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Palindromic width of graph of groups

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Abstract. In this paper, we answer questions raised by Bardakov and Gongopadhyay (*Commun. Algebra* **43(11)** (2015) 4809–4824). We prove that the palindromic width of HNN extension of a group by proper associated subgroups is infinite. We also prove that the palindromic width of the amalgamated free product of two groups via a proper subgroup is infinite (except when the amalgamated subgroup has index two in each of the factors). Combining these results it follows that the palindromic width of the fundamental group of a graph of groups is mostly infinite.

Keywords. Palindromic width; graph of groups; HNN extension; amalgamated free product.

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1. Introduction

Words are basic objects in group theory and they are natural sources to view groups as geometric objects. Using words, one can naturally associate a length to each group element, and the maximum of all such lengths gives the notion of a width. The theory of verbal subgroup, that is subgroup determined by a word (for example, the commutator subgroup), and the verbal width have seen many decisive results in recent time, e.g. see [23]. It is natural to ask for widths given by curious classes of 'non-verbal' words. In this paper, we consider the width that comes from one such class, viz. the palindromic words.

Let G be a group and let S be a generating set with $S^{-1} = S$. A *word-palindrome* or simply, *palindrome* in G is a reduced word in S which reads the same forward and backward. Palindromic words arise naturally in the investigation of combinatorics of words and have been studied widely from several points of view, see [22] for a survey. Palindromes in groups have also appeared in the context of geometry of automorphisms of free groups, for example, see [6,16,17]. Gilman and Keen [18] applied the geometry of palindromes in a two-generator free group to obtain discreteness conditions for two-generator subgroups in $\mathrm{SL}(2, \mathbb{C})$.

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For an element $g \in G$, the *palindromic length*, $l_{\mathcal{P}}(g)$ is the minimum number k such that g can be expressed as a product of k palindromes. Then the *palindromic width* of G with respect to S is defined as

$$pw(G, S) = \sup_{g \in G} l_{\mathcal{P}}(g).$$

Bardakov *et al*. [8] initiated the investigation of palindromic width and proved that the palindromic width of a non-abelian free group is infinite. Recently, there have been a series of work that aims to understand the palindromic widths in several other classes of groups including relatively free groups. Bardakov and Gongopadhyay [3–5] have proved finiteness of palindromic widths of finitely generated free nilpotent groups and certain solvable groups. In [2], finiteness of palindromic width of nilpotent products has been proved. Palindromic widths of wreath products and Grigorchuk groups have been investigated by Fink [13,14]. Riley and Sale [21] have investigated palindromic widths in certain wreath products and solvable groups using finitely supported functions from $\mathbb{Z}^r$ to the given group. Fink and Thom [15] have studied palindromic widths in simple groups and yielded the first examples of groups having finite palindromic widths but infinite commutator widths.

The work has been generalized to free product of groups by Bardakov and Tolstykh in [9]. It has been proved that a free product of two groups, except $\mathbb{Z}_2 * \mathbb{Z}_2$, has infinite palindromic width. In this paper our aim is to investigate the palindromic widths of some other free constructions of groups. We investigate the palindromic width for HNN extensions and amalgamated free products of groups. For HNN extensions, we have the following.

Theorem 1.1. *Let G be a group and let A and B be proper isomorphic subgroups of G and $\phi : A \to B$ be an isomorphism. The HNN extension*

$$G_* = \langle G, t \mid t^{-1}at = \phi(a),\ a \in A \rangle$$

of G with associated subgroups A and B has infinite palindromic width with respect to the generating set $G \cup \{t, t^{-1}\}$.

For amalgamated free product of groups we prove the following theorem that extends the work of Bardakov and Tolstykh cited above.

Theorem 1.2. *Let $G = A *_C B$ be the free product of two groups A and B with amalgamated proper subgroup C and $|A : C| \geq 3$, $|B : C| \geq 2$. Then $pw(G, A \cup B)$ is infinite.*

The above two theorems answer Question 3 and Question 4 in [5], and also Problem 6 and Problem 7 in [7]. Since non-solvable Baumslag–Solitar groups are special cases of the HNN extensions, this also answers Question 2 in [5].

As an application of the above two theorems, we determine the palindromic width for the fundamental group of a graph of groups. We recall that a *graph of groups* (G, Y) consists of a non-empty, connected graph Y, a group G_P for each $P \in \text{vert } Y$ and a group G_e for each $e \in \text{edge } Y$, together with monomorphisms $G_e \to G_{\alpha(e)}$ and $G_e \to G_{\omega(e)}$, where for each edge e, $\alpha(e)$ is the initial vertex and $\omega(e)$ is the final vertex, e.g. [10,24]. We assume that $G_e = G_{\bar{e}}$. For each $P \in \text{vert } Y$, let S_P be the generating set of G_P. Also, let T be a maximal tree in Y. We fix

$$S = \{\cup_{P \in \text{vert } Y} S_P\} \cup \{\text{edge}(Y) - \text{edge}(T)\}$$

to be the *standard generating set* of the fundamental group of $(G, Y), \pi_1(G, Y)$. It is straight forward to see that the fundamental group of any graph of groups has a representation which is an amalgamated free product or a HNN extension. Hence we have the following consequence of Theorems 1.1 and 1.2.

COROLLARY 1.3

Let Y be a non-empty, connected graph. Let $\pi_1(G, Y)$ be the fundamental group of the graph of groups of Y with the standard generating set S. Then the palindromic width of $\pi_1(G, Y)$ is infinite if

(1) *Y is a loop with a vertex P and edge e; and the image of G_e is a proper subgroup of G_P; or*
(2) *Y is a tree and has an oriented edge $e = [P_1, P_2]$ such that removing e, while retaining P_1 and P_2, gives two disjoint graphs Y_1 and Y_2 with $P_i \in$ vert Y_i satisfying the following: extending $G_e \to G_{P_i}$ to $\phi_i : G_e \to \pi_1(G, Y_i)$, $i = 1, 2$, we get $[\pi_1(G, Y_1) : \phi_1(G_e)] \geq 3$ and $[\pi_1(G, Y_2) : \phi_2(G_e)] \geq 2$.*
(3) *Y has an oriented edge $e = [P_1, P_2]$ such that removing the edge, while retaining P_1 and P_2 does not separate Y and gives a new graph Y' satisfying the following: extending $G_e \to G_{P_i}$ to $\phi_i : G_e \to \pi_1(G, Y')$, $i = 1, 2$, we have $\phi_i(G_e) = H_i$ and H_1, H_2 are proper subgroups of $\pi_1(G, Y')$.*

The fundamental group in (1) is an HNN extension of G_P and so, (1) follows from Theorem 1.1. In (2), the fundamental group is an amalgamated free product of $\pi_1(G, Y_1)$ and $\pi_1(G, Y_2)$ with proper amalgamated subgroups $\phi_1(G_e) \cong \phi_2(G_e)$. The result follows from Theorem 1.2. Finally, the fundamental group in (3) is an HNN extension of G', with G' being the fundamental group of the graph of groups corresponding to Y'. Hence, this also follows from Theorem 1.1.

Idea of the proof

Let G be the group under consideration. An element g in G is a *group-palindrome* if g can be represented by a word w such that its reverse $\bar{w}$ also represents g. This notion is weaker than the notion of 'word-palindromes', see [2] for a comparison of these two notions. The set $\mathcal{P}$ of word-palindromes is obviously a subset of $\mathcal{GP}$, the set of group-palindromic words. Thus, for an element g in G, $l_{\mathcal{GP}}(g) \leq l_{\mathcal{P}}(g)$. Consequently, the palindromic width with respect to group-palindromes does not exceed $pw(G, S)$. We shall show that the palindromic width with respect to group-palindromes is infinite and that will establish the main results. To achieve this, we shall use quasi-morphism techniques.

DEFINITION 1.4

Let H be a group. A map $\Delta : H \to \mathbb{R}$ is called a *quasi-homomorphism* if there exists a constant c such that for every $x, y \in H$, $\Delta(xy) \leq \Delta(x) + \Delta(y) + c$.

Quasi-morphisms have wide applications in mathematics and for a brief survey on these objects see [19]. However, our motivation for using them in this paper comes from the work of Bardakov [1] and Dobrynina [11,12] where the authors have proved infiniteness of verbal subgroups of HNN-extensions and amalgamated free products, also see [8,9]. We shall follow their methods here.

We prove Theorem 1.1 in Sect. 2 and Theorem 1.2 in Sect. 3. For both the proofs, a canonical form for a palindromic word will be obtained. Then it will be shown that the underlying quasi-homomorphism Δ will be bounded if G has finite palindromic width. Finally, a sequence of elements in the underlying group will be noted where Δ will be bounded away, thus establishing the infiniteness of the palindromic widths. and proving the theorems.

*N*otation Let f and g be functions over non-zero integers. We write $f =_m g$ to denote that $f(k) = g(k)$ for all values of k except at most m values. Then clearly, $f =_m g$ and $g =_n h$ implies $f =_{m+n} h$. Also $f =_m g$ and $f' =_n g'$ implies $f + f' =_{m+n} g + g'$.

2. Palindromic width for HNN extensions of groups

2.1 *HNN extensions*

Let G be a group and A and B be proper isomorphic subgroups of G with the isomorphism $\phi : A \to B$. Then the HNN extension of G is

$$G_* = \langle G, t \mid t^{-1}at = \phi(a), a \in A\rangle.$$

A sequence $g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, \ldots, g_{n-1}, t^{\epsilon_n}, g_n$, $n \geq 0$, is said to be *reduced* if it does not contain subsequences of the form t^{-1}, g_i, t with $g_i \in$ A or t, g_i, t^{-1} with $g_i \in$ B. By Britton's Lemma, if a sequence $g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, \ldots, g_{n-1}, t^{\epsilon_n}, g_n$ is reduced and $n \geq 1$, then $g = g_0t^{\epsilon_1}g_1t^{\epsilon_2}\ldots g_{n-1}t^{\epsilon_n}g_n$ is not trivial in G_* and we call it a *reduced word*.

Such a representation of a group element of an HNN extension is not unique but the following lemma holds:

Lemma 2.1 [1, *Lemma* 3]. *Let* $g = g_0t^{\epsilon_1}g_1t^{\epsilon_2}\ldots g_{n-1}t^{\epsilon_n}g_n$ *and* $h = h_0t^{\theta_1}h_1t^{\theta_2}\ldots h_{m-1}t^{\theta_m}h_m$ *be reduced words, and suppose* $g = h$ *in* G_*. *Then* $m = n$ *and* $\epsilon_i = \theta_i$ *for* $i = 1, \ldots, n$.

DEFINITION 2.2

The *signature* of $g \in G_*$ is the sequence $sqn(g) = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$, $\epsilon_i \in \{1, -1\}$ for $g = g_0t^{\epsilon_1}g_1t^{\epsilon_2}\ldots g_{n-1}t^{\epsilon_n}g_n$.

By Lemma 2.1, the signature of any $g \in G_*$ is unique, irrespective of the choice of the reduced word.

Let $\sigma = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ be a signature. Then the length of the signature, $|\sigma| = n$. And the inverse signature, σ^{-1}=$(-\epsilon_n, -\epsilon_{n-1}, \ldots, -\epsilon_1)$. So, $sqn(g^{-1}) = (sqn(g))^{-1}$.

Product of two signatures σ and τ, $\sigma\tau$, is obtained by writing τ after σ.

Suppose $\sigma = \sigma_1\rho$ and $\tau = \rho^{-1}\tau_1$ with $|\rho| = r$, then we can define an r-product,

$$\sigma[r]\tau = \sigma_1\tau_1.$$

The following lemma is immediate from the above notions.

Lemma 2.3 [1, *Lemma* 4]. *For any* $g, h \in G_*$, *there exists an integer* $r \geq 0$ *such that* $sqn(gh) = sqn(g)[r]sqn(h)$, *with* $sqn(g) = \sigma_1\rho$ *and* $sqn(h) = \rho^{-1}\tau_1$ *and* $|\rho| = r$.

A reduced expression is called *positive (resp. negative)* if all exponents ϵ_i are positive (resp. negative). Further, if it is either positive or negative then the reduced expression is called *homogeneous*.

2.2 *Proof of Theorem* 1.1

Let $\sigma = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ be the signature of an element $g \in G_*$. We define $p_k(g)$ = number of $+1, +1, \ldots, +1$ sections of length k, $m_k(g)$ = number of $-1, -1, \ldots, -1$ sections of length k, $d_k(g)$= $p_k(g) - m_k(g)$, $r_k(g)$= remainder of $d_k(g)$ divided by 2, and

$$\Delta(g) = \sum_{k=1}^{\infty} r_k(g).$$

Clearly, $p_k(g^{-1}) = m_k(g)$ and so, $d_k(g^{-1}) + d_k(g) = 0$ for all $g \in G_*$.

Lemma 2.4. *For any elements $g, h \in G_*$, $\Delta(gh) \leq \Delta(g) + \Delta(h) + 6$, i.e. Δ is a quasi-homomorphism.*

Proof. The proof follows from [1, Lemma 9]. □

DEFINITION 2.5

Let $g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \ldots g_{n-1} t^{\epsilon_{n-1}} g_n$ be a reduced element in G_*. Put

$$\bar{g} = g_n t^{\epsilon_{n-1}} g_{n-1} t^{\epsilon_{n-2}} \ldots g_1 t^{\epsilon_1} g_0.$$

We say g is a group-palindrome if $\bar{g} = g$ and $\bar{g}$ depends on the reduced form.

Lemma 2.6. *A group-palindrome $g \in G_*$ has the form*

$$g = \begin{cases} g_0 t^{\epsilon_1} g_1 \ldots g_{k-1} t^{\epsilon_k} g'_k t^{\epsilon_k} g_{k-1} \ldots g_1 t^{\epsilon_1} g_0, & \text{if } |sqn(g)| = 2k, \\ g_0 t^{\epsilon_1} g_1 \ldots t^{\epsilon_k} g_k t^{\epsilon_{k+1}} g'_k t^{\epsilon_k} \ldots g_1 t^{\epsilon_1} g_0, & \text{if } |sqn(g)| = 2k+1, \end{cases}$$

where $g'_k = x g_k$, where $x \in A \cup B$.

Proof. Let $g \in G_*$ is a group-palindrome.

Case 1. $|sqn(g)| = 2k + 1$. Let $g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \ldots g_{2k} t^{\epsilon_{2k+1}} g_{2k+1}$. We know that

$$\begin{aligned} g = \bar{g} \ &\Rightarrow g\bar{g}^{-1} = 1 \\ &\Rightarrow g_0 t^{\epsilon_1} g_1 \ldots t^{\epsilon_{2k+1}} g_{2k+1} g_0^{-1} t^{-\epsilon_1} g_1^{-1} \ldots t^{-\epsilon_{2k+1}} g_{2k+1}^{-1} = 1. \end{aligned}$$

The left side is reducible. So we have $g_{2k+1} g_0^{-1} = x_0$, where $x_0 \in A$ (or $x_0 \in B$) such that $t^{\epsilon_{2k+1}} x_0 t^{-\epsilon_1} = y_0$, with $y_0 \in B$ (or $y_0 \in A$) and $\epsilon_{2k+1} = \epsilon_1 = -1$ (or 1) which implies

$$g_0 t^{\epsilon_1} g_1 \ldots t^{\epsilon_{2k}} g_{2k} y_0 g_1^{-1} t^{-\epsilon_2} \ldots g_{2k+1}^{-1} = 1.$$

Since $y_0 \in B$ (or $y_0 \in A$), $g_{2k} y_0 g_1{}^{-1} = y_1$, $y_1 \in B$ (or $y_1 \in A$) such that $t^{\epsilon_{2k}} y_1 t^{-\epsilon_2} = x_1$, where $x_1 \in A$ (or $x_1 \in B$) and $\epsilon_{2k} = \epsilon_2 = 1$ (or -1) which implies

$$g_0 t^{\epsilon_1} g_1 \ldots t^{\epsilon_{2k-1}} g_{2k-1} x_1 g_2^{-1} t^{-\epsilon_2} \ldots g_{2k+1}^{-1} = 1.$$

Since $x_1 \in A$ (or $x_1 \in B$), $g_{2k-1} x_1 g_2{}^{-1} = x_2$, $x_2 \in A$ (or $x_2 \in B$) such that $t^{\epsilon_{2k-1}} x_2 t^{-\epsilon_3} = y_2$, where $y_2 \in B$ (or $y_2 \in A$) and $\epsilon_{2k-1} = \epsilon_2 = -1$ (or 1).

In general, we get $g_{2k-i} x_i g_{i+1}^{-1} = x_{i+1}$, $x_i, x_{i+1} \in A$ (or B) such that $t^{\epsilon_{2k-i}} x_{i+1} t^{-\epsilon_{i+2}} = y_{i+1}$, where $y_{i+1} \in B$ (or A) and $\epsilon_{2k-i} = \epsilon_{i+2}$, where $0 \leq i \leq k-1$.

In the expression $g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \ldots g_k t^{\epsilon_{k+1}} g_{k+1} t^{\epsilon_{k+2}} g_{k+2} \ldots t^{\epsilon_{2k+1}} g_{2k+1}$, we put $g_{2k+1} = x_0 g_0$, and for $0 \leq i \leq k-1$, $g_{2k-i} = x_{i+1} g_{i+1} x_i^{-1}$ and $\epsilon_{2k-i} = \epsilon_{i+2}$ which implies

$$g = g_0 \ldots g_k t^{\epsilon_{k+1}} x_k g_k x_{k-1}^{-1} t^{\epsilon_k} y_{k-1} g_{k-1} y_{k-2}^{-1} t^{\epsilon_{k-1}} \ldots y_1 g_1 y_0{}^{-1} t^{\epsilon_1} x_0 g_0$$

We know $t^{\epsilon_{i+2}} x_{i+1} = y_{i+1} t^{\epsilon_{i+2}}$ (or $t^{\epsilon_{i+2}} y_{i+1} = x_{i+1} t^{\epsilon_{i+2}}$) for $-1 \leq i \leq k-2$, which implies

$$g = g_0 \ldots g_k t^{\epsilon_{k+1}} x_k g_k x_{k-1}^{-1} x_{k-1} t^{\epsilon_k} g_{k-1} y_{k-2}^{-1} y_{k-2} t^{\epsilon_{k-1}} \ldots x_1 t^{\epsilon_2} g_1 y_0{}^{-1} y_0 t^{\epsilon_1} g_0$$
$$\Rightarrow g = g_0 \ldots g_k t^{\epsilon_{k+1}} x_k g_k t^{\epsilon_k} g_{k-1} t^{\epsilon_{k-1}} \ldots t^{\epsilon_2} g_1 t^{\epsilon_1} g_0$$

Therefore,

$$g = g_0 \ldots g_k t^{\epsilon_{k+1}} g'_k t^{\epsilon_k} g_{k-1} t^{\epsilon_{k-1}} g_{k-2} \ldots g_1 t^{\epsilon_1} g_0;$$

where $g'_k = x_k g_k$.

Case 2. $|sqn(g)| = 2k$. Let $g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \ldots g_{2k-1} t^{\epsilon_{2k}} g_{2k}$. We know, $g = \bar{g}$. This implies

$$g_0 t^{\epsilon_1} g_1 \ldots t^{\epsilon_{2k}} g_{2k} g_0^{-1} t^{-\epsilon_1} g_1^{-1} \ldots t^{-\epsilon_{2k}} g_{2k}^{-1} = 1$$

The left side is reducible. So we have $g_{2k} g_0^{-1} = x_0$, where $x_0 \in A$ (or $x_0 \in B$) such that $t^{\epsilon_{2k}} x_0 t^{-\epsilon_1} = y_0$, with $y_0 \in B$ (or $y_0 \in A$) and $\epsilon_{2k} = \epsilon_1 = -1$ (or 1), which implies

$$g_0 t^{\epsilon_1} g_1 \ldots t^{\epsilon_{2k-1}} g_{2k-1} y_0 g_1^{-1} t^{-\epsilon_2} \ldots t^{-\epsilon_{2k}} g_{2k}^{-1} = 1.$$

Since $y_0 \in B$ (or $y_0 \in A$), $g_{2k-1} y_0 g_1{}^{-1} = y_1$, $y_1 \in B$ (or $y_1 \in A$) such that $t^{\epsilon_{2k-1}} y_1 t^{-\epsilon_2} = x_1$, where $x_1 \in A$ (or $x_1 \in B$) and $\epsilon_{2k-1} = \epsilon_2$ implies

$$g_0 t^{\epsilon_1} g_1 \ldots t^{\epsilon_{2k-2}} g_{2k-2} x_1 g_2^{-1} t^{-\epsilon_3} \ldots t^{-\epsilon_{2k}} g_{2k}^{-1} = 1.$$

Similarly, since $x_1 \in A$ (or $x_1 \in B$), $g_{2k-2} x_1 g_2{}^{-1} = x_2$, $x_2 \in A$ (or $x_2 \in B$) such that $t^{\epsilon_{2k-2}} x_2 t^{-\epsilon_3} = y_2$, where $y_2 \in B$ (or $y_2 \in A$) and $\epsilon_{2k-2} = \epsilon_3$ which implies

$$g_0 t^{\epsilon_1} g_1 \ldots t^{\epsilon_{2k-3}} g_{2k-3} y_2 g_3^{-1} t^{-\epsilon_4} \ldots t^{-\epsilon_{2k}} g_{2k}^{-1} = 1.$$

In general, we get $g_{2k-i} x_{i-1} g_i^{-1} = x_i$, $x_{i-1}, x_i \in A$ (or B) such that $t^{\epsilon_{2k-i}} x_i t^{-\epsilon_{i+1}} = y_i$, where $y_i \in B$ (or A) and $\epsilon_{2k-i} = \epsilon_{i+1}$, where $1 \leq i \leq k$.

In $g = g_0 t^{\epsilon_1} g_1 \dots t^{\epsilon_{2k}} g_{2k}$, we put $g_{2k} = x_0 g_0$ and for $1 \le i \le k$, $g_{2k-i} = x_i g_i x_{i-1}^{-1}$ and $\epsilon_{2k-i} = \epsilon_{i+1}$ which implies

$$\begin{aligned} g &= g_0 t^{\epsilon_1} g_1 \dots t^{\epsilon_k} x_k g_k x_{k-1}^{-1} t^{\epsilon_k} y_{k-1} g_{k-1} y_{k-2}^{-1} t^{\epsilon_{k-1}} \\ &\quad \dots t^{\epsilon_3} x_2 g_2 x_1^{-1} t^{\epsilon_2} y_1 g_2 y_0^{-1} t^{\epsilon_1} x_0 g_0 \end{aligned}$$

Putting $t^{\epsilon_{i+1}} x_i = y_i t^{\epsilon_{i+1}}$ for $-1 \le i \le k-1$ implies

$$\begin{aligned} g &= g_0 t^{\epsilon_1} g_1 \dots t^{\epsilon_k} x_k g_k x_{k-1}^{-1} x_{k-1} t^{\epsilon_k} g_{k-1} y_{k-2}^{-1} y_{k-2} t^{\epsilon_{k-1}} \\ &\quad \dots y_2 t^{\epsilon_3} g_2 x_1^{-1} x_1 t^{\epsilon_2} g_2 y_0^{-1} y_0 t^{\epsilon_1} g_0 \\ \Rightarrow g &= g_0 t^{\epsilon_1} g_1 \dots t^{\epsilon_k} x_k g_k t^{\epsilon_k} g_{k-1} t^{\epsilon_{k-1}} \dots t^{\epsilon_3} g_2 t^{\epsilon_2} g_2 t^{\epsilon_1} g_0 \end{aligned}$$

Therefore,

$$g = g_0 \dots g_k' t^{\epsilon_{k-1}} g_{k-1} \dots g_1 t^{\epsilon_1} g_0,$$

where $g_k' = x_k g_k$. □

Lemma 2.7. *Let $g \in G_*$ be a product of k group-palindromes, say $g = p_1 p_2 \dots p_k$. Then,* $\Delta(g) \le 7k - 6$.

Proof. Let p be a group-palindrome in G_* of non-zero length. Then p can be represented as $p = uv\bar{u}$, where v is the maximal homogeneous palindromic sub-word in p and $\bar{u}$ is u written in reverse.

For example, if $p = g_0 t^{\epsilon_1} g_1 \dots t^{-1} g_i t g_{i+1} t \dots t \bar{g}_{i+1} t \bar{g}_i t^{-1} \bar{g}_{i-1} \dots \bar{g}_1 t^{\epsilon_1} \bar{g}_0$, then

$$\begin{aligned} u &= g_0 t^{\epsilon_1} g_1 \dots t^{-1}, \\ v &= g_i t g_{i+1} t \dots t \bar{g}_{i+1} t \bar{g}_i, \\ \bar{u} &= t^{-1} \bar{g}_{i-1} \dots \bar{g}_1 t^{\epsilon_1} \bar{g}_0. \end{aligned}$$

Then for every k, $d_k(u) = d_k(\bar{u})$. As v is homogeneous, if k' is the length of $sqn(v)$, then

$$p_{k'}(p) = 2p_{k'}(u) + p_{k'}(v), \text{ or } m_{k'}(p) = 2m_{k'}(u) + m_{k'}(v).$$

For all other k, $p_k(p) = 2p_k(u)$ and $m_k(p) = 2m_k(u)$.

Therefore,

$$r_{k'}(p) = 1, \text{ and } r_k(p) = 0 \text{ for all other } k.$$

Thus,

$$\Delta(p) = 1.$$

If $p \in G$, then $\Delta(p) = 0$. So, $\Delta(p) \le 1$.

Then, if $g \in G_*$ is a product of k group-palindromes, say $g = p_1 p_2 \dots p_k$, then

$$\begin{aligned} \Delta(g) &= \Delta(p_1 p_2 \dots p_k) \\ &\le \Delta(p_1) + \Delta(p_2) + \cdots + \Delta(p_k) + 6(k-1) \le 7k - 6. \end{aligned} \tag{2.1}$$

This completes the proof. □

2.3 *Proof of Theorem* 1.1

Now we prove that Δ is not bounded from above. For that purpose, we produce the following sequence of reduced words $\{a_i\}$, for which $\Delta(a_i)$ is increasing.

Let $a_1 = g_0 t g_1 t^{-1} g_2 t g_3$.

Then $d_1(a_1) = 1$, so $\Delta(a_1) = 1$.

For $a_2 = g_0 t g_1 t^{-1} g_2 t g_3 t^{-1} g_4 t^{-1} g_5 t g_6 t g_7 t^{-1} g_8 t^{-1} g_9$.

$d_1(a_2) = 1,\ d_2(a_2) = -1$, so, $\Delta(a_2) = 2$.

$$a_3 = g_0 t g_1 t^{-1} g_2 t g_3 t^{-1} g_4 t^{-1} g_5 t g_6 t g_7 t^{-1} g_8 t^{-1} g_9 t g_{10} t g_{11} t g_{12} t^{-1} g_{13} t^{-1} g_{14} t^{-1} g_{15} t g_{16} t g_{17} t g_{18}.$$

Then, $d_1(a_3) = 1, d_2(a_3) = -1, d_3(a_3) = 1$, so, $\Delta(a_3) = 3$.

For each $a_i = g_0 t g_1 t^{-1} g_2 t \ldots$, we have $g_j \in G$ and since a_i is reduced, for subwords of the form $t^{\epsilon} g_i t^{-\epsilon}$, $g_i \notin A$ if $\epsilon = -1$ and $g_i \notin B$ if $\epsilon = 1$.

Given a_i, we construct a_{i+1} by attaching a segment with signature of length $3(i+1)$. In general,

$$a_n = g_0 t g_1 t^{-1} g_2 t g_3 \ldots g_{N-2n} t^{\mp 1} \ldots g_{N-n-1} t^{\mp 1} g_{N-n} t^{\pm 1} g_{N-n+1} t^{\pm 1} \ldots g_{N-1} t^{\pm 1} g_N;$$

where $N = \frac{3n(n+1)}{2}$; and

$$sqn(a_n) = (1, -1, 1, -1, -1, 1, 1, -1, -1, \ldots, \underbrace{\mp 1, \ldots, \mp 1}_{n-1 \text{ times}}, \underbrace{\pm 1, \ldots, \pm 1}_{n \text{ times}}, \underbrace{\mp 1, \ldots, \mp 1}_{n \text{ times}}, \underbrace{\pm 1, \ldots, \pm 1}_{n \text{ times}})$$

$$\Delta(a_n) = n.$$

Then, by (2.1), we get that the palindromic width of G_* is infinite. This proves Theorem 1.1.

3. Palindromic width for amalgamated free products

First recall the notion of free product with amalgamation. Let $A = \langle a_1, \ldots | R_1, \ldots \rangle$ and $B = \langle b_1, \ldots | S_1, \ldots \rangle$ be groups. Let $C_1 \subset A$ and $C_2 \subset B$ be subgroups such that there exists an isomorphism $\phi : C_1 \to C_2$. Then the *free product of A and B*, amalgamating the subgroups C_1 and C_2 by the isomorphism ϕ is the group

$$G = \langle A, B \,|\, c = \phi(c), c \in C_1 \rangle.$$

We can view G as the quotient of the free product $A * B$ by the normal subgroup generated by $\{c\phi(c)^{-1} | c \in C_1\}$. The subgroups A and B are called factors of G, and since C_1 and C_2 are identified in G, we will denote them both by C.

We shall divide the proof of Theorem 1.2 into two cases.

3.1 *Case* 1

For a non-trivial $a \in A \cup B$ such that $CaC \neq Ca^{-1}C$. We shall prove the following:

Lemma 3.1. *Let $G = A *_C B$ be the free product of two groups A and B with amalgamated subgroup C. Let $|A : C| \geq 3$, $|B : C| \geq 2$ and there exists an element $a \in A \cup B$ for which $CaC \neq Ca^{-1}C$. Then $pw(G, \{A \cup B\})$ is infinite.*

To prove this, we shall use the quasi-homomorphism constructed in [11,12]. We recall the construction here.

3.1.1 *Quasi-homomorphisms*

DEFINITION 3.2

A sequence $x_1, \ldots, x_n$, $n \geq 0$, is said to be *reduced* if

(1) Each x_i is in one of the factors.
(2) Successive x_i, x_{i+1} come from different factors.
(3) If $n > 1$, no x_i is in C.
(4) If $n = 1$, $x_1 \neq 1$.

For the normal form of elements in free products with amalgamation, see for eg. [10,20], if $x_1, \ldots, x_n$ is a reduced sequence, $n \geq 1$, then the product $x_1 \ldots x_n \neq 1$ is in G and it is called a reduced word. Such a representation of a group element is not unique but the following proposition holds:

PROPOSITION 3.3

Let $g = x_1 \ldots x_n$ and $h = y_1 \ldots y_m$ be reduced words such that $g = h$ in G. Then $m = n$.

Proof. Since $g = h$, we have

$$1 = x_1 \ldots x_n y_m^{-1} \ldots y_2^{-1} y_1^{-1}.$$

Since g and h are reduced, we require $x_n y_m^{-1}$ to belong to C. To reduce it further we need $x_{n-1} x_n y_m^{-1} y_{m-1}^{-1}$ to be in C and so on. Hence, $m = n$. □

DEFINITION 3.4

Let $g = x_1 \ldots x_n$ be a reduced word of $g \in G$. The elements x_k are said to be *syllables* of g. Then the *length* of g is the number of syllables of g and it is denoted by $l(g)$. Here, for $g = x_1 \ldots x_n$, $l(g) = n$.

DEFINITION 3.5

Let $a \in A$ such that $CaC \neq Ca^{-1}C$. Let $g \in G$, and $g = x_1 x_2 \ldots x_n$ be a reduced word representing it. Then the *special form* of g associated to this reduced word is obtained by replacing x_i by $ua^{\epsilon}u'$, whenever $x_i = ua^{\epsilon}u'$ for some $u, u' \in C$ and $\epsilon \in \{+1, -1\}$, in the following way:

(1) When $i = 1$, $x_1 = ua^{\epsilon}u'$, we write $g = ua^{\epsilon}x_2' \ldots x_n$, where $x_2' = u'x_2$.
(2) When $2 \leq i \leq n-1$, $x_i = ua^{\epsilon}u'$, we write $g = x_1 x_2 \ldots x_{i-1}' a^{\epsilon} x_{i+1}' \ldots x_n$, where $x_{i-1}' = x_{i-1}u$ and $x_{i+1}' = u'x_{i+1}$.
(3) When $i = n$ and $x_n = ua^{\epsilon}u'$, where $\epsilon \in \{+1, -1\}$ and $u, u' \in C$, we replace x_n by $g = x_1 x_2 \ldots x_{n-1}' a^{\epsilon} u'$, where $x_{n-1}' = x_{n-1}u$.

An *a-segment* of length $2k-1$ is a segment of the reduced word of the following form

$$ax_1 \dots x_{2k-1}a,$$

where $x_j \neq a$ for $j = 1, \dots, 2k-1$ such that the length of $x_1 \dots x_{2k-1}$ is $2k-1$.

Similarly, an a^{-1}-*segment* of length $2k-1$ is a segment of the reduced word of the following form

$$a^{-1}x_1 \dots x_{2k-1}a^{-1},$$

where $x_j \neq a^{-1}$ for $j = 1, \dots, 2k-1$ such that the length of $x_1 \dots x_{2k-1}$ is $2k-1$.

For $g \in G$ expressed in special form, we define $p_k(g)$ = number of a-segments of length $2k-1$, $m_k(g)$ = number of a^{-1}-segments of length $2k-1$, $d_k(g)= p_k(g) - m_k(g)$, $r_k(g)=$ remainder of $d_k(g)$ divided by 2, and

$$\Delta(g) = \sum_{k=1}^{\infty} r_k(g) \tag{3.1}$$

Clearly, $p_k(g^{-1}) = m_k(g)$ and so, $d_k(g^{-1}) + d_k(g) = 0$ for all $g \in$ G.

Lemma 3.6. *Δ is well-defined on special forms.*

Proof. Let $g \in G$ and $x_1x_2 \dots x_n$ and $y_1y_2 \dots y_n$ be two reduced forms of g. Now, $x_1x_2 \dots x_n = y_1y_2 \dots y_n$ implies $x_1 \dots x_n y_n^{-1} \dots y_1^{-1} = 1$. Then $x_n y_n^{-1} = c_n \in C$. Further $x_{n-1}c_n y_{n-1}^{-1} = c_{n-1} \in C$ and so on. In general, for $1 \leq i \leq n$, $x_i c_{i+1} y_i^{-1} = c_i \in C$.

So, if $x_i = ua^{\epsilon}u'$ for some $u, u' \in C$ and $\epsilon \in \{+1, -1\}$, $ua^{\epsilon}u'c_{i+1}y_i^{-1} = c_i$. This gives $va^{\epsilon}v' = y_i$, where $v = c_i^{-1}u \in C$ and $v' = u'c_{i+1} \in C$.

Thus, for any $k \in \mathbb{N}$, number of a^{ϵ} segments of length $2k-1$, for $\epsilon \in \{+1, -1\}$ is independent of the special form of $g \in G$. Thus, Δ is well-defined on special forms. □

Lemma 3.7. *For any elements $g, h \in G$, $\Delta(gh) \leq \Delta(g) + \Delta(h) + 9$, i.e., Δ is a quasi-homomorphism.*

Proof. See the proof of [11, Lemma 2]. □

3.1.2 *Normal form of palindromes*

DEFINITION 3.8

Let $g = x_1 \dots x_n$ be a reduced word of $g \in G$. Let $\bar{g}$ be the word obtained by writing g in the reverse order, i.e. $\bar{g} = x_n \dots x_1$. This is a non-trivial element of G. We say g is a group-palindrome if $\bar{g} = g$.

Lemma 3.9. *A group-palindrome $g \in G$ has the form*

$$g = x_1x_2 \dots x_k x'_{k+1} x_k x_{k-1} \dots x_1,$$

where $x'_{k+1} = x_{k+1}c$ with $c \in C$.

Proof. Let g is a group-palindrome in G.

Case 1. $l(g) = 2k + 1$. Let

$$g = x_1 x_2 \ldots x_k x_{k+1} \ldots x_{2k} x_{2k+1}.$$

We know that $g = \bar{g}$ implies

$$x_1 x_2 \ldots x_k x_{k+1} \ldots x_{2k} x_{2k+1} = x_{2k+1} x_{2k} \ldots x_2 x_1$$
$$\Rightarrow x_1 x_2 \ldots x_{2k} x_{2k+1} {x_1}^{-1} {x_2}^{-1} \ldots {x_{2k}}^{-1} {x_{2k+1}}^{-1} = 1.$$

Since the expression on the left side is reducible, $x_{2k+1} {x_1}^{-1} = c_1$; for $c_1 \in C$. This implies, $x_{2k+1} = c_1 x_1$. Thus,

$$x_1 x_2 \ldots x_{2k} c_1 {x_2}^{-1} \ldots {x_{2k-1}}^{-1} {x_{2k}}^{-1} = 1.$$

Further, $x_{2k} c_1 {x_2}^{-1} = c_2$; for $c_2 \in C \Rightarrow x_{2k} = c_2 x_2 c_1^{-1}$ which implies

$$x_1 x_2 \ldots x_{2k-1} c_2 {x_3}^{-1} \ldots {x_{2k-1}}^{-1} {x_{2k}}^{-1} = 1.$$

In general, we get $x_{2k-i} = c_{i+2} x_{i+2} c_{i+1}^{-1}$ for $0 \leq i \leq k - 2$. Then

$$g = x_1 x_2 \ldots x_k x_{k+1} \ldots x_{2k-1} x_{2k}.$$
$$\Rightarrow g = x_1 x_2 \ldots x_k x_{k+1} c_k x_k c_{k-1}^{-1} c_{k-1} x_{k-1} c_{k-2}^{-1} \ldots c_3 x_3 c_2^{-1} c_2 x_2 c_1^{-1} c_1 x_1$$
$$\Rightarrow g = x_1 x_2 \ldots x_k x_{k+1} c_k x_k \ldots x_3 x_2 x_1.$$

Therefore, $g = x_1 x_2 \ldots x'_{k+1} x_k \ldots x_3 x_2 x_1$, where $x'_{k+1} = x_{k+1} c_k$.

Case 2. $l(g) = 2k$. Let

$$g = x_1 x_2 \ldots x_k x_{k+1} \ldots x_{2k-1} x_{2k}.$$

We know that $g = \bar{g}$ implies

$$x_1 x_2 \ldots x_k x_{k+1} \ldots x_{2k-1} x_{2k} = x_{2k} x_{2k-1} \ldots x_2 x_1$$
$$\Rightarrow x_1 x_2 \ldots x_{2k-1} x_{2k} {x_1}^{-1} {x_2}^{-1} \ldots {x_{2k-1}}^{-1} {x_{2k}}^{-1} = 1.$$

We know the expression on the left side is reducible. So, $x_{2k} {x_1}^{-1} = c_1$; for $c_1 \in C$ $\Rightarrow x_{2k} = c_1 x_1$ which implies

$$x_1 x_2 \ldots x_{2k-1} c_1 {x_2}^{-1} \ldots {x_{2k-1}}^{-1} {x_{2k}}^{-1} = 1.$$

Further, $x_{2k-1} c_1 {x_2}^{-1} = c_2$; for $c_2 \in C \Rightarrow x_{2k-1} = c_2 x_2 c_1^{-1}$ which implies

$$x_1 x_2 \ldots x_{2k-2} c_2 {x_3}^{-1} \ldots {x_{2k-1}}^{-1} {x_{2k}}^{-1} = 1.$$

In general, we get $x_{2k-i} = c_{i+1} x_{i+1} c_i^{-1}$ for $1 \leq i \leq k - 1$. In particular, for $i = k$, $x_k = c_{k+1} x_{k+1} c_k^{-1}$. This is a contradiction as the consecutive syllables lie in different factors in a reduced word.

Thus, for $g \in G$ with $l(g) = 2k$, g cannot be a group-palindrome.

Lemma 3.10. *Let $g \in G$ be a product of k group-palindromes, say $g = p_1 p_2 \dots p_k$. Then,* $\Delta(g) \leq 12k - 9$.

Proof. Let p be a group-palindrome in G of non-zero length. Then p can be expressed as $p = hu\bar{h}$, where $\bar{h}$ is h written in reverse and u is of the form x_i' from Lemma 3.9. Then, for every k, $d_k(h) = d_k(\bar{h})$.

If $u = a$, we have

$$p_k(p) =_2 2p_k(h),$$
$$m_k(p) =_1 2m_k(h).$$

Then we get

$$d_k(p) =_3 2d_k(h).$$

If $u = a^{-1}$, as above, we get $d_k(p) =_3 2d_k(h)$. If $u \neq a, a^{-1}$, we get $d_k(p) =_2 2d_k(h)$. So, in general we have,

$$d_k(p) =_3 2d_k(h).$$

Thus,

$$r_k(p) =_3 0.$$

and $\Delta(p) \leq 3$.

Thus, if $g \in G$ is a product of k group-palindromes, say $g = p_1 p_2 \dots p_k$, then

$$\Delta(g) = \Delta(p_1 p_2 \dots p_k) \leq \Delta(p_1) + \Delta(p_2) + \dots + \Delta(p_k) + 9(k-1) \leq 12k - 9. \tag{3.2}$$

This completes the proof. □

3.1.3 *Proof of Lemma* 3.1

Now we prove that Δ is not bounded from above. For that purpose, we produce the following sequence $\{g_i\}$ for which $\Delta(g_i)$ is increasing.

Let $b \in B$ but not in C.

Let $g_1 = baba^{-1}ba$. Then, $p_1(g_1) = 0$, $p_2(g_1) = 1$ and $p_k(g_1) = 0$ for all other k; $m_k(g_1) = 0$ for all k; $d_2(g_1) = 1$ and $d_k(g_1) = 0$ for all other k. So, $\Delta(g_1) = 1$.

Let $g_2 = baba^{-1}baba^{-1}ba^{-1}ba$. Then $p_1(g_2) = 0$, $p_2(g_2) = p_3(g_2) = 1$ and $p_k(g_2) = 0$ for all other k, and, $m_1(g_2) = m_2(g_2) = 1$ and $m_k(g_2) = 0$ for all other k. So, $\Delta(g_2) = 2$.

Let $g_3 = baba^{-1}baba^{-1}ba^{-1}baba^{-1}ba^{-1}ba^{-1}ba$. Then $p_1(g_3) = 0$, $p_2(g_3) = p_3(g_3) = p_4(g_3) = 1$ and $p_k(g_3) = 0$ for all other k; $m_1(g_3) = 3$, $m_2(g_3) = 2$ and $m_k(g_3) = 0$ for all other k. So, $\Delta(g_3) = 4$.

In general, for

$$g_n = baba^{-1}ba(ba^{-1})^2 \dots ba(ba^{-1})^{n-1}ba(ba^{-1})^n ba,$$

$p_1(g_n) = 0$, $p_k(g_n) = 1$ for $1 < k < n+1$; $m_1(g_n) = \frac{n(n-1)}{2}$, $m_2(g_n) = n-1$ and for $k \neq 1, 2$, $m_k(g_n) = 0$. Thus we have

$$\Delta(g_n) = r_1 + r_2 + (n-1),$$

where r_1 is the remainder of $\frac{n(n-1)}{2}$ divided by 2 and r_2 is that of n divided by 2. So,

$$\Delta(g_n) \geq n-1.$$

Then, by (3.2), we get that the palindromic width of G is infinite. This proves Lemma 3.1.

3.2 *Case* 2

For a non-trivial $a \in A \cup B$ such that $CaC = Ca^{-1}C$.

Lemma 3.11. *Let $G = A*_C B$ be the free product of two groups A and B with amalgamated subgroup C. Let $|A : C| \geq 3$, $|B : C| \geq 2$ and there exists an element $a \in A \cup B$ for which $CaC = Ca^{-1}C$. Then $pw(G, \{A \cup B\})$ is infinite.*

This lemma follows using similar methods as in Lemma 3.1. Since the arguments will only be a slight modification of the ones used in Lemma 3.1, we shall skip the details here. Following [11], we can define a quasi-homomorphism Δ, same as (3.1), and it follows that:

Lemma 3.12. *For any elements $g, h \in G$, $\Delta(gh) \leq \Delta(g) + \Delta(h) + 9$.*

For a proof of the above lemma, see [11,12].

If $g \in G$ is a product of k group-palindromes, say $g = p_1 p_2 \ldots p_k$, then using Lemma 3.12, a version of Lemma 3.10 holds, and we have

$$\Delta(g) = \Delta(p_1 p_2 \ldots p_k) \leq \Delta(p_1) + \Delta(p_2) + \cdots + \Delta(p_k) + 9(k-1) \leq 12k - 9. \tag{3.3}$$

And finally, for the same sequence used in Section 3.1, using (3.3), we get that the palindromic width of G is infinite.

3.3 *Proof of Theorem* 1.2

The result follows by combining Lemmas 3.1 and 3.11.

3.3.1 *The index two case*

So far we have shown that the palindromic width of $G = A*_C B$, when $|A : C| \geq 3$, $|B : C| \geq 2$, is infinite. Let us now consider the case when $|A : C| \leq 2$, $|B : C| \leq 2$.

PROPOSITION 3.13

Let $G = A*_C B$ be the free product of two groups A and B with amalgamated subgroup C and $|A : C| \leq 2$, $|B : C| \leq 2$. Let S and T be the generating sets of A and B respectively. If $pw(C, \{S, T\})$ is finite, then $pw(G, \{A \cup B\})$ is finite.

Proof. We only need to consider the case of $|A : C| = 2, |B : C| = 2$. Then C is a normal subgroup of both A and B.

Let T_A and T_B be the sets of right coset represntatives of C in A and C in B respectively. Here, $T_A \cong \mathbb{Z}_2$ and $T_B \cong \mathbb{Z}_2$.

Every element $g \in G$ can be expressed uniquely as a C-normal form. A C-normal form of g is a sequence $(x_0, x_1, ..., x_n)$, where $g = x_0 x_1 ... x_n$ with $x_0 \in C, x_i \in T_A \setminus \{1\} \sqcup T_B \setminus \{1\}$ and consecutive x_i, x_{i+1} lie in distinct sets. Existence and uniqueness of such a C-normal form follows from [10, Theorem 11.3].

So, for $g = x_0 x_1 ... x_n$, where $(x_0, x_1, ..., x_n)$ is the C-normal form of g, clearly, $x_1 x_2 ... x_n \in T_A * T_B \cong \mathbb{Z}_2 * \mathbb{Z}_2$.

This implies that $pw(x_1 x_2 ... x_n) \leq 2$. Therefore, $pw(g) \leq pw(x_0) + pw(x_1 x_2 ... x_n) \leq 3$. □

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