numbers. Therefore, from (7) and Lemma 2.6, we obtain

$$\begin{aligned}\frac{\pi^2}{6} A_k &= \sum_{m_1,\ldots,m_{k-1}} \sum_{1 < a_1 < \cdots < a_{k-1}} \frac{1}{\psi(m_1^{a_1} \cdots m_{k-1}^{a_{k-1}})} \\ &= \sum_{m_1,\ldots,m_{k-1}} \frac{m_1 \cdots m_{k-1}}{\psi(m_1 \cdots m_{k-1})} \sum_{1 < a_1 < \cdots < a_{k-1}} \frac{1}{m_1^{a_1} \cdots m_{k-1}^{a_{k-1}}} \\ &= \sum_{m_1,\ldots,m_{k-1}} \frac{1}{\psi(m_1 \cdots m_{k-1})} \prod_{j=1}^{k-1} \frac{1}{m_j \cdots m_{k-1} - 1},\end{aligned}$$

where, here and in the rest of the proof, in summation subscripts $m_1, \ldots, m_{k-1}$ are meant to be pairwise coprime, squarefree and greater than 1. At this point, it is enough to prove that

$$\sum_{n = m_1 \cdots m_{k-1}} \prod_{j=1}^{k-1} \frac{1}{m_j \cdots m_{k-1} - 1} = \rho_k(n)$$

for all squarefree numbers $n > 1$. We proceed by induction on k. For $k = 2$, the claim is true since

$$\frac{1}{n-1} = \frac{\rho_1(1)}{n-1} = \frac{1}{n-1} \sum_{\substack{d \mid n \\ d < n}} \rho_1(d) = \rho_2(n),$$

for all squarefree numbers $n > 1$. Assuming that the claim is true for k, we shall prove it for $k + 1$. We have

$$\begin{aligned}\sum_{n=m_1\cdots m_k}\prod_{j=1}^{k}\frac{1}{m_j\cdots m_k-1} &= \frac{1}{n-1}\sum_{m_1\,|\,n}\ \sum_{n/m_1=m_2\cdots m_k}\prod_{j=2}^{k}\frac{1}{m_j\cdots m_k-1}\\ &= \frac{1}{n-1}\sum_{m_1\,|\,n}\rho_k(n/m_1)\\ &= \frac{1}{n-1}\sum_{\substack{d\,|\,n\\ d<n}}\rho_k(d) = \rho_{k+1}(n),\end{aligned}$$

for all squarefree numbers $n > 1$, as desired. The proof is complete.

5. Proof of Theorem 1.4

We have to count the number of positive integers $n \le x$ such that $g(n) = k$. As in the proof of Theorem 1.1, every n can be written in a unique way as $n = m\ell$, where m is a squarefree number, ℓ is a powerful number, and $(m, \ell) = 1$. If $m = 1$, then $n = \ell$ is powerful and by Lemma 2.1, belongs to a set of cardinality $O(x^{1/2})$. If $m > 1$, then

$$\omega(m) = \omega(n) - \omega(\ell) = g(n) + f(n) - f(\ell) - g(\ell) = k + 1 - g(\ell).$$

In particular, $1 \le \omega(m) \le k+1$. Assume x sufficiently large, and put $y := (\log x)^2$. Then, by Lemma 2.2, the number of $n \le x$ such that $\ell > y$ is at most

$$\sum_{\substack{\ell\in\mathcal{P}\\ \ell>y}}\frac{x}{\ell} \ll \frac{x}{y^{1/2}} = \frac{x}{\log x}.$$

Therefore,

$$M_k(x) := \#\{n \le x : g(n) = k\} = \sum_{s=1}^{k+1}\ \sum_{\substack{\ell\in\mathcal{P}(y)\\ g(\ell)=k+1-s}} Q_s\left(\frac{x}{\ell};\ell\right) + O\left(\frac{x}{\log x}\right). \tag{12}$$

For each nonnegative integer r, put

$$B_r := \sum_{\substack{\ell\in\mathcal{P}\\ g(\ell)=r}}\frac{1}{\ell}.$$

Note that, in light of Lemma 2.2, the series defining B_r converges and, more precisely,

$$\sum_{\substack{\ell\in\mathcal{P}(y)\\ g(\ell)=r}}\frac{1}{\ell} = B_r + O\left(\frac{1}{y^{1/2}}\right) = B_r + O\left(\frac{1}{\log x}\right). \tag{13}$$

Clearly, we can assume x sufficiently large so that $x/y \ge 3$ and $y \le x^{\delta/(1+\delta)}$, for some fixed $0 < \delta < 1$. Hence, applying Lemma 2.5, we obtain

$$Q_s\left(\frac{x}{\ell};\ell\right) = \frac{x(\log\log(x/\ell))^{s-1}}{\ell(s-1)!\log(x/\ell)}\left(1 + O_k\left(\frac{\log\log\log(\ell+15)}{\log\log(x/\ell)}\right)\right)$$

$$= \frac{x(\log\log x)^{s-1}}{\ell(s-1)!\log x}\left(1+O_k\left(\frac{\log \ell}{\log x}\right)\right)\left(1+O_k\left(\frac{\log\log\log(\ell+15)}{\log\log x}\right)\right)$$
$$= \frac{x(\log\log x)^{s-1}}{\ell(s-1)!\log x}\left(1+O_k\left(\frac{\log(\ell+1)}{\log\log x}\right)\right),$$

for all positive integers $s \le k+1$ and $\ell \le y$. Consequently,

$$\sum_{\substack{\ell\in\mathcal{P}(y)\\ g(\ell)=k+1-s}} Q_s\left(\frac{x}{\ell};\ell\right)$$
$$= \frac{x(\log\log x)^{s-1}}{(s-1)!\log x}\sum_{\substack{\ell\in\mathcal{P}(y)\\ g(\ell)=k+1-s}}\frac{1}{\ell}\left(1+O_k\left(\frac{\log(\ell+1)}{\log\log x}\right)\right)$$
$$= \frac{x(\log\log x)^{s-1}}{(s-1)!\log x}\left(B_{k+1-s}+O\left(\frac{1}{\log x}\right)+O_k\left(\frac{1}{\log\log x}\right)\right)$$
$$= \frac{x(\log\log x)^{s-1}}{(s-1)!\log x}\left(B_{k+1-s}+O_k\left(\frac{1}{\log\log x}\right)\right), \tag{14}$$

where we used (13) and the fact that the series

$$\sum_{\ell\in\mathcal{P}}\frac{\log(\ell+1)}{\ell}$$

converges. Thus, putting together (12) and (14), and noting that $B_0 = B$, we obtain

$$M_k(x) = \frac{Bx(\log\log x)^k}{k!\log x}\left(1+O_k\left(\frac{1}{\log\log x}\right)\right),$$

as desired. The proof is complete.

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