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A set of generators for the Picard modular group $SU(2, 1, \mathcal{O}_2)$

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Abstract. In this work, we find a system of generators for the Picard modular group $SU(2, 1, \mathcal{O}_2)$. This system contains five transformations, three translations a rotation and an involution.

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1. Introduction

Let $\mathcal{O}_d$ be the ring of algebraic integers in the imaginary quadratic number field $K = Q(\sqrt{-d})$, where d is a positive and square free integer. According to [5], the elements of the ring $\mathcal{O}_d$ can be described as follows:

$$\mathcal{O}_d = \begin{cases} \mathbb{Z}[i\sqrt{d}] & \text{if } d \equiv 1, 2 \pmod 4 \\ \mathbb{Z}[\frac{1+i\sqrt{d}}{2}] & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

It is well known that the ring $\mathcal{O}_d$ is Euclidean for positive square free integer d if and only if $d = 1, 2, 3, 7, 11$; see [7].

The subgroups of $SU(2, 1)$ with entries in $\mathcal{O}_d$ are called Picard modular groups and are denoted by $SU(2, 1, \mathcal{O}_d)$. They are a natural generalization of Bianchi groups $PSL_2(\mathcal{O}_d)$; the simplest arithmetically defined discrete groups.

It is interesting to get a system of generators for Picard modular groups. In [1], the authors proved that the Picard modular group with Gaussian integers acting on the two-dimensional complex hyperbolic space can be generated by four transformations. Also, in [2], the authors obtained a finite system of generators for $SU(2, 1, \mathbb{Z}[i])$ in a geometric way. Analogously, generators of the Picard modular group $SU(2, 1, \mathbb{Z}[\omega])$, where ω is a cubic root of unity, were studied by Falbel and Parker in [3] and by Wang *et al.* in [8]. Zhao in [9] obtained a system of generators for the Picard modular group $SU(2, 1, \mathbb{Z}[i\sqrt{2}])$ in a geometric way. In this work, we decompose any element of the Picard modular group $SU(2, 1, \mathbb{Z}[i\sqrt{2}])$ as a product of the generators, three translations a rotation and an involution, in a non-geometric way.

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2. Preliminaries

In this section, by taking a suitable Hermitian form and making a choice of section for $\mathbb{C}^{2,1}$, we get an unbounded hyperquadric model for the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$. The definitions can be found in [1,4,6].

Let $\mathbb{C}^{2,1}$ be the complex vector space of (complex) dimension 3 equipped with a non-degenerate infinite Hermitian form $\langle\ ,\ \rangle : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}$ of signature (2,1) given by $\langle z, w\rangle = w^* C z = -z_1\bar{w_3} + z_2\bar{w_2} - z_3\bar{w_1}$, where z and w are column vectors in $\mathbb{C}^3$, w^* is the Hermitian transpose of w and C is the nonsingular Hermitian matrix

$$C = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

with two positive and one negative eigenvalues. It can be shown that for each $z \in \mathbb{C}^{2,1}$, $\langle z, z\rangle$ is real. Thus, we may define the subsets V_-, V_0 and V_+ of $\mathbb{C}^{2,1}$ by

$$\begin{aligned} V_- &= \{z \in \mathbb{C}^{2,1} \mid \langle z, z\rangle < 0\}, \\ V_0 &= \{z \in \mathbb{C}^{2,1} - \{0\} \mid \langle z, z\rangle = 0\}, \\ V_+ &= \{z \in \mathbb{C}^{2,1} \mid \langle z, z\rangle > 0\}. \end{aligned}$$

We say that $z \in \mathbb{C}^{2,1}$ is negative, null or positive if z is in V_-, V_0 or V_+ respectively. Let $\mathbb{P}\colon \quad \mathbb{C}^{2,1} - \{0\} \quad \to \mathbb{CP}^2$ denote the standard projection map defined by $\mathbb{P}(z) = [z]$, where $[z]$ is the equivalence class of z in the complex projective plane $\mathbb{CP}^2$. In the chart of $\mathbb{C}^{2,1}$ with $z_1 \neq 0$, the projective map $\mathbb{P}$ is given by

$$\mathbb{P}\colon \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \to \begin{bmatrix} \frac{z_2}{z_1} \\ \frac{z_3}{z_1} \end{bmatrix}.$$

The projective model of complex hyperbolic plane is defined as the collection of negative lines in $\mathbb{C}^{2,1}$ and its boundary is defined as the collection of null lines. In other words, $\mathbb{H}^2_{\mathbb{C}}$ is $\mathbb{P}(V_-)$ and the boundary of that $\partial\mathbb{H}^2_{\mathbb{C}}$ is $\mathbb{P}(V_0)$. We define the unbounded hyperquadric model of the complex hyperbolic space by taking the section defined by $z_1 = 1$. In other words, if we take column vectors

$$z = \begin{pmatrix} 1 \\ z_1 \\ z_2 \end{pmatrix}$$

in $\mathbb{C}^{2,1}$, then $z \in \mathbb{H}^2_{\mathbb{C}}$ provided

$$\langle z, z\rangle = -\bar{z_2} + z_1\bar{z_1} - z_2 < 0.$$

In other words,

$$\mathfrak{Re}(z_2) > \frac{1}{2}|z_1|^2.$$

Thus $z = (z_1, z_2)$ is in a domain in $\mathbb{C}^2$ whose boundary is the paraboloid defined by

$$\mathfrak{Re}(z_2) = \frac{1}{2}|z_1|^2.$$

For a point $z = (z_1, z_2) \in \mathbb{C}^2$, we define the standard lift of z to be the point $z \in \mathbb{C}^{2,1}$ given by

$$z = \begin{pmatrix} 1 \\ z_1 \\ z_2 \end{pmatrix}.$$

It is clear that $\mathbb{P}(z) = z$. Therefore the standard lift enables us to give a well defined inverse of $\mathbb{P}$ whose domain is $\mathbb{C}^2$. We extend this definition to include the point ∞. We define the standard lift of ∞ to be

$$q_\infty = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{C}^{2,1}.$$

We freely pass between points of $\mathbb{C}^2 \cup \{\infty\}$ and their standard lifts.

The metric on $\mathbb{H}^2_{\mathbb{C}}$, called the Bergman metric, is given by

$$\mathrm{d}s^2 = \frac{-4}{\langle z, z\rangle^2} \det \begin{pmatrix} \langle z, z\rangle & \langle dz, z\rangle \\ \langle z, dz\rangle & \langle dz, dz\rangle \end{pmatrix}.$$

Alternatively, the Bergman metric is given by the distance function $\rho(.,.)$ defined by the formula

$$\cosh^2\left(\frac{\rho(z, w)}{2}\right) = \frac{\langle z, w\rangle \ \langle w, z\rangle}{\langle z, z\rangle \ \langle w, w\rangle}.$$

The holomorphic isometry group of $\mathbb{H}^2_{\mathbb{C}}$ is the group $PU(2, 1)$ of complex linear transformations, which preserve the above Hermitian form. Recall that $PU(2, 1)$ is the projectivisation of the special unitary group $SU(2, 1)$. The matrix $G = (g_{jk})^3_{j,k=1} \in SU(2, 1)$ satisfies the condition

$$G^* C G = C, \tag{1}$$

where G^* denotes the conjugate transpose of the matrix G and the determinant of the matrix G is normalized to be equal to 1.

For every $z \in \partial\mathbb{H}^2_{\mathbb{C}} \cup \{\infty\}$, the stabilizer subgroup (isotropy subgroup) Γ_z of z in $SU(2, 1)$ contains all the elements that leaves z fixed, that is,

$$\Gamma_z \equiv \{g \in SU(2, 1) \ | \ g(z) = z\}.$$

The following lemma plays a key role in the rest of the work.

Lemma 2.1. *Suppose that* $P = (p_{jk})^3_{j,k=1} \in SU(2, 1)$. *Then the following items are equivalent*:

(*i*) P *leaves* ∞ *fixed*,
(*ii*) P *is lower triangular*,
(*iii*) $p_{13} = 0$.

Proof. Let $P = (p_{ij})^3_{i,j=1} \in SU(2, 1)$. Then it is quite easy to see that P fixes ∞ if and only if $p_{13} = p_{23} = 0$. Moreover since P satisfies the condition 1, by computing and comparing the entries in the third row and second column, we obtain $-\bar{p_{33}} p_{12} = 0$. In addition, by computing and comparing the third row and first column, we get $-\bar{p_{33}} p_{11} = -1$ and hence we conclude that $p_{33} = 0$. Thus $p_{12} = 0$. Therefore, P is lower triangular.

Clearly, if P is lower triangular, then $p_{13} = 0$. Now let $p_{13} = 0$. By computing and comparing the entries in the third row and third column of 1, we obtain

$$-\bar{p}_{33} p_{13} + \bar{p}_{23} p_{23} - \bar{p}_{13} p_{33} = 0.$$

It follows immediately that $p_{23} = 0$. Similarly, comparing the entries in the third row and second column gives the equation

$$-\bar{p}_{33} p_{12} + \bar{p}_{23} p_{22} - \bar{p}_{13} p_{32} = 0$$

Therefore $\bar{p}_{33} p_{12} = 0$. However, $p_{13} = p_{23} = 0$ and also $\det P = 1$ conclude that $p_{33} = 0$. Thus P leaves ∞ fixed. □

The three important classes of the group Γ_∞ are Heisenberg translations, dilations and rotations.

The Heisenberg translation by $a = (a_1, a_2) \in \partial\mathbb{H}^2_{\mathbb{C}}$ is defined as follows:

$$N_{(a_1,a_2)}(z_1, z_2) = (z_1 + a_1, z_2 + a_2 + z_1\bar{a}_1).$$

Due to $\partial\mathbb{H}^2_{\mathbb{C}} = \{z \in \mathbb{C}^2 |\ \mathfrak{Re} z_2 = \frac{1}{2}|z_1|^2\}$, we can write $a = (a_1, a_2) = (\gamma, \frac{1}{2}|\gamma|^2 + ir)$ with $\gamma \in \mathbb{C}$, $r \in \mathbb{R}$. Then the Heisenberg translation by a is given by

$$N_{(\gamma, \frac{1}{2}|\gamma|^2+ir)}(z_1, z_2) = \left(z_1 + \gamma, z_2 + \frac{1}{2}|\gamma|^2 + ir + z_1\bar{\gamma}\right).$$

The corresponding matrix representation is

$$N_a \equiv \begin{pmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_2 & \bar{a}_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ \frac{1}{2}|\gamma|^2 + ir & \bar{\gamma} & 1 \end{pmatrix}.$$

The product of the Heisenberg translations N_a and N_b with $a, b \in \partial\mathbb{H}^2_{\mathbb{C}}$ gives the Heisenberg translation as

$$N_a \circ N_b = N_{(a_1+b_1, a_2+b_2+\bar{a}_1 b_1)}. \tag{2.2}$$

The inverse of the Heisenberg translation N_a is $N_a^{-1} = N_{(-a_1, -a_2+|a_1|^2)}$. It is easy to check that both of the elements $(a_1 + b_1, a_2 + b_2 + \bar{a}_1 b_1)$ and $(-a_1, -a_2 + |a_1|^2)$ are in $\partial\mathbb{H}^2_{\mathbb{C}}$.

We need the following lemma in the sequel.

Lemma 2.2. *Let $N_{(\gamma, \frac{1}{2}|\gamma|^2+ir)}$ be the Heisenberg translation by $(\gamma, \frac{1}{2}|\gamma|^2 + ir) \in \partial\mathbb{H}^2_{\mathbb{C}}$ with $\gamma \in \mathbb{C}$, $r \in \mathbb{R}$. Then for every $k \in \mathbb{Z}$, we have*

$$N^k_{(\gamma, \frac{1}{2}|\gamma|^2+ir)} = N_{(k\gamma, k^2(\frac{1}{2}|\gamma|^2)+i(kr))}.$$

Proof. First we prove that the lemma is true for $k = 2$. According to the relation (2.2) we can write

$$\begin{aligned} N^2_{(\gamma,\frac{1}{2}|\gamma|^2+ir)} &= N_{(\gamma,\frac{1}{2}|\gamma|^2+ir)} \circ N_{(\gamma,\frac{1}{2}|\gamma|^2+ir)} \\ &= N_{(2\gamma,4(\frac{1}{2}|\gamma|^2)+i(2r))}. \end{aligned}$$

Now let the relation be true for k. We show that it is true for $k+1$ as well.

$$\begin{aligned} N^{k+1}_{(\gamma,\frac{1}{2}|\gamma|^2+ir)} &= N^k_{(\gamma,\frac{1}{2}|\gamma|^2+ir)} \circ N_{(\gamma,\frac{1}{2}|\gamma|^2+ir)} \\ &= N_{(k\gamma,k^2(\frac{1}{2}|\gamma|^2)+i(kr))} \circ N_{(\gamma,\frac{1}{2}|\gamma|^2+ir)} \\ &= N_{((k+1)\gamma,(k^2+1)(\frac{1}{2}|\gamma|^2)+i(k+1)r+k|\gamma|^2)} \\ &= N_{((k+1)\gamma,(k^2+1+2k)(\frac{1}{2}|\gamma|^2)+i(k+1)r)} \\ &= N_{((k+1)\gamma,(k+1)^2(\frac{1}{2}|\gamma|^2)+i(k+1)r)}. \end{aligned}$$

This proves our lemma. □

The Heisenberg dilation by parameter $\delta > 0$ is given by

$$A_\delta(z_1, z_2) = (\delta z_1, \delta^2 z).$$

Its matrix representation is

$$A_\delta \equiv \begin{pmatrix} \frac{1}{\delta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \delta \end{pmatrix}.$$

Also, the Heisenberg rotation in the first variable by $e^{i\varphi}$ is given by

$$M_{e^{i\varphi}} = (e^{i\varphi} z_1, z_2),$$

where $\varphi \in \mathbb{R}$. There are three matrices

$$M_\beta \equiv \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta^{-2} & 0 \\ 0 & 0 & \beta \end{pmatrix},$$

where $\beta = e^{-\frac{i}{3}(\varphi-2k\pi)}$ for $k = 0, 1, 2$ corresponding to the same rotation.

Another isometry of $\mathbb{H}^2_{\mathbb{C}}$ is the holomorphic involution

$$J(z_1, z_2) = \left(\frac{z_1}{z_2}, \frac{1}{z_2}\right)$$

that maps ∞ into $(0, 0)$ and the matrix representation of J is

$$J \equiv \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

According to the Langlands decomposition, any element P of the stabilizer subgroup ∞ in $SU(2,1)$ can be decomposed as a product of a suitable Heisenberg translation N_a, a dilation A_δ, and a rotation M_β as follows:

$$\begin{aligned} P &= \begin{pmatrix} p_{11} & 0 & 0 \\ p_{21} & p_{22} & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \\ &= N_{(\gamma, \frac{1}{2}|\gamma|^2+ir)} A_\delta M_\beta = \begin{pmatrix} \frac{\beta}{\delta} & 0 & 0 \\ \frac{\beta\gamma}{\delta} & \beta^{-2} & 0 \\ \frac{\beta}{\delta}(\frac{1}{2}|\gamma|^2+ir) & \bar{\gamma}\beta^{-2} & \beta\delta \end{pmatrix}. \end{aligned}$$

3. Main results

In this section, as our main result, we obtain a system of generators for the stabilizer subgroup of ∞ in $SU(2,1,\mathcal{O}_2)$ and then using the fact that the ring $\mathcal{O}_2$ is Euclidean, we obtain a system of generators for $SU(2,1,\mathcal{O}_2)$ using an approximation method.

Lemma 3.1. *A matrix* $P = (p_{ij})_{i,j=1}^3$ *of* $SU(2,1)$ *is in* $\Gamma_\infty(2,1,\mathcal{O}_2)$, *the stabilizer subgroup of* ∞ *in* $SU(2,1,\mathcal{O}_2)$, *if and only if the parameters in the Langlands decomposition of* P *satisfy the following conditions*:

$$\delta = 1,\ \beta = \pm 1,\quad \gamma \in \mathbb{Z}[i\sqrt{2}],\quad |\gamma|^2 \in 2\mathbb{Z},\quad r \in \sqrt{2}\mathbb{Z}.$$

Also, *the real part of* γ *is even.*

Proof. Consider the Langlands decomposition. Using the Langlands decomposition we have $p_{11} = \dfrac{\beta}{\delta}$, $p_{33} = \beta\delta$, where $|\beta| = 1$ and $\delta > 0$. By considering that the entries of P are in the ring $\mathbb{Z}[i\sqrt{2}]$, we can write $p_{11} = \dfrac{\beta}{\delta} = a_1 + b_1\sqrt{2}i$ and $p_{33} = \beta\delta = a_2 + b_2\sqrt{2}i$, where a_1, a_2, b_1 and b_2 are integers. Therefore $|p_{11}| = \dfrac{1}{\delta} = \sqrt{a_1^2 + 2b_1^2} \geq 1$ and $|p_{33}| = \delta = \sqrt{a_2^2 + 2b_2^2} \geq 1$. The inequalities result in $\delta = 1$. Since the only units in the ring $\mathbb{Z}[i\sqrt{2}]$ are ± 1, we have $\beta = \pm 1$. In addition, $\dfrac{p_{31}}{\beta} = \dfrac{1}{2}|\gamma|^2 + ir$ and $\dfrac{p_{21}}{\beta} = \gamma$ belong to the ring $\mathbb{Z}[i\sqrt{2}]$. We may write $\dfrac{1}{2}|\gamma|^2 + ir = a_3 + b_3\sqrt{2}i$ with $a_3, b_3 \in \mathbb{Z}$. By comparing the real and the imaginary parts of the equality, we get $|\gamma|^2 \in 2\mathbb{Z}$ and $r \in \sqrt{2}\mathbb{Z}$. Moreover since $\gamma \in \mathbb{Z}[i\sqrt{2}]$ and $|\gamma|^2 \in 2\mathbb{Z}$, the real part of γ should be even. □

The following proposition is one of the main results.

PROPOSITION 3.2

$\Gamma_\infty(2,1,\mathcal{O}_2)$, *the stabilizer subgroup of* ∞ *in the Picard modular group* $SU(2,1,\mathcal{O}_2)$, *is generated by the Heisenberg translations*

$$N_{(2,2)} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}, \quad N_{(\sqrt{2}i,1)} = \begin{pmatrix} 1 & 0 & 0 \\ \sqrt{2}i & 1 & 0 \\ 1 & -\sqrt{2}i & 1 \end{pmatrix},$$

$$N_{(0,\sqrt{2}i)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sqrt{2}i & 0 & 1 \end{pmatrix} \tag{3.1}$$

and the rotation M_{-1} (*the rotation by* π),

$$M_{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.2}$$

Proof. Let P be an element in the stabilizer subgroup of ∞ in the Picard modular group $SU(2,1,\mathcal{O}_2)$. According to Lemma 3.1, there is no dilation component in its Langlands decomposition, that is,

$$P = N_{(\gamma,\frac{1}{2}|\gamma|^2+ir)}M_\beta = \begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ \frac{1}{2}|\gamma|^2+ir & \bar{\gamma} & 1 \end{pmatrix} \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta^{-2} & 0 \\ 0 & 0 & \beta \end{pmatrix}. \tag{3.3}$$

Since $\gamma \in \mathbb{Z}[i\sqrt{2}]$ and $r \in \sqrt{2}\mathbb{Z}$, we can write

$$N_{(\gamma,\frac{1}{2}|\gamma|^2+ir)} = N_{(a+b\sqrt{2}i,\frac{1}{2}(a^2+2b^2)+ir)},$$

where $a, b \in \mathbb{Z}$ and $r \in \sqrt{2}\mathbb{Z}$.

By (2.2), the Heisenberg translation $N_{(a+b\sqrt{2}i,\frac{1}{2}(a^2+2b^2)+ir)}$ splits as

$$N_{(a+b\sqrt{2}i,\frac{1}{2}(a^2+2b^2)+ir)} = N_{(a,\frac{1}{2}a^2)} \circ N_{(b\sqrt{2}i,b^2)} \circ N_{(0,ir-ab\sqrt{2}i)}. \tag{3.4}$$

Since a is an even integer and b is an integer, according to Lemma 2.2, the first two factors in (3.4) can be written as

$$N_{(a,\frac{1}{2}a^2)} = N_{(2(\frac{a}{2}),2(\frac{a^2}{4}))} = N_{(2,2)}^{\frac{a}{2}}, \tag{3.5}$$

$$N_{(b\sqrt{2}i,b^2)} = N_{(\sqrt{2}i,1)}^{b}. \tag{3.6}$$

Moreover, since $r \in \sqrt{2}\mathbb{Z}$ we get $\dfrac{r}{\sqrt{2}} - ab \in \mathbb{Z}$. Therefor the third factor in (3.4) can be written as follows:

$$N_{(0,ir-ab\sqrt{2}i)} = N_{(0,\sqrt{2}i)}^{\frac{r}{\sqrt{2}}-ab}. \tag{3.7}$$

According to Lemma 3.1, $\beta = \pm 1$. Therefore the rotation component of P in (3.3) is one of the following matrices:

$$M_{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly $M_{-1}^2 = M_1$. Thus the stabilizer subgroup of infinity in the Picard modular group $SU(2, 1, \mathcal{O}_2)$ is generated by the translations $N_{(2,2)}$, $N_{(\sqrt{2}i,1)}$, $N_{(0,\sqrt{2}i)}$ and the rotation M_{-1}. □

And finally we have the theorem that contains our main goal.

Theorem 3.3. *The Picard modular group $SU(2, 1, \mathcal{O}_2)$ is generated by the Heisenberg translations*

$$N_{(2,2)} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}, \quad N_{(\sqrt{2}i,1)} = \begin{pmatrix} 1 & 0 & 0 \\ \sqrt{2}i & 1 & 0 \\ 1 & -\sqrt{2}i & 1 \end{pmatrix},$$

$$N_{(0,\sqrt{2}i)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sqrt{2}i & 0 & 1 \end{pmatrix},$$

the rotation by π,

$$M_{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and the involution

$$J = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Proof. Let $G = (g_{jk})_{j,k=1}^3$ be an element of the group $SU(2, 1, \mathcal{O}_2)$ and does not belong to the stabilizer subgroup of infinity. In this case, according to Lemma 2.1, $g_{13} \neq 0$ and also G maps ∞ to $\left(\frac{g_{23}}{g_{13}}, \frac{g_{33}}{g_{13}}\right)$. Since G preserves the boundary $\partial\mathbb{H}_{\mathbb{C}}^2$, then $G(\infty)$ is in $\partial\mathbb{H}_{\mathbb{C}}^2$ and we have

$$\mathfrak{Re}\frac{g_{33}}{g_{13}} = \frac{1}{2}\left|\frac{g_{23}}{g_{13}}\right|^2.$$

Consider the Heisenberg translation $N_{G(\infty)}$ that maps $(0, 0)$ to $G(\infty)$. Note that if $|g_{13}| \neq 1$, then the translation $N_{G(\infty)}$ is not necessarily in the Picard modular group $SU(2, 1, \mathcal{O}_2)$. However we know that the transformation defined by

$$JN^{-1}_{G(\infty)}G$$

belongs to the stabilizer subgroup ∞ in $SU(2,1,\mathcal{O}_2)$. We successively approximate $N^{-1}_{G(\infty)}$ by Heisenberg translation in the Picard modular group to decrease the value $|g_{13}|^2 \in \mathbb{Z}$ until it becomes 0. If we show this translation by G_n then according to Lemma 2.1, G_n belongs to the stabilizer subgroup Γ_∞ and therefore G can be expressed as a product of the generators of (3.1), (3.2) and the involution J. The approximation steps use the fact that the ring $\mathcal{O}_2$ is Euclidean. To begin the approximation procedure, let

$$\begin{aligned}
G_1 &\equiv JN_{(\gamma,\frac{1}{2}|\gamma|^2+ir)}G \\
&= \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ \frac{1}{2}|\gamma|^2+ir & \bar{\gamma} & 1 \end{pmatrix} G \\
&= \begin{pmatrix} -\frac{1}{2}|\gamma|^2-ir & \bar{\gamma} & -1 \\ -\gamma & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} G
\end{aligned}$$

for some γ and r. So $g^{(1)}_{13}$, the entry in the upper right corner of $G_1 = (g^{(1)}_{jk})$, is equal to

$$\begin{aligned}
g^{(1)}_{13} &\equiv -\left(\frac{1}{2}|\gamma|^2+ir\right)g_{13} - \bar{\gamma}g_{23} - g_{33} \\
&= -g_{13}\left(\frac{1}{2}|\gamma|^2 + ir + \bar{\gamma}\frac{g_{23}}{g_{13}} + \frac{g_{33}}{g_{13}}\right) \\
&= -g_{13}\left[\frac{1}{2}|\gamma|^2 + \mathfrak{Re}\left(\bar{\gamma}\frac{g_{23}}{g_{13}}\right) + \mathfrak{Re}\left(\frac{g_{33}}{g_{13}}\right)\right] \\
&\quad - ig_{13}\left[r + \mathfrak{Im}\left(\bar{\gamma}\frac{g_{23}}{g_{13}}\right) + \mathfrak{Im}\left(\frac{g_{33}}{g_{13}}\right)\right] \\
&\equiv -g_{13}(I_1 + iI_2).
\end{aligned}$$

We can simplify I_1 to

$$\begin{aligned}
I_1 &= \frac{1}{2}|\gamma|^2 + \mathfrak{Re}\left(\bar{\gamma}\frac{g_{23}}{g_{13}}\right) + \mathfrak{Re}\left(\frac{g_{33}}{g_{13}}\right) \\
&= \frac{1}{2}\left(|\gamma|^2 + 2\mathfrak{Re}\left(\bar{\gamma}\frac{g_{23}}{g_{13}}\right) + \left|\frac{g_{23}}{g_{13}}\right|^2\right) \\
&= \frac{1}{2}\left(\gamma + \frac{g_{23}}{g_{13}}\right)\left(\bar{\gamma} + \frac{\bar{g}_{23}}{\bar{g}_{13}}\right) \\
&= \frac{1}{2}\left|\gamma + \frac{g_{23}}{g_{13}}\right|^2.
\end{aligned}$$

Let $\frac{g_{23}}{g_{13}} = x + iy$ and $Y = \frac{y}{\sqrt{2}}$ with $x, y \in \mathbb{R}$. Then there is an integer $-b$ in the interval $\left[Y - \frac{1}{2}, Y + \frac{1}{2}\right]$ such that $|Y + b| \leq \frac{1}{2}$. Therefore,

$$|y + \sqrt{2}b| \leq \frac{\sqrt{2}}{2}. \tag{3.8}$$

Also, there is an integer $-a$ in the interval $\left[x - \frac{1}{2}, x + \frac{1}{2}\right]$ such that

$$|x + a| \leq \frac{1}{2}. \tag{3.9}$$

Set $\gamma = a + b\sqrt{2}i$, where a, b are the above integers. Then

$$\gamma + \frac{g_{23}}{g_{13}} = (a + x) + (y + b\sqrt{2})i$$

and the relations (3.8) and (3.9) give the upper bound for $|I_1|$ as follows:

$$|I_1| \leq \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2\right) = \frac{1}{2}\left(\frac{1}{4} + \frac{2}{4}\right) = \frac{3}{8}. \tag{3.10}$$

Now let

$$X = \frac{1}{\sqrt{2}}\left(\mathfrak{Im}\left(\bar{\gamma}\frac{g_{23}}{g_{13}}\right) + \mathfrak{Im}\left(\frac{g_{33}}{g_{13}}\right)\right).$$

Since $X \in \mathbb{R}$, then there is an integer $-m$ such that $|X + m| \leq \frac{1}{2}$. Therefore,

$$\left|\mathfrak{Im}\left(\bar{\gamma}\frac{g_{23}}{g_{13}}\right) + \mathfrak{Im}\left(\frac{g_{33}}{g_{13}}\right) + \sqrt{2}m\right| \leq \frac{\sqrt{2}}{2}.$$

Thus considering $r = \sqrt{2}m$, we have

$$|I_2| = \left|Im\left(\bar{\gamma}\frac{g_{23}}{g_{13}}\right) + Im\left(\frac{g_{33}}{g_{13}}\right) + r\right| \leq \frac{\sqrt{2}}{2}. \tag{3.11}$$

Using (3.10) and (3.11), we can estimate $g_{13}^{(1)}$ as follows:

$$\begin{aligned}|g_{13}^{(1)}|^2 &= |g_{13}|^2|I_1 + iI_2|^2 = |g_{13}|^2(|I_1|^2 + |I_2|^2) \leq |g_{13}|^2\left(\frac{9}{64} + \frac{2}{4}\right)\\ &= \frac{41}{64}|g_{13}|^2 < |g_{13}|^2.\end{aligned}$$

Since the entries of g are in $\mathcal{O}_2$, then repeating this approximation procedure finitely many times we can reduce the matrix transformation G to the matrix of a transformation G_n with $g_{13}^{(n)} = 0$. Thus we prove our main result. $\square$

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