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A ternary diophantine inequality with prime numbers of a special type

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Abstract. We consider the inequality

$$|p_1^c + p_2^c + p_3^c - N| < (\log N)^{-E},$$

where $1 < c < \frac{281}{250}$, N is a sufficiently large real number and $E > 0$ is an arbitrarily large constant. We prove that the above inequality has a solution in primes p_1, p_2, p_3 such that each of the numbers $p_1 + 2$, $p_2 + 2$, $p_3 + 2$ has at most $[\frac{1475}{562-500c}]$ prime factors, counted with the multiplicity. This result constitutes an improvement upon that of Tolev.

Keywords. Diophantine inequality; exponential sum; prime.

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1. Introduction

In 1952, Piatetski-Shapiro [12] considered the following problem which is analogous to the Waring–Goldbach problem. Assume that $c > 1$ is not an integer and let ε be a small positive number. If r is a sufficiently large integer (depending only on c) then the inequality

$$|p_1^c + p_2^c + \cdots + p_r^c - N| < \varepsilon \tag{1.1}$$

is solvable in prime numbers $p_1, p_2, \ldots, p_r$ for sufficiently large real number N. More precisely, if the least r such that (1.1) is solvable in prime numbers for every $\varepsilon > 0$ is denoted by $H(c)$, then it is proved in [12] that

$$\limsup_{c\to\infty} \frac{H(c)}{c\log c} \le 4.$$

Piatetski-Shapiro also proved that if $1 < c < \frac{3}{2}$, then $H(c) \le 5$. On the other hand, the Vinogradov-Goldbach theorem [17] suggested that at least for c close to 1, one should expect $H(c) \le 3$. The first result in this direction was obtained by Tolev [14], who showed that the inequality

$$|p_1^c + p_2^c + p_3^c - N| < \Delta(N) \tag{1.2}$$

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with $\lim_{N\to\infty}\Delta(N)=0$ is solvable in primes p_1, p_2, p_3, provided that $1<c<\frac{27}{26}$. Afterwards, several authors sharpened Tolev's result by enlarging the range for c (see [1–4,9,10,15]).

Suppose that r is a natural number. Let P_r denote an almost-prime with at most r prime factors, counted with the multiplicity. In 1973, Chen [5] established that there exist infinitely many primes p such that $p+2\in P_2$.

Motivated by Tolev [15] and Chen [5], it is reasonable to conjecture that if c is close to 1, then the inequality (1.2) with a suitable $\Delta(N)$ satisfying $\Delta(N)\to 0$ as $N\to\infty$, is solvable in primes p_i $(i=1,2,3)$ such that $p_i+2\in P_r$ for a fixed $r\geq 2$. An attempt to establish a result of this type was made by Tolev [16]. Let $E>0$ be an arbitrarily large constant and $1<c<\frac{15}{14}$. He showed that the inequality

$$|p_1^c+p_2^c+p_3^c-N|<(\log N)^{-E}, \tag{1.3}$$

has a solution in primes p_1, p_2, p_3 such that each of the numbers $p_i+2,\ i=1,2,3$ has at most $[\frac{369}{180-168c}]$ prime factors, counted with the multiplicity.

In this paper, we shall prove the following sharper result.

Theorem. *Let $1<c<\frac{281}{250}$ and let N be a sufficiently large real number. Then the inequality (1.3) has a solution in primes p_1, p_2, p_3 such that each of the numbers p_i+2, $i=1,2,3$ has at most $[\frac{1475}{562-500c}]$ prime factors, counted with the multiplicity.*

It is worth mentioning that the crucial point of this paper lies in the use of Lemma 4.8. In contrast to the previous work, Lemma 4.8 leads to a stronger estimate for a cubic moment of an exponential sum associated to indicator function of primes and some sieve weights. This enables us to improve the limit of the exponent c to $\frac{281}{250}$.

2. Notation and some lemmas

Let N be a sufficiently large real number and let ε be an arbitrarily small positive number, which may not be the same at each occurrence. The letter p, with or without subscript, is reserved for a prime number. As usual, $\varphi(n)$ and $\mu(n)$ denote the Euler's function and Möbius function. $[x]$ denotes the integer part of the real number x. With χ, we denote a Dirichlet's character and $\sum_{\chi(q)^*}$ means that the summation is taken over the primitive Dirichlet's characters modulo q. Write $e(\alpha)=e^{2\pi i\alpha}$. Put

$$\begin{aligned}
&1<c<\frac{281}{250},\ \delta=\frac{281}{250}-c,\ \xi=\frac{4c}{5}-\frac{2}{5},\ \eta=\frac{20}{59}\delta,\\
&X=N^{\frac{1}{c}},\ z=X^{\eta},\ D=X^{\delta},\ \tau=X^{\xi-c},\ \Delta=(\log N)^{-E},\\
&r=[(\log X)^2],\ \Xi=(\log X)^{E+3},\ P(z)=\prod_{2<p<z}p.
\end{aligned}$$

Lemma 2.1. *There exists a function $\theta(y)$ which is r times continuously differentiable such that*

$$\begin{aligned}
\theta(y)&=1\ \ for\ |y|\leq\frac{3\Delta}{4},\\
0<\theta(y)&<1\ \ for\ \frac{3\Delta}{4}<|y|\leq\Delta,
\end{aligned}$$

$$\theta(y) = 0 \ for \ |y| \ge \Delta,$$

and its Fourier transform

$$\Theta(x) = \int_{-\infty}^{\infty} e(-xy)\theta(y)\mathrm{d}y \tag{2.1}$$

satisfies

$$|\Theta(x)| \le \min\left(\frac{7\Delta}{4}, \frac{1}{\pi|x|}, \frac{1}{\pi|x|}\left(\frac{8r}{2\pi|x|\Delta}\right)^r\right). \tag{2.2}$$

Proof. See [13].

Lemma 2.2. *Let $\lambda^{\pm}(d)$ be the Rosser's functions of level D. Then we have the following properties*:

(1) *For any positive integer d, we have*

$$|\lambda^{\pm}(d)| \le 1, \ \lambda^{\pm}(d) = 0 \ \ if \ d > D \ \ or \ \ \mu(d) = 0. \tag{2.3}$$

(2) *If n is a positive integer, then*

$$\sum_{d|n} \lambda^{-}(d) \le \sum_{d|n} \mu(d) \le \sum_{d|n} \lambda^{+}(d). \tag{2.4}$$

(3) *Let*

$$\mathfrak{B} = \prod_{2<p<z} (1 - \frac{1}{p-1}), \quad \mathfrak{N}^{\pm} = \sum_{d|P(z)} \frac{\lambda^{\pm}(d)}{\varphi(d)}, \quad s = \frac{\log D}{\log z}. \tag{2.5}$$

Then we have

$$\frac{1}{\log X} \ll \mathfrak{B} \ll \frac{1}{\log X}, \tag{2.6}$$

$$\mathfrak{B} \le \mathfrak{N}^{+} \le \mathfrak{B}(F(s) + O(\log^{-\frac{1}{3}} D)), \tag{2.7}$$

$$\mathfrak{B} \ge \mathfrak{N}^{-} \ge \mathfrak{B}(f(s) + O(\log^{-\frac{1}{3}} D)), \tag{2.8}$$

where

$$f(s) = 2e^{\gamma} s^{-1} \log(s-1), \ \ F(s) = 2e^{\gamma} s^{-1} \ \ for \ \ 2 \le s \le 3 \tag{2.9}$$

and γ stands for the Euler constant.

Proof. This is [16, Lemma 3 and (124)].

Lemma 2.3. *Let*

$$\Lambda_i = \sum_{d|(p_i+2, P(z))} \mu(d), \quad \Lambda_i^{\pm} = \sum_{d|(p_i+2, P(z))} \lambda^{\pm}(d), \ \ i = 1, 2, 3. \tag{2.10}$$

Then we have

$$\Lambda_1\Lambda_2\Lambda_3 \ge \Lambda_1^{-}\Lambda_2^{+}\Lambda_3^{+} + \Lambda_1^{+}\Lambda_2^{-}\Lambda_3^{+} + \Lambda_1^{+}\Lambda_2^{+}\Lambda_3^{-} - 2\Lambda_1^{+}\Lambda_2^{+}\Lambda_3^{+}. \tag{2.11}$$

Proof. See [16, Lemma 4].

3. Outline of the method

Consider the sum

$$\Gamma = \sum_{\substack{\frac{X}{4}<p_1,p_2,p_3<X \\ |p_1^c+p_2^c+p_3^c-N|<\Delta \\ (p_i+2,P(z))=1, i=1,2,3}} (\log p_1)(\log p_2)(\log p_3). \tag{3.1}$$

From Lemma 2.1, we have

$$\Gamma \geq \Gamma^* := \sum_{\substack{\frac{X}{4}<p_1,p_2,p_3\leq X \\ (p_i+2,P(z))=1, (i=1,2,3)}} (\log p_1)(\log p_2)(\log p_3)\theta(p_1^c+p_2^c+p_3^c-N). \tag{3.2}$$

By applying Lemma 2.3 and changing the order of summation, we can find that

$$\begin{aligned}\Gamma^* &= \sum_{\frac{X}{4}<p_1,p_2,p_3\leq X} (\log p_1)(\log p_2)(\log p_3)\Lambda_1\Lambda_2\Lambda_3\theta(p_1^c+p_2^c+p_3^c-N) \\ &\geq \sum_{\frac{X}{4}<p_1,p_2,p_3\leq X} (\log p_1)(\log p_2)(\log p_3)\theta(p_1^c+p_2^c+p_3^c-N) \\ &\quad \times (\Lambda_1^-\Lambda_2^+\Lambda_3^+ + \Lambda_1^+\Lambda_2^-\Lambda_3^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^- - 2\Lambda_1^+\Lambda_2^+\Lambda_3^+) \\ &= \sum_{\frac{X}{4}<p_1,p_2,p_3\leq X} (\log p_1)(\log p_2)(\log p_3)\theta(p_1^c+p_2^c+p_3^c-N) \\ &\quad \times (3\Lambda_1^-\Lambda_2^+\Lambda_3^+ - 2\Lambda_1^+\Lambda_2^+\Lambda_3^+) \\ &= 3\Gamma_1 - 2\Gamma_2,\end{aligned} \tag{3.3}$$

where

$$\Gamma_1 = \sum_{\frac{X}{4}<p_1,p_2,p_3\leq X} (\log p_1)(\log p_2)(\log p_3)\Lambda_1^-\Lambda_2^+\Lambda_3^+\theta(p_1^c+p_2^c+p_3^c-N), \tag{3.4}$$

$$\Gamma_2 = \sum_{\frac{X}{4}<p_1,p_2,p_3\leq X} (\log p_1)(\log p_2)(\log p_3)\Lambda_1^+\Lambda_2^+\Lambda_3^+\theta(p_1^c+p_2^c+p_3^c-N). \tag{3.5}$$

Let

$$\begin{aligned}L^{\pm}(x) &= \sum_{\frac{X}{4}<p\leq X} (\log p)e(xp^c) \sum_{d|(p+2,P(z))} \lambda^{\pm}(d) \\ &= \sum_{d|P(z)} \lambda^{\pm}(d) \sum_{\substack{\frac{X}{4}<p\leq X \\ d|p+2}} (\log p)e(xp^c).\end{aligned} \tag{3.6}$$

By Fourier's inversion formula,

$$\theta(t) = \int_{-\infty}^{\infty} \Theta(x)e(xt)\mathrm{d}x, \tag{3.7}$$

we can deduce from (3.4)–(3.5) that

$$\begin{aligned}\Gamma_1 &= \sum_{\frac{X}{4}<p_1,p_2,p_3\le X} (\log p_1)(\log p_2)(\log p_3)\Lambda_1^-\Lambda_2^+\Lambda_3^+ \\ &\quad \times \int_{-\infty}^{\infty} \Theta(x)e(x(p_1^c+p_2^c+p_3^c-N))\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \Theta(x)L^-(x)L^+(x)^2 e(-Nx)\mathrm{d}x \end{aligned} \tag{3.8}$$

and

$$\Gamma_2 = \int_{-\infty}^{\infty} \Theta(x)L^+(x)^3 e(-Nx)\mathrm{d}x. \tag{3.9}$$

We divide Γ_1 and Γ_2 into three parts as follows:

$$\Gamma_1 = \Gamma_1^{(1)} + \Gamma_1^{(2)} + \Gamma_1^{(3)}, \tag{3.10}$$

where

$$\begin{aligned}\Gamma_1^{(1)} &= \int_{|x|<\tau} \Theta(x)L^-(x)L^+(x)^2 e(-Nx)\mathrm{d}x, \\ \Gamma_1^{(2)} &= \int_{\tau\le|x|<\Xi} \Theta(x)L^-(x)L^+(x)^2 e(-Nx)\mathrm{d}x, \\ \Gamma_1^{(3)} &= \int_{|x|\ge\Xi} \Theta(x)L^-(x)L^+(x)^2 e(-Nx)\mathrm{d}x. \end{aligned} \tag{3.11}$$

and

$$\Gamma_2 = \Gamma_2^{(1)} + \Gamma_2^{(2)} + \Gamma_2^{(3)}, \tag{3.12}$$

where

$$\begin{aligned}\Gamma_2^{(1)} &= \int_{|x|<\tau} \Theta(x)L^+(x)^3 e(-Nx)\mathrm{d}x, \\ \Gamma_2^{(2)} &= \int_{\tau\le|x|<\Xi} \Theta(x)L^+(x)^3 e(-Nx)\mathrm{d}x, \\ \Gamma_2^{(3)} &= \int_{|x|\ge\Xi} \Theta(x)L^+(x)^3 e(-Nx)\mathrm{d}x. \end{aligned} \tag{3.13}$$

From (3.2)–(3.3), (3.10) and (3.12), we have

$$\Gamma \ge (3\Gamma_1^{(1)} - 2\Gamma_2^{(1)}) + 3\Gamma_1^{(2)} + 3\Gamma_1^{(3)} - 2\Gamma_2^{(2)} - 2\Gamma_2^{(3)}. \tag{3.14}$$

In the following, we will prove

$$3\Gamma_1^{(1)} - 2\Gamma_2^{(1)} \gg \frac{\Delta X^{3-c}}{\log^3 X}, \qquad |\Gamma_1^{(2)}| + |\Gamma_1^{(3)}| + |\Gamma_2^{(2)}| + |\Gamma_2^{(3)}| \ll \Delta X^{3-c-\varepsilon}. \tag{3.15}$$

4. Some preliminary lemmas

In this section, we state some preliminary lemmas which are required in this paper. We define

$$I(x) = \int_{\frac{X}{4}}^{X} e(xt^c)\mathrm{d}t, \quad L(x) = \sum_{d \le D} \lambda(d) \sum_{\substack{\frac{X}{4} < p \le X \\ d \mid p+2}} (\log p) e(xp^c), \tag{4.1}$$

where $\lambda(d)$ are real numbers satisfying

$$|\lambda(d)| \le 1, \ \ \lambda(d) = 0 \ \text{ if } \ 2|d \ \text{ or } \ \mu(d) = 0. \tag{4.2}$$

Lemma 4.1. *For $d \le D$ and $0 < |x| \le 2\Xi$, we have*

$$\sum_{\frac{X+8}{4d} < h \le \frac{X+2}{d}} e((hd-2)^c x) \ll X^{\frac{c}{2}+\varepsilon} + \frac{X^{1-c}}{|x|d}.$$

Proof. If $|x|dX^{c-1} < \frac{1}{100c}$, then by [7, Theorem 2.1], we have

$$\sum_{\frac{X+8}{4d} < h \le \frac{X+2}{d}} e((hd-2)^c x) \ll \frac{X^{1-c}}{|x|d}. \tag{4.3}$$

If $\frac{1}{100c} \le |x|dX^{c-1} \le \frac{100}{c}$, then it follows from [7, Theorem 2.2] that

$$\begin{aligned}\sum_{\frac{X+8}{4d} < h \le \frac{X+2}{d}} e((hd-2)^c x) &\ll (|x|d^2X^{c-2})^{\frac{1}{2}} \frac{X}{d} + \frac{1}{(|x|d^2X^{c-2})^{\frac{1}{2}}} \\ &\ll |x|^{\frac{1}{2}} X^{\frac{c}{2}} + \left(\frac{X}{d}\right)^{\frac{1}{2}} \ll X^{\frac{c}{2}+\varepsilon}. \end{aligned} \tag{4.4}$$

If $|x|dX^{c-1} \ge \frac{100}{c}$, then the method of exponent pairs (see [8, §2.3]) may be applied. With the exponent pair $(\frac{1}{2}, \frac{1}{2})$, we can get

$$\sum_{\frac{X+8}{4d} < h \le \frac{X+2}{d}} e((hd-2)^c x) \ll (X^{c-1}d|x|)^{\frac{1}{2}} \left(\frac{X}{d}\right)^{\frac{1}{2}} \ll |x|^{\frac{1}{2}} X^{\frac{c}{2}} \ll X^{\frac{c}{2}+\varepsilon}. \tag{4.5}$$

Now Lemma 4.1 follows from (4.3)–(4.5).

Lemma 4.2. *Let*

$$T(x) = \sum_{d \le D} \sum_{\substack{\frac{X}{4} < n \le X \\ d \mid n+2}} e(n^c x). \tag{4.6}$$

Then for $0 < |x| \le 2\Xi$, we have

$$T(x) \ll X^{\frac{c}{2}+\varepsilon} D + \frac{X^{1-c}}{x} \log X. \tag{4.7}$$

Proof. It is easy to see that

$$\begin{aligned} T(x) &= \sum_{d\le D} \sum_{\frac{X+8}{4d}<h\le\frac{X+2}{d}} e((hd-2)^c x) \\ &\ll \max_{D_1\le D} (\log X) \sum_{D_1\le d\le 2D_1} \left| \sum_{\frac{X+8}{4d}<h\le\frac{X+2}{d}} e((hd-2)^c x) \right|. \end{aligned} \tag{4.8}$$

By Lemma 4.1, we get

$$\begin{aligned} &\sum_{D_1\le d\le 2D_1} \left| \sum_{\frac{X+8}{4d}<h\le\frac{X-2}{d}} e((hd+2)^c x) \right| \\ &\ll \sum_{D_1\le d\le 2D_1} \left(X^{\frac{c}{2}+\varepsilon} + \frac{X^{1-c}}{|x|D_1} \right) \\ &\ll X^{\frac{c}{2}+\varepsilon} D + \frac{X^{1-c}}{|x|}. \end{aligned} \tag{4.9}$$

Now Lemma 4.2 follows from (4.8)–(4.9).

Lemma 4.3. *Suppose that χ is a primitive Dirichlet's character modulo d. Let $N(T,\sigma,\chi)$ be the number of zeros $\rho(\chi)=\beta(\chi)+i\gamma(\chi)$ of Dirichlet's L-function $L(s,\chi)$ such that $|\gamma(\chi)|\le T$ and $\sigma\le\beta(\chi)\le 1$. If $T\ge 2$, then we have*

$$N(T,0,\chi) \ll T\log(\mathrm{d}T). \tag{4.10}$$

If $Q\ge 1$, $T\ge 2$, then the sum

$$\sum_{d\le Q} \sum_{\chi(d)^*} N(T,\sigma,\chi) \ll (Q^2T)^{\frac{5(1-\sigma)}{2}} (\log(QT))^{14}. \tag{4.11}$$

Proof. Equation (4.10) can be found in [6, Ch. 16]. Equation (4.11) follows easily from [11, Theorem 12.2].

Lemma 4.4. *Let*

$$\mathcal{L}(T,Q,X) = \sum_{d\le Q} \sum_{\chi(d)^*} \sum_{|\gamma(\chi)|\le T} X^{\beta(\chi)}, \tag{4.12}$$

where the summation in the inner most sum is taken over the non-trivial zeros $\rho(\chi)=\beta(\chi)+i\gamma(\chi)$ of Dirichlet's L-function $L(s,\chi)$ such that $|\gamma(\chi)|\le T$. Then we have

$$\mathcal{L}(T,Q,X) \ll X\log^{15} X + X^{\frac{1}{2}} Q^{\frac{5}{2}} T^{\frac{5}{4}} \log^{15} X.$$

Proof. Let $J = 1 + [\log X]$ and $\sigma_j = \frac{1}{2} + \frac{j}{2J}$. By Lemma 4.3, we get

$$\begin{aligned}\mathcal{L}(T, Q, X) &\ll X^{\frac{1}{2}} \sum_{d \le Q} \sum_{\chi(d)^*} \sum_{\substack{\rho(\chi)=\beta(\chi)+i\gamma(\chi)\\ |\gamma(\chi)|\le T\\ \beta(\chi)\le \frac{1}{2}}} 1 \\ &\quad + \sum_{0\le j\le J} \sum_{d \le Q} \sum_{\chi(d)^*} \sum_{\substack{\rho(\chi)=\beta(\chi)+i\gamma(\chi)\\ |\gamma(\chi)|\le T\\ \sigma_j\le\beta(\chi)\le\sigma_{j+1}}} X^{\sigma_j} \\ &\ll X^{\frac{1}{2}} Q^2 T \log X + (\log X) \max_{0\le j\le J} X^{\sigma_j} \sum_{d \le Q} \sum_{\chi(d)^*} \sum_{\substack{\rho(\chi)=\beta(\chi)+i\gamma(\chi)\\ |\gamma(\chi)|\le T\\ \sigma_j\le\beta(\chi)}} 1 \\ &\ll X^{\frac{1}{2}} Q^2 T \log X + (\log X)^{15} \max_{\frac{1}{2}\le\sigma\le 1} X^{\sigma} (Q^2T)^{\frac{5(1-\sigma)}{2}}. \end{aligned} \tag{4.13}$$

We write the second term in the form

$$X^{\sigma}(Q^2T)^{\frac{5(1-\sigma)}{2}} = e^{h(\sigma)}, \tag{4.14}$$

where

$$h(\sigma) = \left(\log X - \frac{5}{2}\log(Q^2T)\right)\sigma + \frac{5}{2}\log(Q^2T). \tag{4.15}$$

For $\frac{1}{2} \le \sigma \le 1$, it is easy to see that $h(\sigma) \le h(1)$ when $Q^2T \le X^{\frac{2}{5}}$ and $h(\sigma) \le h(\frac{1}{2})$ when $Q^2T > X^{\frac{2}{5}}$. Hence

$$\max_{\frac{1}{2}\le\sigma\le 1} X^{\sigma}(Q^2T)^{\frac{5(1-\sigma)}{2}} \ll e^{h(1)} + e^{h(\frac{1}{2})} \ll X + X^{\frac{1}{2}} Q^{\frac{5}{2}} T^{\frac{5}{4}}. \tag{4.16}$$

From (4.13) and (4.16), we can obtain Lemma 4.4.

Lemma 4.5. *Let $I(x)$ and $L(x)$ be defined by* (4.1). *Then for $|x| \le \tau$, we have*

$$L(x) = \sum_{d\le D} \frac{\lambda(d)}{\varphi(d)} I(x) + O\left(\frac{X}{\log^{\frac{A}{3}} X}\right),$$

where $A > 0$ is an arbitrarily large constant.

Proof. For $|x| \le X^{-c}(\log X)^A$, by the same argument in [16, Lemma 10, (62)–(63)], we can obtain

$$L(x) = \sum_{d\le D} \frac{\lambda(d)}{\varphi(d)} I(x) + O\left(\frac{X}{\log^{A} X}\right). \tag{4.17}$$

For $X^{-c}(\log X)^A \leq |x| \leq \tau$, it follows from [16, (70), (73), (77), (78) and (80)] that

$$L(x) - \sum_{d \leq D} \frac{\lambda(d)}{\varphi(d)} I(x) \ll \frac{X}{\log^{A-3} X} + \mathcal{R}_1 \log^3 X + \mathcal{R}_2 \log^3 X, \tag{4.18}$$

where

$$\mathcal{R}_1 \ll (\log X)^2 \max_{\substack{1 \leq Q \leq D \\ |x|X^c \leq L \leq |x|X^c D(\log X)^A}} \frac{\mathcal{L}(L, Q, X)}{QL}, \tag{4.19}$$

$$\mathcal{R}_2 \ll \frac{(\log X)}{(|x|X^c)^{\frac{1}{2}}} \max_{1 \leq Q \leq D} \frac{\mathcal{L}(4\pi c|x|X^c, Q, X)}{Q} \tag{4.20}$$

and $\mathcal{L}(L, Q, X)$ is defined by (4.12). By Lemma 4.4, we have

$$\begin{aligned} \mathcal{R}_1 &\ll \max_{\substack{1 \leq Q \leq D \\ |x|X^c \leq L \leq |x|X^c D(\log X)^A}} \frac{X \log^{17} X}{QL} + X^{\frac{1}{2}} Q^{\frac{3}{2}} L^{\frac{1}{4}} \log^{17} X \\ &\ll \frac{X}{\log^{A-17} X} + X^{\frac{1}{2}+\frac{7}{4}\delta+\frac{1}{4}\xi} \log^{17} X \ll \frac{X}{\log^{A-17} X} \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} \mathcal{R}_2 &\ll \max_{1 \leq Q \leq D} \frac{X \log^{16} X}{Q(|x|X^c)^{\frac{1}{2}}} + X^{\frac{1}{2}} Q^{\frac{3}{2}} (|x|X^c)^{\frac{3}{4}} \log^{16} X \\ &\ll \frac{X}{\log^{\frac{A}{2}-16} X} + X^{\frac{1}{2}+\frac{3}{2}\delta+\frac{3}{4}\xi} \log^{16} X \ll \frac{X}{\log^{\frac{A}{2}-16} X}. \end{aligned} \tag{4.22}$$

Now combining (4.17)–(4.18) and (4.21)–(4.22), we can obtain Lemma 4.5.

Lemma 4.6. *We have*

(i)
$$\int_{|x| \leq \tau} |L(x)|^2 \mathrm{d}x \ll X^{2-c} \log^6 X,$$

(ii)
$$\int_{|x| \leq \tau} |I(x)|^2 \mathrm{d}x \ll X^{2-c} \log^4 X,$$

(iii)
$$\int_{|x| \leq \Xi} |L(x)|^2 \mathrm{d}x \ll X \Xi \log^6 X.$$

Proof. See [16, Lemma 11].

Lemma 4.7. *We have*

$$\max_{\tau \leq |x| \leq \Xi} |L(x)| \ll X^{\frac{3}{2}-\frac{c}{2}-\varepsilon}.$$

Proof. By [16, (104), (106)–(107) and (122)], we get

$$|L(x)| \ll X^{\varepsilon}(X^{\frac{1}{3}+\frac{c}{2}}D|x|^{\frac{1}{2}} + X^{1-\frac{c}{2}}|x|^{-\frac{1}{2}} + X^{\frac{3}{4}+\frac{c}{6}}D^{\frac{2}{3}}|x|^{\frac{1}{6}} + X^{\frac{5}{6}} + X^{1-\frac{c}{6}}D^{\frac{1}{3}}|x|^{-\frac{1}{6}} + X^{1-\frac{c}{4}}|x|^{-\frac{1}{4}}). \tag{4.23}$$

Since $X^{\xi-c} \le |x| \le (\log X)^{E+3}$ and $D = X^{\delta}$, we can deduce from (4.23) that

$$|L(x)| \ll X^{2\varepsilon}(X^{\frac{1}{3}+\frac{c}{2}+\delta} + X^{\frac{3}{4}+\frac{c}{6}+\frac{2\delta}{3}} + X^{1-\frac{\xi}{6}+\frac{\delta}{3}}) \ll X^{\frac{3}{2}-\frac{c}{2}-\varepsilon}. \tag{4.24}$$

Lemma 4.8. *We have*

$$\int_{\tau<|x|\le\Xi} |L(x)|^3 \mathrm{d}x \ll X^{3-c-\varepsilon}.$$

Proof. Let $G(x) = \overline{L(x)}|L(x)|$. Then we have

$$\begin{aligned}
&\left|\int_{\tau<|x|\le\Xi} |L(x)|^3 \mathrm{d}x\right| \\
&= \left|\sum_{d\le D} \lambda(d) \sum_{\substack{\frac{X}{4}\le p\le X \\ d|p+2}} (\log p) \int_{\tau<|x|\le\Xi} e(xp^c)G(x)\mathrm{d}x\right| \\
&\le (\log X) \sum_{d\le D} \sum_{\substack{\frac{X}{4}\le p\le X \\ d|p+2}} \left|\int_{\tau<|x|\le\Xi} e(xp^c)G(x)\mathrm{d}x\right| \\
&\le (\log X) \sum_{d\le D} \sum_{\substack{\frac{X}{4}\le n\le X \\ d|n+2}} \left|\int_{\tau<|x|\le\Xi} e(xn^c)G(x)\mathrm{d}x\right|.
\end{aligned} \tag{4.25}$$

Applying Cauchy's inequality, we can deduce from (4.25) that

$$\begin{aligned}
&\left|\int_{\tau<|x|\le\Xi} |L(x)|^3 \mathrm{d}x\right|^2 \\
&\ll X(\log^3 X) \sum_{d\le D} \sum_{\substack{\frac{X}{4}\le n\le X \\ d|n+2}} \left|\int_{\tau<|x|\le\Xi} e(xn^c)G(x)\mathrm{d}x\right|^2
\end{aligned}$$

$$\ll X(\log^3 X) \int_{\tau<|y|\le\Xi} \overline{G(y)}\mathrm{d}y \int_{\tau<|x|\le\Xi} G(x)T(x-y)\mathrm{d}x$$
$$\ll X(\log^3 X) \int_{\tau<|y|\le\Xi} |G(y)|\mathrm{d}y \int_{\tau<|x|\le\Xi} |G(x)||T(x-y)|\mathrm{d}x, \tag{4.26}$$

where $T(x)$ is defined by (4.6). Now

$$\int_{\tau<|x|\le\Xi} |G(x)||T(x-y)|\mathrm{d}x$$
$$\ll \int_{\substack{\tau<|x|\le\Xi \\ |x-y|\le X^{-c}}} |G(x)||T(x-y)|\mathrm{d}x + \int_{\substack{\tau<|x|\le\Xi \\ X^{-c}<|x-y|\le 2\Xi}} |G(x)||T(x-y)|\mathrm{d}x. \tag{4.27}$$

By the trivial bound $|T(x-y)| \le T(0) \le X\log X$ and Lemma 4.7, we get

$$\int_{\substack{\tau<|x|\le\Xi \\ |x-y|\le X^{-c}}} |G(x)||T(x-y)|\mathrm{d}x$$
$$\ll X(\log X) \max_{\tau<|x|\le\Xi} |G(x)| \int_{|x-y|\le X^{-c}} 1\,\mathrm{d}x$$
$$\ll X^{1-c}(\log X) \max_{\tau<|x|\le\Xi} |L(x)|^2$$
$$\ll X^{4-2c-\varepsilon}. \tag{4.28}$$

From Lemma 4.2, Lemma 4.6(iii) and Lemma 4.7, we obtain

$$\int_{\substack{\tau<|x|\le\Xi \\ X^{-c}<|x-y|\le 2\Xi}} |G(x)||T(x-y)|\mathrm{d}x$$
$$\ll \int_{\substack{\tau<|x|\le\Xi \\ X^{-c}<|x-y|\le 2\Xi}} |G(x)| \left(X^{\frac{c}{2}+\delta+\varepsilon} + \frac{X^{1-c}\log X}{|x-y|} \right) \mathrm{d}x$$
$$\ll X^{1-c}\log X \max_{\tau<|x|\le\Xi} |L(x)|^2 \int_{X^{-c}<|x-y|\le 2\Xi} \frac{1}{|x-y|}\mathrm{d}x$$
$$+ X^{\frac{c}{2}+\delta+\varepsilon} \int_{|x|\le\Xi} |L(x)|^2\mathrm{d}x$$
$$\ll X^{4-2c-\varepsilon} + X^{1+\frac{c}{2}+\delta+2\varepsilon} \ll X^{4-2c-\varepsilon}. \tag{4.29}$$

By (4.27)–(4.29), we have

$$\int_{\tau<|x|\le\Xi} |G(x)||T(x-y)|\mathrm{d}x \ll X^{4-2c-\varepsilon}. \tag{4.30}$$

Then we can deduce from (4.26) and (4.30) that

$$\begin{aligned}\left|\int_{\tau<|x|\le\Xi} |L(x)|^3\mathrm{d}x\right|^2 &\ll X(\log X)^3 X^{4-2c-\varepsilon}\int_{|x|\le\Xi} |L(x)^2|\mathrm{d}x\\ &\ll X^{6-2c-\frac{\varepsilon}{2}}, \end{aligned}\tag{4.31}$$

where Lemma 4.6(iii) is used. Now Lemma 4.8 follows easily from (4.31).

5. The evaluation of $\Gamma_1^{(1)}$ and $\Gamma_2^{(1)}$

From Lemma 4.5, we know that if $|x|<\tau$, we have

$$L^{\pm}(x) = \mathfrak{N}^{\pm} I(x) + O\left(\frac{X}{\log^{\frac{A}{3}} X}\right), \tag{5.1}$$

where $\mathfrak{N}^{\pm}$ and $I(x)$ are defined by (2.5) and (4.1). Then we can deduce from (5.1) and Cauchy's inequality that

$$\begin{aligned}&L^-(x)(L^+(x))^2 - \mathfrak{N}^-(\mathfrak{N}^+)^2 I(x)^3\\ &= (L^-(x)-\mathfrak{N}^- I(x))(\mathfrak{N}^+)^2 I(x)^2 + L^-(x)(L^+(x)-\mathfrak{N}^+ I(x))\mathfrak{N}^+ I(x)\\ &\quad + L^-(x)L^+(x)(L^+(x)-\mathfrak{N}^+ I(x))\\ &\ll X(\log X)^{3-\frac{A}{3}}(|I(x)|^2+|L^-(x)|^2+|L^+(x)|^2),\end{aligned}\tag{5.2}$$

where the obvious estimate (see [16, (85)])

$$\mathfrak{N}^{\pm} \ll \log X \tag{5.3}$$

is used. Let

$$\mathcal{J}_0 = \int_{|x|<\tau} \Theta(x)e(-Nx)I(x)^3\mathrm{d}x,\quad \mathcal{J} = \int_{-\infty}^{\infty} \Theta(x)e(-Nx)I(x)^3\mathrm{d}x. \tag{5.4}$$

By (5.2), (2.2) and Lemma 4.6(i) (ii), we obtain

$$\begin{aligned}&\Gamma_1^{(1)} - \mathfrak{N}^-(\mathfrak{N}^+)^2\mathcal{J}_0\\ &\ll \Delta X(\log X)^{3-\frac{A}{3}}\int_{|x|<\tau}\left(|I(x)|^2+|L^-(x)|^2+|L^+(x)|^2\right)\mathrm{d}x\\ &\ll \Delta X^{3-c}(\log X)^{9-\frac{A}{3}}.\end{aligned}\tag{5.5}$$

It follows from [16, (88) and (89)] that

$$\mathcal{J} \gg \Delta X^{3-c}, \qquad |\mathcal{J}_0 - \mathcal{J}| \ll \Delta X^{3-c-2\xi}. \tag{5.6}$$

Thus combining (5.3), (5.5) and (5.6), we can get

$$\Gamma_1^{(1)} = \mathfrak{N}^-(\mathfrak{N}^+)^2\mathcal{J} + O(\Delta X^{3-c}(\log X)^{-\frac{A}{4}}). \tag{5.7}$$

By the same argument, we can obtain

$$\Gamma_2^{(1)} = (\mathfrak{N}^+)^3\mathcal{J} + O(\Delta X^{3-c}(\log X)^{-\frac{A}{4}}). \tag{5.8}$$

6. The estimation of $\Gamma_1^{(2)}$ and $\Gamma_2^{(2)}$

By Hölder's inequality, (2.2) and Lemma 4.8, we get

$$\begin{aligned}\Gamma_1^{(2)} &\ll \Delta \int\limits_{\tau<|x|\le\Xi} |L^-(x)||L^+(x)|^2\mathrm{d}x \\ &\ll \Delta \left(\int\limits_{\tau<|x|\le\Xi} |L^-(x)|^3\mathrm{d}x\right)^{\frac{1}{3}} \left(\int\limits_{\tau<|x|\le\Xi} |L^+(x)|^3\mathrm{d}x\right)^{\frac{2}{3}} \\ &\ll \Delta X^{3-c-\varepsilon}.\end{aligned} \tag{6.1}$$

Similarly,

$$\Gamma_2^{(2)} \ll \Delta \left(\int\limits_{\tau<|x|\le\Xi} |L^+(x)|^3\mathrm{d}x\right) \ll \Delta X^{3-c-\varepsilon}. \tag{6.2}$$

7. The estimation of $\Gamma_1^{(3)}$ and $\Gamma_2^{(3)}$

By [16, (42)], we have

$$|\Gamma_1^{(3)}| + |\Gamma_2^{(3)}| \ll 1. \tag{7.1}$$

8. Proof of the theorem

Combining (3.14), (5.7)–(5.8), (6.1)–(6.2) and (7.1), we obtain

$$\begin{aligned}\Gamma &\ge |3\Gamma_1^{(1)} - 2\Gamma_2^{(1)}| + O(|\Gamma_1^{(2)}| + |\Gamma_2^{(2)}| + |\Gamma_1^{(3)}| + |\Gamma_2^{(3)}|) \\ &= \left|3\mathfrak{N}^- - 2\mathfrak{N}^+\right| (\mathfrak{N}^+)^2\mathcal{J} + O(\Delta X^{3-c}(\log X)^{-\frac{A}{4}}).\end{aligned} \tag{8.1}$$

Then by (2.7)–(2.8) and (8.1), we have

$$\Gamma \ge (3f(s) - 2F(s))\mathfrak{B}^3\mathcal{J} + O(\Delta X^{3-c}(\log X)^{-\frac{A}{4}}). \tag{8.2}$$

Note that $s = \frac{\log D}{\log z} = \frac{59}{20}$ and $3f(\frac{59}{20}) - 2F(\frac{59}{20}) \ge \frac{e^\gamma}{500}$. So we can deduce from (2.6), (5.6) and (8.2) that

$$\Gamma \gg \Delta X^{3-c}(\log X)^{-3}. \tag{8.3}$$

From (8.3), we conclude that the inequality (1.3) would have a solution in primes p_1, p_2, p_3 such that each of the numbers $p_i + 2$, $i = 1, 2, 3$ has at most $[\frac{1475}{562-500c}]$ prime factors. Now the proof of the theorem is completed.

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