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Exponential sums of squares of Fourier coefficients of cusp forms

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Abstract. We prove nontrivial estimates for linear sums of squares of Fourier coefficients of holomorphic and Maass cusp forms twisted by additive characters. For holomorphic forms f, we show that if $|\alpha - a/q| \leq 1/q^2$ with $(a, q) = 1$, then for any $\varepsilon > 0$,

$$\sum_{n \leqslant X} \lambda_f(n)^2 e(n\alpha) \ll_{f,\varepsilon} X^{\frac{4}{5}+\varepsilon} \quad \text{for } X^{\frac{1}{5}} \ll q \ll X^{\frac{4}{5}}.$$

Moreover, for any $\varepsilon > 0$, there exists a set $S \subset (0, 1)$ with $\mu(S) = 1$ such that for every $\alpha \in S$, there exists $X_0 = X_0(\alpha)$ such that the above inequality holds true for any $\alpha \in S$ and $X \geqslant X_0(\alpha)$. A weaker bound for Maass cusp forms is also established.

Keywords. Cusp forms; exponential sums; diophantine approximation.

Mathematics Subject Classification. 11F30.

1. Introduction

In this paper, we study linear sums of squares of the Fourier coefficients of holomorphic and Maass cusp forms twisted by additive characters. Let f be a fixed eigencusp form (holomorphic cuspidal eigenform of even integral weight or a Hecke–Maass cusp form of weight zero) for the full modular group $SL(2, \mathbb{Z})$. Define

$$S(X, \alpha) = \sum_{n \leqslant X} \lambda_f(n)^2 e(n\alpha). \tag{1.1}$$

Note that $S(X, \alpha)$ is periodic in α with period 1. If α is an integer, then

$$S(X, \alpha) = \sum_{n \leqslant X} \lambda_f(n)^2 \sim C_f X \tag{1.2}$$

by the Rankin–Selberg method (see [8] and [7]). Hence, the trivial bound

$$S(X, \alpha) = O(X)$$

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can not be improved for every α. So let us consider $0 < \alpha < 1$. By diophantine approximation, there exist positive integers a, q such that $q \leqslant X$, $(a, q) = 1$ and

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{1}{q^2}. \tag{1.3}$$

Theorem 1.1. *Suppose f is a holomorphic cuspidal eigenform and α satisfies (1.3) for some positive integers a, q such that $q \leqslant X$, $(a, q) = 1$. Then for any $\varepsilon > 0$, we have the bound*

$$S(X, \alpha) \ll_{f,\varepsilon} X^{\varepsilon} \left(X^{\frac{4}{5}} \log q + \frac{X \log X}{q} + q \log q \right), \tag{1.4}$$

where the implied constant depends on the form f and ε.

Remark 1. The above estimate is worse than the trivial bound $O(X)$ for very small or very large values of q in terms of X. In remaining cases, we have improved the estimates as follows. If $X^{\delta} \ll q \ll X^{1-\delta}$ for some $\delta \in (0, 1)$, then for any $\varepsilon > 0$, we have

$$S(X, \alpha) \ll_{f,\varepsilon} X^{\max(\delta, 1-\delta, \frac{4}{5})+\varepsilon}. \tag{1.5}$$

For example, if

$$\alpha = \rho = \frac{1 + \sqrt{5}}{2},$$

the golden ratio, then we know that for every $X > 2$, there is a fraction $\frac{a}{q}$ with $(a, q) = 1$ satisfying (1.3) and the inequality $\sqrt{X} < q < 2\sqrt{X}$. Thus $S(X, \rho) = O(X^{\frac{4}{5}+\varepsilon})$ for any $\varepsilon > 0$.

Consequently, we have the following result.

COROLLARY 1

Suppose f is a holomorphic cuspidal eigenform. Then for any $\varepsilon > 0$, there exists a set $S \subset (0, 1)$ with $\mu(S) = 1$ such that for every $\alpha \in S$, there exists $X_0 = X_0(\alpha)$ such that

$$S(X, \alpha) \ll_{f,\varepsilon} X^{4/5+\varepsilon} \quad \text{for all } X \geqslant X_0.$$

Here $\mu(.)$ denotes the Lebesgue measure.

For Maass cusp forms, we have the following result.

Theorem 1.2. *Suppose f is a Hecke–Maass cusp form and α satisfies (1.3) for some positive integers a, q such that $q \leqslant X$ and $(a, q) = 1$. Then for any $\varepsilon > 0$, we have*

$$S(X, \alpha) \ll_{f,\varepsilon} X^{7/47+\varepsilon} \left(X^{\frac{32}{47}} \log q + \frac{X \log X}{q} + q \log q \right), \tag{1.6}$$

Remark 2. Thus, we have better estimates of $S(X, \alpha)$ than the trivial bound $O(X)$, if $X^{7/47+\delta} \ll q \ll X^{40/47-\delta}$ for any arbitrarily small $\delta > 0$. As in the previous case, here we obtain $S(X, \rho) = O(X^{\frac{39}{47}+\varepsilon})$, for any $\varepsilon > 0$.

Moreover, we have the following result.

COROLLARY 2

Suppose f is a Hecke–Maass cusp form. Then for any $\varepsilon > 0$, there exists a set $S \subset (0, 1)$ with $\mu(S) = 1$ such that for every $\alpha \in S$, there exists $X_0 = X_0(\alpha)$ such that

$$S(X, \alpha) \ll_{f,\varepsilon} X^{39/47+\varepsilon} \quad \textit{for all } X \geqslant X_0.$$

2. Preliminaries

Let f be a fixed holomorphic cusp form of even integral weight or a Maass cusp form (of weight zero) for the full modular group $SL(2, \mathbb{Z})$. Suppose, for $n \geq 1$, $\lambda_f(n)$ denotes the normalized n-th Fourier coefficient of the form f. Here, the normalization is done in such a way that when f is an eigenfunction of all the Hecke operators, then the Fourier coefficients $\lambda_f(n)$ coincides with the normalized eigenvalue of the n-th Hecke operator. With this normalization, the Ramanujan conjecture predicts $|\lambda_f(p)| \leqslant 2$, for all primes p. This is known in the case of holomorphic cusp forms for all levels by the work of Deligne [1]. Thus we have

$$\lambda_f(n) \ll_\varepsilon n^\varepsilon, \tag{2.1}$$

for all $\varepsilon > 0$. For a Maass cusp form, the best-known bound to date is the following:

$$\lambda_f(n) \ll_\varepsilon n^{7/64+\varepsilon} \tag{2.2}$$

for all $\varepsilon > 0$, by the work of Kim and Sarnak [4].

The normalized Fourier coefficients satisfy the relation

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right). \tag{2.3}$$

From the above equation one can infer that

$$\lambda_f(mn) = \sum_{d|(m,n)} \mu(d)\lambda_f\left(\frac{m}{d}\right)\lambda_f\left(\frac{n}{d}\right). \tag{2.4}$$

Now we recall a result of Miller [6, Theorem 1.1], which he deduced by an application of the Voronoi summation formula for $SL(3, \mathbb{Z})$. Let $a_F(q, n)$ denote the Fourier coefficients of a cusp form F for $SL(3, \mathbb{Z})$. Then for any $\epsilon > 0$, he has shown that

$$\sum_{n \leqslant X} a_F(q, n)e(n\alpha) \ll X^{\frac{3}{4}+\epsilon}, \tag{2.5}$$

uniformly in $\alpha \in \mathbb{R}$, with the implied constant depending upon q, ϵ and the cusp form. As an application of Miller's theorem (by taking $q = 1$), we obtain the following result [2, Lemma 6.3] by using the identity:

$$L(s, \mathrm{sym}^2 f) = \zeta(2s) \sum_{n=1}^{\infty} \lambda_f(n^2) n^{-s}.$$

Lemma 2.1. *Let f be a holomorphic or a Maass cusp form for $SL(2, \mathbb{Z})$ with normalized Fourier coefficients $\lambda_f(n)$. Then we have*

$$\sum_{n \leqslant X} \lambda_f(n^2) e(n\alpha) \ll_{f,\varepsilon} X^{\frac{3}{4}+\epsilon}, \tag{2.6}$$

uniformly in $\alpha \in \mathbb{R}$, with the implied constant depending upon q, ϵ and the form f.

3. Proofs of the Theorems 1.1 and 1.2

Using the Hecke multiplicativity relation (2.3), we can write

$$S(X, \alpha) = \sum\sum_{lm \leqslant X} \lambda_f(m^2) e(lm\alpha). \tag{3.1}$$

Now we divide the sum according to the ranges of l.

$$\begin{aligned} S(X, \alpha) &= \sum_{l \leqslant L} \sum_{m \leqslant \frac{X}{l}} \lambda_f(m^2) e(lm\alpha) + \sum_{l > L} \sum_{m \leqslant \frac{X}{l}} \lambda_f(m^2) e(lm\alpha) \\ &= S_1(X, \alpha) + S_2(X, \alpha) \text{ (say)}, \end{aligned} \tag{3.2}$$

where L will be chosen later. Lemma 2.1 tells us that

$$S_1(X, \alpha) = \sum_{l \leqslant L} \sum_{m \leqslant \frac{X}{l}} \lambda_f(m^2) e(lm\alpha) \ll_{f,\varepsilon} \sum_{l \leqslant L} \left(\frac{X}{l}\right)^{\frac{3}{4}+\varepsilon} \ll_{f,\varepsilon} L^{\frac{1}{4}} X^{\frac{3}{4}+\varepsilon}. \tag{3.3}$$

Now we estimate $S_2(X, \alpha)$. Note that

$$S_2(X, \alpha) = \sum_{m < \frac{X}{L}} \lambda_f(m^2) \sum_{L < l \leqslant \frac{X}{m}} e(lm\alpha). \tag{3.4}$$

We recall the following result on the Weyl sum for a linear polynomial $f(x) = \alpha x$, which is a sum over geometric progression.

Lemma 3.1. *Suppose $||\alpha||$ denotes the distance of α to the nearest integer. Then we have*

$$\left| \sum_{n \leqslant N} e(\alpha n) \right| \leqslant \min\left\{ N, \frac{1}{2||\alpha||} \right\}, \tag{3.5}$$

if α is not an integer.

Proof. The proof follows directly from the fact (see [3, Chapter 8])

$$\sum_{n\leqslant N} e(\alpha n) = \frac{\sin \pi\alpha N}{\sin \pi\alpha} e\left(\frac{\alpha}{2}(N+1)\right).$$

□

So from (3.4) and (3.5), we have

$$S_2(X,\alpha) \ll \sum_{m<\frac{X}{L}} |\lambda_f(m^2)| \min\left\{\frac{X}{m}, \frac{1}{||m\alpha||}\right\}. \tag{3.6}$$

To prove Theorem 1.1, we use Deligne's bound (2.1) for holomorphic cusp forms. In case of Maass cusp forms, the estimate (2.2) will help us to prove Theorem 1.2.

Following the idea in proving [3, Theorem 13.6], we estimate the sums over m, that appear in (3.6). For a given α, we can choose integers a, q with $(a, q) = 1$ such that $\left|\alpha - \frac{a}{q}\right| \leqslant \frac{1}{q^2}$. Consequently, as m varies over integers in each interval of length $\frac{q}{2}$, $||m\alpha||$'s are all distinct and are spaced by at least $\frac{1}{2q}$. Hence, there exists at the most one m for which $||m\alpha|| < \frac{1}{2q}$. In particular, one can check that if $m \in [1, \frac{q}{2}]$, then $||m\alpha|| \geqslant \frac{1}{2q}$.

Therefore, if I be any interval of length $\frac{q}{2}$, then for each integer $k \in [1, q/2-1]$, there exists an integer $m \in I$ such that $||m\alpha|| \geq \frac{k}{2q}$. However, for $I = I_0 = [1, q/2]$, we have a stronger result. Precisely, for each integer $k \in [1, q/2]$, there exists an integer $m \in [1, q/2]$ such that $||m\alpha|| \geq \frac{k}{2q}$.

These observations will help us to estimate $S_2(X,\alpha)$ for both holomorphic and Maass cusp forms. We divide the interval $1 \leqslant m < \frac{X}{L}$ into subintervals of length $\frac{q}{2}$ as follows. Note that the last subinterval will be of length at most $\frac{q}{2}$. But that does not make any difference as the following estimates hold true for such incomplete sums too.

First note that

$$\sum_{m\leqslant\frac{q}{2}} \min\left\{\frac{X}{m}, \frac{1}{||m\alpha||}\right\} \leqslant \sum_{m\leqslant\frac{q}{2}} \frac{1}{||m\alpha||} \leqslant 2q \sum_{k\leqslant\frac{q}{2}} \frac{1}{k} \ll q\log q.$$

Similarly, in the remaining part we use the same bound except for at most one m in each subinterval. Thus, we obtain

$$\sum_{r\frac{q}{2}<m\leqslant(r+1)\frac{q}{2}} \min\left\{\frac{X}{m}, \frac{1}{||m\alpha||}\right\} \leqslant 2q\log q + \frac{2X}{rq}.$$

Now summing over $1 \leqslant r \leqslant [\frac{2X}{Lq}]$, we have the following upper bounds:

$$\sum_{m<\frac{X}{L}} \min\left\{\frac{X}{m}, \frac{1}{||m\alpha||}\right\} \ll q\log q + \frac{X\log X}{q} + \frac{X\log q}{L}. \tag{3.7}$$

For holomorphic cusp form f, the above estimate (3.7) together with (3.6) implies that

$$S_2(X,\alpha) \ll X^{\varepsilon}\left(q\log q + \frac{X\log X}{q} + \frac{X\log q}{L}\right), \tag{3.8}$$

by the Deligne's bound (2.1). Hence, combining with (3.3), we have the bound claimed in Theorem 1.1 by choosing $L = X^{\frac{1}{5}}$.

For Maass cusp form f, we have the analogous estimates. From (3.6) and (3.7), it follows that

$$S_2(X,\alpha) \ll \left(\frac{X}{L}\right)^{7/32+\varepsilon}\left(q\log q + \frac{X\log X}{q} + \frac{X\log q}{L}\right), \tag{3.9}$$

by (2.2). Similarly, combining with (3.3) and choosing $L = X^{\frac{15}{47}}$, we conclude Theorem 2.

4. Proofs of the Corollaries 1 and 2

Here we will prove Corollary 1, and the proof of Corollary 2 is identical. From (1.5), one notes that for any $\varepsilon > 0$, we have

$$S(X,\alpha) \ll_{\varepsilon} X^{\frac{4}{5}+\varepsilon} \quad \text{if } X^{\frac{1}{5}} \ll q \ll X^{\frac{4}{5}},$$

where f is a holomorphic cuspidal eigenform.

We recall some basic facts about continued fractions. For any $\alpha = [a_1, a_2, a_3, \ldots] \in (0,1)$, the n-th convergent of α is given by $[a_1, a_2, \ldots, a_n] = p_n(\alpha)/q_n(\alpha)$ (say). These convergents give rational approximations for α which are the best ones in terms of the size of the denominators. Precisely, we have

$$\left|\alpha - \frac{p_n}{q_n}\right| \leqslant \frac{1}{q_n q_{n+1}} \leqslant \frac{1}{a_{n+1}q_n^2} \leqslant \frac{1}{q_n^2} \quad \text{for all } n \geqslant 1.$$

Moreover, we know that almost all (in the sense of Lebesgue) irrational numbers α satisfies

$$q_n(\alpha)^{1/n} \to e^{\pi^2/12\log 2} = 3.275\cdots,$$

which is known as Levy's constant (see [5]). In other words, for almost all irrational numbers α, there exists $n_0 \in \mathbb{N}$ such that

$$(3.27)^n \leqslant q_n(\alpha) \leqslant (3.28)^n \quad \text{for all } n \geqslant n_0. \tag{4.1}$$

Now let S denote the set of all such irrational numbers $\alpha \in (0,1)$ satisfying (4.1). We also note that $S \subset (0,1)$ is of full measure. Therefore, for a given $\alpha \in S$ and sufficiently large X, the interval $[X^{\frac{1}{5}}, X^{\frac{4}{5}}]$ contains at least one denominator $q_n(\alpha)$. Hence we are through.

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