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Upper bound for the first nonzero eigenvalue related to the p-Laplacian

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Abstract. Let M be a closed hypersurface in $\mathbb{R}^n$ and Ω be a bounded domain such that $M = \partial\Omega$. In this article, we obtain an upper bound for the first nonzero eigenvalue of the following problems:

(1) *Closed eigenvalue problem*:

$$\Delta_p u = \lambda_p \, |u|^{p-2} \, u \quad \text{on } M.$$

(2) *Steklov eigenvalue problem*:

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \mu_p \, |u|^{p-2} \, u & \text{on } M. \end{cases}$$

These bounds are given in terms of the first nonzero eigenvalue of the usual Laplacian on the geodesic ball of the same volume as of Ω.

Keywords. p-Laplacian; closed eigenvalue problem; Steklov eigenvalue problem; center-of-mass.

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1. Introduction

The p-Laplace operator, defined as $\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u)$, is the nonlinear generalization of the usual Laplace operator.

Many interesting results, providing sharp upper bounds for the first nonzero eigenvalue of the usual Laplacian ($p = 2$) have been obtained. For a compact n-dimensional manifold M isometrically immersed in $\mathbb{R}^m$, Bleecker and Weiner [2] obtained a sharp upper bound for the first nonzero eigenvalue of Laplacian in terms of the second fundamental form of M. Under the same assumptions, Reilly [11] improved this estimate and gave an upper bound for the first nonzero Laplacian eigenvalue in terms of the higher order mean curvatures of M. This result was later extended to submanifolds of the simply connected space forms in various ways [8,9]. These bounds are extrinsic in nature as they

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depend either on the length of the second fundamental form or on the mean curvature vector of M, and these quantities depend on the way manifold is embedded into another manifold.

Later, Santhanam [12,13] obtained some sharp bounds of the first nonzero Laplace eigenvalue in terms of the intrinsic quantities. Let M be a hypersurface in a rank-1 symmetric space. In [12], a sharp upper bound for the first nonzero eigenvalue of M was obtained in terms of the integral of the first nonzero eigenvalue of the geodesic spheres centered at the centre of gravity of M.

For a closed hypersurface M bounding a convex domain Ω in the simply connected space form $\mathbb{M}(k)$, $k = 0$ or 1, Santhanam [13] proved that

$$\frac{\lambda_1(M)}{\lambda_1(S(R))} \leq \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(S(R))},$$

where $S(R)\,(= \partial B(R))$ is a geodesic sphere of radius $R > 0$ such that $\mathrm{Vol}(B(R)) = \mathrm{Vol}(\Omega)$. A similar result was also obtained for $k = -1$.

The result of [11] was generalized for p-Laplacian by Du and Mao [4] for a compact Riemannian manifold M immersed into an Euclidean space, a unit sphere and a projective space. Later, Chen and Wei extended this result to all space forms [3].

In this article, we extend the results of [13] to the p-Laplacian for a closed hypersurface $M \subset \mathbb{R}^n$. In particular, we consider the closed eigenvalue problem:

$$\Delta_p u = \lambda_p \, |u|^{p-2} \, u \quad \text{on } M, \tag{1}$$

where M is a closed hypersurface in $\mathbb{R}^n$. This problem has discrete spectrum. We find an upper bound for the first nonzero eigenvalue of this problem.

Let M be a closed hypersurface in $\mathbb{R}^n$ and Ω be the bounded domain such that $M = \partial\Omega$. Consider the following problem:

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= \mu u \quad \text{on } \partial\Omega, \end{aligned}$$

where ν is the outward unit normal on the boundary $\partial\Omega$ and μ is a real number. This problem is known as the Steklov eigenvalue problem and was introduced by a Russian mathematician, V. A. Steklov. This problem is important as the set of eigenvalues of the Steklov problem is same as the set of eigenvalues of the well-known Dirichlet–Neumann map.

There are several results which estimate the first nonzero eigenvalue μ_1 of the Steklov eigenvalue problem [1,5–7,10]. The first result is an isoperimetric upper bound for μ_1 and was given by Weinstock [15] in 1954. He proved that among all simply connected planar domains with analytic boundary of fixed perimeter, the circle maximizes μ_1. Using the Weinstock inequality, Escobar [6] proved that for a fixed volume, among all bounded simply connected domain in 2-dimensional simply connected space forms, geodesic balls maximize the first nonzero Steklov eigenvalue. This result was extended to non-compact rank-1 symmetric spaces in [1]. In this article, we prove a result similar to [1] for a bounded domain in $\mathbb{R}^n$. The Steklov problem for p-Laplacian is given by

$$\Delta_p u = 0 \quad \text{in } \Omega, \tag{2}$$

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \mu_p\, |u|^{p-2}\, u \quad \text{on } M, \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ such that $M = \partial\Omega$ and ν is outward unit normal on M. This problem has a nondecreasing and unbounded sequence of positive eigenvalues [14].

The main ingredient to prove the results of this article is the variational characterization of eigenvalues. We choose suitable test functions for the variational characterization to obtain these eigenvalue bounds.

In section 2, we state our main results. In section 3, we state some basic facts about the first nonzero eigenvalues of problems (1) and (2), and prove some results which will be required in the later sections. Followed by this, in sections 4, 5 and 6, we provide the proof of the results stated in section 2.

2. Statement of the results

We state a variation of centre of mass theorem. This is crucial for our proof of main results.

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $M = \partial\Omega$. Then for every real number $p > 1$, there exists a point $t \in \mathbb{R}^n$ depending on p and the normal coordinate system $(X_1, X_2, \ldots, X_n)$ centered at t such that for $1 \leq i \leq n$,*

$$\int_M |X_i|^{p-2} X_i = 0.$$

Now we state our main results.

The following theorem provides an upper bound for the first nonzero eigenvalue $\lambda_{1,p}$ of the closed eigenvalue problem (1).

Theorem 2. *Let M be a closed hypersurface in $\mathbb{R}^n$ bounding a bounded domain Ω. Let $R > 0$ be such that $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B(R))$, where $B(R)$ is a ball of radius R. Then the first nonzero eigenvalue $\lambda_{1,p}$ of the closed eigenvalue problem (1) satisfies*

$$\lambda_{1,p} \leq n^{\frac{|p-2|}{2}}\, \lambda_1(S(R))^{\frac{p}{2}} \left(\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(S(R))}\right). \tag{4}$$

Further, for $p = 2$, the bound (4) is sharp and the equality holds if and only if M is a geodesic sphere of radius R [13]. Moreover, if equality holds in (4), then M is a geodesic sphere of radius R and $p = 2$.

In case of the Steklov eigenvalue problem, we have the following upper bound for the first nonzero eigenvalue.

Theorem 3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary M and $R > 0$ be such that $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B(R))$, where $B(R)$ is a ball of radius R. Then the first nonzero eigenvalue $\mu_{1,p}$ of problem (2) satisfies the following inequalities:*

(1) *For* $1 < p < 2$,

$$\mu_{1,p} \leq \frac{1}{R^{p-1}}. \tag{5}$$

(2) *For* $p \geq 2$,

$$\mu_{1,p} \leq \frac{n^{p-2}}{R^{p-1}}. \tag{6}$$

For $p = 2$, *equality holds in* (5) *and* (6) *iff* M *is a geodesic sphere of radius* R [1]. *Furthermore, if equality holds in* (5) *and* (6), *then* M *is a geodesic sphere of radius* R *and* $p = 2$.

3. Preliminaries

In this section, we state some basic facts about the first nonzero eigenvalue of the eigenvalue problems (1) and (2). We will also prove some results that are needed in subsequent sections.

Let $\lambda_{1,p}$ and $\mu_{1,p}$ be the first nonzero eigenvalues of the closed and the Steklov eigenvalue problems, respectively. Then the variational characterization for $\lambda_{1,p}$ and $\mu_{1,p}$ is given by

$$\lambda_{1,p} = \inf \left\{ \frac{\int_M \|\nabla^M u\|^p}{\int_M |u|^p} : \int_M |u|^{p-2}u = 0, u(\neq 0) \in C^1(M) \right\},$$
$$\mu_{1,p} = \inf \left\{ \frac{\int_\Omega \|\nabla u\|^p}{\int_M |u|^p} : \int_M |u|^{p-2}u = 0, u(\neq 0) \in C^1(\Omega) \right\}.$$

Let M be a closed hypersurface in $\mathbb{R}^n$ and Ω be a bounded domain in $\mathbb{R}^n$ such that $M = \partial\Omega$. Fix a point $q \in \mathbb{R}^n$. Then for every point $s \in M$, the line joining q and s may intersect M at some point other than s. For every point $s \in M$, let $r(s) = d(q, s)$ and for every $u \in \mathbb{S}^{n-1}$, let $\beta(u) = \max\{\beta > 0 | \, q + \beta u \in M\}$. We take $\beta(u) = 0$ if there does not exist any β such that $q + \beta u \in M$. Let $A = \{q + \beta(u)u |\, u \in \mathbb{S}^{n-1}$ and $\beta(u) > 0\}$. Then $A \subseteq M$. The following lemma is a variation of Lemma 1 in [13].

Lemma 4. *Fix a point* $q \in \mathbb{R}^n$. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with smooth boundary* M *and* $R > 0$ *be such that* $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B(R))$, *where* $B(R)$ *is a ball of radius* R *centered at* q. *Then for* $p > 1$,

$$\int_M r^p(s)\,\mathrm{d}s \geq R^p \mathrm{Vol}(S(R)). \tag{7}$$

Further, equality holds in (7) *iff* M *is a geodesic sphere of radius* R *centered at* q.

Proof. For a point $s \in A$, let γ_s be the unique unit speed geodesic joining q and s with $\gamma_s(0) = q$. Let $u = \gamma_s'(0)$ and $t_s(u) = d(q, s)$. Let $\theta(s)$ be the angle between the outward unit normal $\nu(s)$ to M and the radial vector $\partial r(s)$. Let du be the spherical volume density of the unit sphere $\mathbb{S}^{n-1}$. Then

$$\begin{aligned}\int_M r^p(s)\,\mathrm{d}s &\geq \int_A r^p(s)\,\mathrm{d}s\\ &= \int_{\mathbb{S}^{n-1}} (t_s(u))^p \sec\theta(s)\,(t_s(u))^{n-1}\,\mathrm{d}u\\ &\geq \int_{\mathbb{S}^{n-1}} (t_s(u))^{n+p-1}\,\mathrm{d}u\\ &= (n+p-1)\int_{\mathbb{S}^{n-1}}\int_0^{t_s(u)} r^{n+p-2}\mathrm{d}r\,du\\ &\geq (n+p-1)\int_\Omega r^{p-1}\mathrm{d}V\end{aligned}$$

and

$$\begin{aligned}\int_\Omega r^{p-1}\mathrm{d}V &= \int_{\Omega\cap B(R)} r^{p-1}\mathrm{d}V + \int_{\Omega\setminus\Omega\cap B(R)} r^{p-1}\mathrm{d}V\\ &= \int_{B(R)} r^{p-1}\mathrm{d}V - \int_{B(R)\setminus\Omega\cap B(R)} r^{p-1}\mathrm{d}V + \int_{\Omega\setminus\Omega\cap B(R)} r^{p-1}\mathrm{d}V\\ &\geq \int_{B(R)} r^{p-1}\mathrm{d}V - \int_{B(R)\setminus\Omega\cap B(R)} r^{p-1}\mathrm{d}V + \int_{\Omega\setminus\Omega\cap B(R)} R^{p-1}\mathrm{d}V\\ &= \int_{B(R)} r^{p-1}\mathrm{d}V + \int_{B(R)\setminus\Omega\cap B(R)} (R^{p-1}-r^{p-1})\mathrm{d}V\\ &\geq \int_{B(R)} r^{p-1}\mathrm{d}V\\ &= \int_{\mathbb{S}^{n-1}}\int_0^R r^{n+p-2}\mathrm{d}r\,\mathrm{d}u\\ &= \int_{\mathbb{S}^{n-1}} \frac{R^{n+p-1}}{n+p-1}\mathrm{d}u\\ &= \frac{R^p}{n+p-1}\mathrm{Vol}(S(R)).\end{aligned} \tag{8}$$

In (8), we used the fact that $R \leq r$ in $(\Omega\setminus\Omega\cap B(R))$. Further, equality holds in (7) if and only if $\sec\theta(s) = 1$ for all points $s \in M$ and $\mathrm{Vol}\,(B(R)\setminus\Omega\cap B(R)) = 0$. Note that $\sec\theta(s) = 1$ if and only if $\theta(s) = 0$ for all points $s \in M$. Therefore, the outward unit normal $\nu(s) = \partial r(s)$ for all points $s \in M$. This shows that $\Omega = B(q, R)$ and M is a geodesic sphere of radius R. □

Lemma 5. *Let $n \in \mathbb{N}$ and $y_1, y_2, \ldots, y_n$ be non-negative real numbers. Then, for every real number $\gamma \geq 1$,*

$$(y_1 + y_2 + \cdots + y_n)^\gamma \geq y_1^\gamma + y_2^\gamma + \cdots + y_n^\gamma. \tag{9}$$

Proof. Let $n \in \mathbb{N}$ and $y_1, y_2, \ldots, y_n$ be non-negative real numbers. Let $\gamma \geq 1$. Then inequality (9) can be written as

$$\left(\frac{y_1}{y_1 + y_2 + \cdots + y_n}\right)^\gamma + \left(\frac{y_2}{y_1 + y_2 + \cdots + y_n}\right)^\gamma + \cdots + \left(\frac{y_n}{y_1 + y_2 + \cdots + y_n}\right)^\gamma \leq 1.$$

Therefore, it is enough to show that $a_1^\gamma + a_2^\gamma + \cdots + a_n^\gamma \leq 1$ for non-negative real numbers a_i such that $a_1 + a_2 + \cdots + a_n = 1$. Since $0 \leq a_i \leq 1$ and $\gamma \geq 1$, then $a_i^\gamma \leq a_i$. Therefore, $a_1^\gamma + a_2^\gamma + \cdots + a_n^\gamma \leq a_1 + a_2 + \cdots + a_n = 1$. This proves the lemma. □

Next we estimate $\sum_{i=1}^n \|\nabla^M x_i\|^2$. For a Riemannian geometric proof of the following lemma, see [9].

Lemma 6. *Let M be a closed hypersurface in $\mathbb{R}^n$ and Ω be a bounded domain such that $M = \partial\Omega$. For a fixed point $t \in \mathbb{R}^n$, let $(x_1, x_2, \ldots, x_n)$ be the normal coordinate system centered at t. Then for every point $q \in M$,*

$$\sum_{i=1}^n \|\nabla^M x_i(q)\|^2 = n - 1.$$

Proof. For every point $q \in M$, observe that $\|\nabla x_i(q)\| = 1$, $1 \leq i \leq n$. Let ν be the outward unit normal to M. Then

$$\begin{aligned}\sum_{i=1}^n \|\nabla^M x_i(q)\|^2 &= \sum_{i=1}^n (\|\nabla x_i(q)\|^2 - \langle \nabla x_i, \nu\rangle^2(q)) \\ &= \sum_{i=1}^n \|\nabla x_i(q)\|^2 - \|\nu\|^2 \\ &= n - 1.\end{aligned}$$

□

4. Proof of Theorem 1

Proof. Given a point $x \in \mathbb{R}^n$, we write $(x_1, \ldots, x_n)$, the standard Euclidean coordinate system centered at the origin. For $1 < p < \infty$, define a function $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(t_1, \ldots, t_n) = \frac{1}{p} \int_M \sum_{i=1}^n |x_i - t_i|^p \, \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

The function f is non-negative on $\mathbb{R}^n$. Let α be its infimum. Then it is easy to see that there exists $t \in \mathbb{R}^n$ such that $f(t_1, \ldots, t_n) = \alpha$. Since f attains its minimum at $t = (t_1, \ldots, t_n)$, we have $(\nabla f)_t = 0$. Therefore for each $1 \leq i \leq n$,

$$\langle \nabla f, e_i \rangle_{(t_1, \ldots, t_n)} = \int_M |X_i|^{p-2} X_i = 0,$$

where $\{e_i \mid 1 \leq i \leq n\}$ is the standard orthonormal basis of $\mathbb{R}^n$ and $X_i = (x_i - t_i)$, $1 \leq i \leq n$. This proves the theorem. □

We will use the above theorem to show the existence of a point $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, such that the coordinate functions with respect to t are test functions for the variational characterization of the eigenvalue problems (1) and (2).

5. Proof of Theorem 2

Proof. Let M be a closed hypersurface in $\mathbb{R}^n$ and Ω be the bounded domain such that $M = \partial\Omega$. Let $R > 0$ be such that Vol(Ω)= Vol($B(R)$). The variational characterization for $\lambda_{1,p}$ is given by

$$\lambda_{1,p} = \inf \left\{ \frac{\int_M \|\nabla^M u\|^p}{\int_M |u|^p} : \int_M |u|^{p-2}u = 0, u(\neq 0) \in C^1(M) \right\}.$$

By Theorem 1, there exists a point $t \in \mathbb{R}^n$ such that

$$\int_M |x_i|^{p-2} x_i = 0 \qquad \text{for } 1 \leq i \leq n,$$

where $(x_1, \ldots, x_n)$ denotes the normal coordinate system centered at t. Therefore, for all $p > 1$,

$$\lambda_{1,p} \int_M \sum_{i=1}^n |x_i|^p \leq \int_M \sum_{i=1}^n \|\nabla^M x_i\|^p \quad \text{for } 1 \leq i \leq n. \tag{10}$$

Now, we divide the proof of the theorem into the following two cases:

Case 1. When $1 < p \leq 2$. Since $|\frac{x_i}{r}| \leq 1$, it follows that

$$|x_i|^p = r^p \left|\frac{x_i}{r}\right|^p \geq r^p \left|\frac{x_i}{r}\right|^2 \quad \text{for } 1 \leq i \leq n. \tag{11}$$

Therefore,

$$r^p = r^p \sum_{i=1}^n \left|\frac{x_i}{r}\right|^2 \leq r^p \sum_{i=1}^n \left|\frac{x_i}{r}\right|^p = \sum_{i=1}^n |x_i|^p.$$

For $1 < p < 2$, using Hölder's inequality,

$$\sum_{i=1}^n \|\nabla^M x_i\|^p \leq \left(\sum_{i=1}^n \|\nabla^M x_i\|^2 \right)^{\frac{p}{2}} n^{\frac{2-p}{2}}.$$

Using Lemma 6 in the above expression, we have

$$\sum_{i=1}^{n} \|\nabla^M x_i\|^p \le (n-1)^{\frac{p}{2}} n^{\frac{2-p}{2}}.$$

Observe that the above inequality is also true for $p = 2$. By substituting the above values in inequality (10), we get

$$\lambda_{1,p} \int_M r^p \le (n-1)^{\frac{p}{2}} \, n^{\frac{2-p}{2}} \, \mathrm{Vol}(M). \tag{12}$$

By substituting $\int_M r^p \ge R^P \, \mathrm{Vol}(S(R))$ in the above inequality, we get

$$\lambda_{1,p} \, R^P \, \mathrm{Vol}(S(R)) \le (n-1)^{\frac{p}{2}} \, n^{\frac{2-p}{2}} \, \mathrm{Vol}(M).$$

As a consequence, we have

$$\lambda_{1,p} \le n^{\frac{2-p}{2}} \, \lambda_1(S(R))^{\frac{p}{2}} \left(\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(S(R))} \right).$$

This proves Theorem 2 for $1 < p \le 2$.

If equality holds in (4) then we will have equality in Lemma 4 and (11), which implies that M is a geodesic sphere of radius R and $p = 2$.

Case 2. When $p \ge 2$. It follows from Hölder's inequality that

$$\sum_{i=1}^{n} |x_i|^2 \le \left(\sum_{i=1}^{n} (|x_i|^2)^{\frac{p}{2}} \right)^{\frac{2}{p}} n^{\frac{p-2}{p}}.$$

Therefore,

$$n^{\frac{2-p}{2}} r^p \le \sum_{i=1}^{n} |x_i|^p. \tag{13}$$

Observe that equality holds in the above inequality for $p = 2$, so (13) holds for $p \ge 2$. Now we estimate $\sum_{i=1}^{n} \|\nabla^M x_i\|^p$. Since $\frac{p}{2} \ge 1$ and $\|\nabla^M x_i\|^2 \ge 0$, for each $1 \le i \le n$, it follows from Lemma 5 and Lemma 6 that

$$\begin{aligned} \sum_{i=1}^{n} \|\nabla^M x_i\|^p &= \sum_{i=1}^{n} (\|\nabla^M x_i\|^2)^{\frac{p}{2}} \\ &\le \left(\sum_{i=1}^{n} \|\nabla^M x_i\|^2 \right)^{\frac{p}{2}} \\ &= (n-1)^{\frac{p}{2}}. \end{aligned} \tag{14}$$

By substituting the above values in (10), we get

$$\lambda_{1,p}\, n^{\frac{2-p}{2}} \int_M r^p \leq (n-1)^{\frac{p}{2}} \operatorname{Vol}(M). \tag{15}$$

By substituting $\int_M r^p \geq R^p \operatorname{Vol}(S(R))$ from Lemma 4, we have

$$\lambda_{1,p}\, n^{\frac{2-p}{2}}\, R^p \operatorname{Vol}(S(R)) \leq (n-1)^{\frac{p}{2}} \operatorname{Vol}(M).$$

Therefore,

$$\lambda_{1,p} \leq n^{\frac{p-2}{2}}\, \lambda_1(S(R))^{\frac{p}{2}} \left(\frac{\operatorname{Vol}(M)}{\operatorname{Vol}(S(R))}\right).$$

This proves Theorem 2 for $p \geq 2$.

If equality holds in (4), then equality holds in (7) and also in (13). Equality in (7) implies that M is a geodesic sphere of radius R and equality in (13) holds if and only if $p = 2$. Otherwise, $p > 2$ and equality in (13) implies that $|x_i| = c$, for some constant c and $1 \leq i \leq n$. Then each point of M will be of the form $(\pm c, \pm c, \pm c, \ldots, \pm c)$, for some constant c. This contradicts our assumption that M is the boundary of a bounded domain Ω.

6. Proof of Theorem 3

Proof. Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary M and $R > 0$ be such that $\operatorname{Vol}(\Omega) = \operatorname{Vol}(B(R))$, where $B(R)$ is a ball of radius R. By Theorem 1, there exists a point $t \in \mathbb{R}^n$ such that

$$\int_M |x_i|^{p-2} x_i = 0, \qquad \text{for all } 1 \leq i \leq n,$$

where $(x_1, x_2, \ldots, x_n)$ denotes the normal coordinate system centered at t. By considering each x_i as a test function, we have

$$\mu_{1,p} \int_M \sum_{i=1}^{n} |x_i|^p \leq \int_\Omega \sum_{i=1}^{n} \|\nabla x_i\|^p. \tag{16}$$

Now we consider the following two cases to prove the theorem:

Case 1. When $1 < p \leq 2$. By similar argument as in (11), we get

$$r^p \leq \sum_{i=1}^{n} |x_i|^p.$$

By Hölder's inequality,

$$\sum_{i=1}^{n} \|\nabla x_i\|^p \leq \left(\sum_{i=1}^{n} \|\nabla x_i\|^2\right)^{\frac{p}{2}} n^{\frac{2-p}{2}} = n.$$

By substituting above values in (16), we get

$$\mu_{1,p} \int_M r^p \leq n \ \mathrm{Vol}(\Omega).$$

Since $\int_M r^p \geq R^P \ \mathrm{Vol}(S(R))$, we have

$$\mu_{1,p} \ R^p \ \mathrm{Vol}(S(R)) \leq n \ \mathrm{Vol}(\Omega).$$

Since $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B(R))$ and $\frac{\mathrm{Vol}(B(R))}{\mathrm{Vol}(S(R))} = \frac{R}{n}$, we get

$$\mu_{1,p} \leq \frac{1}{R^{p-1}}.$$

Case 2. When $p \geq 2$. From (13), we have

$$n^{\frac{2-p}{2}} r^p \leq \sum_{i=1}^{n} |x_i|^p \quad \text{for all } \ p \geq 2.$$

By Lemma 5, we have

$$\begin{aligned} \sum_{i=1}^{n} \|\nabla x_i\|^p &\leq \sum_{i=1}^{n} (\|\nabla x_i\|^2)^{\frac{p}{2}} \\ &\leq \left(\sum_{i=1}^{n} \|\nabla x_i\|^2\right)^{\frac{p}{2}} \\ &\leq n^{\frac{p}{2}}. \end{aligned}$$

By substituting above values in (16), we get

$$\mu_{1,p} \, n^{\frac{2-p}{2}} \int_M r^p \leq n^{\frac{p}{2}} \ \ \mathrm{Vol}(\Omega).$$

We use Lemma 4 again to get

$$\mu_{1,p} \, n^{\frac{2-p}{2}} R^p \ \mathrm{Vol}(S(R)) \leq n^{\frac{p}{2}} \ \ \mathrm{Vol}(\Omega).$$

Since $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B(R))$ and $\frac{\mathrm{Vol}(B(R))}{\mathrm{Vol}(S(R))} = \frac{R}{n}$, the above equation becomes

$$\mu_{1,p} \leq \frac{n^{p-2}}{R^{p-1}}.$$

Equality case will be the same as in Theorem 2. This completes the proof. □

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