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Rigidity of Bott–Samelson–Demazure–Hansen variety for F_4 and G_2

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Abstract. Let G be a simple algebraic group of adjoint type over $\mathbb{C}$, whose root system is of type F_4. Let T be a maximal torus of G and B be a Borel subgroup of G containing T. Let w be an element of the Weyl group W and $X(w)$ be the Schubert variety in the flag variety G/B corresponding to w. Let $Z(w, \underline{i})$ be the Bott–Samelson–Demazure–Hansen variety (the desingularization of $X(w)$) corresponding to a reduced expression $\underline{i}$ of w. In this article, we study the cohomology modules of the tangent bundle on $Z(w_0, \underline{i})$, where w_0 is the longest element of the Weyl group W. We describe all the reduced expressions of w_0 in terms of a Coxeter element such that $Z(w_0, \underline{i})$ is rigid (see Theorem 7.1). Further, if G is of type G_2, there is no reduced expression $\underline{i}$ of w_0 for which $Z(w_0, \underline{i})$ is rigid (see Theorem 8.2).

Keywords. Bott–Samelson–Demazure–Hansen variety; coexeter element; tangent bundle.

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1. Introduction

Let G be a simple algebraic group of adjoint type over the field $\mathbb{C}$ of complex numbers. We fix a maximal torus T of G and let $W = N_G(T)/T$ denote the Weyl group of G with respect to T. We denote by R the set of roots of G with respect to T and by $R^+ \subset R$ a set of positive roots. Let B^+ be the Borel subgroup of G containing T with respect to R^+. Let w_0 denote the longest element of the Weyl group W. Let B be the Borel subgroup of G opposite to B^+ determined by T, i.e. $B = n_{w_0} B^+ n_{w_0}^{-1}$, where n_{w_0} is a representative of w_0 in $N_G(T)$. Note that the roots of B is the set $R^- := -R^+$ of negative roots. We use the notation $\beta < 0$ for $\beta \in R^-$. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ denote the set of all simple roots in R^+, where n is the rank of G. For simplicity of notation, the simple reflection s_{α_i} corresponding to a simple root α_i is denoted by s_i. For $w \in W$, let $X(w) := \overline{BwB/B}$ denote the Schubert variety in G/B corresponding to w. Given a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ of w, with the corresponding tuple $\underline{i} := (i_1, \ldots, i_r)$, we denote by $Z(w, \underline{i})$ the desingularization of the Schubert variety $X(w)$, which is now known as the Bott–Samelson–Demazure–Hansen variety. This was first introduced by Bott and Samelson in a differential geometric and

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topological context (see [2]). Demazure in [4] and Hansen in [6] independently adapted the construction in algebro-geometric situation, which explains the reason for the name. For the sake of simplicity, we will denote any Bott–Samelson–Demazure–Hansen variety by BSDH variety.

The construction of the BSDH variety $Z(w, \underline{i})$ depends on the choice of the reduced expression $\underline{i}$ of w. In [13], the automorphism groups of these varieties were studied. There, the following vanishing results of the tangent bundle $T_{Z(w,\underline{i})}$ on $Z(w, \underline{i})$ were proved (see [13, section 3]):

(1) $H^j(Z(w, \underline{i}), T_{Z(w,\underline{i})}) = 0$ for all $j \geq 2$.
(2) If G is simply laced, then $H^j(Z(w, \underline{i}), T_{Z(w,\underline{i})}) = 0$ for all $j \geq 1$.

As a consequence, it follows that the BSDH varieties are rigid for simply laced groups and their deformations are unobstructed, in general (see [5, section 3]). The above vanishing result is independent of the choice of the reduced expression $\underline{i}$ of w. While computing the first cohomology module $H^1(Z(w, \underline{i}), T_{Z(w,\underline{i})})$ for non simply laced group, we observed that this cohomology module very much depended on the choice of a reduced expression $\underline{i}$ of w.

It is a natural question to ask for which reduced expressions $\underline{i}$ of w, the cohomology module $H^1(Z(w, \underline{i}), T_{Z(w,\underline{i})})$ does vanish? In [14], a partial answer is given to this question for $w = w_0$ when $G = PSp(2n, \mathbb{C})$. In [16], a partial answer is given to this question for $w = w_0$ when $G = PSO(2n + 1, \mathbb{C})$. In this article, we give partial answers to this question for $w = w_0$ when G is of type F_4, G_2.

Recall that a Coxeter element is an element of the Weyl group having a reduced expression of the form $s_{i_1} s_{i_2} \cdots s_{i_n}$ such that $i_j \neq i_l$ whenever $j \neq l$ (see [10, p. 56, section 4.4]). Note that for any Coxeter element $c \in W$, the Weyl group corresponding to the root system of type F_4 (respectively, G_2), there is a decreasing sequence of integers $4 \geq a_1 > a_2 > \cdots > a_k = 1$ (respectively, $2 \geq a_1 > \cdots > a_k = 1$) such that $c = \prod_{j=1}^{k} [a_j, a_{j-1} - 1]$, where $a_0 := 5$ (respectively, $a_0 := 3$), $[i, j] := s_i s_{i+1} \cdots s_j$ for $i \leq j$.

In this paper, we prove the following theorems.

Theorem 1.1. *Assume that G is of type F_4. Then $H^j(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) = 0$ for all $j \geq 1$ if and only if $a_1 \neq 3$ or $a_2 \neq 2$.*

Theorem 1.2. *Assume that G is of type G_2. Then $H^1(Z(w_0, \underline{i_r}), T_{(w_0,\underline{i_r})}) \neq 0$ for $r = 1, 2$.*

By the above results, we conclude that if G is of type F_4 (respectively, G_2) and $\underline{i} = (\underline{i}^1, \underline{i}^2, \underline{i}^3, \underline{i}^4, \underline{i}^5, \underline{i}^6)$ (respectively, $\underline{i} = (\underline{i}^1, \underline{i}^2)$) is a reduced expression of w_0 as above, then the BSDH variety $Z(w_0, \underline{i})$ is rigid (respectively, non rigid).

The organization of the paper is as follows: In section 2, we recall some preliminaries on BSDH varieties. We deal with G which is of type F_4 in sections 3, 4, 5, 6 and 7. In section 3, we prove $H^1(w, \alpha_j) = 0$ for $j = 1, 2$ and $w \in W$. In section 4 (respectively, section 5), we compute the weight spaces of H^0 (respectively, H^1) of the relative tangent bundle of BSDH varieties associated to some elements of the Weyl group. In section 6, we prove surjectivity results of some maps from the cohomology module of the tangent bundle on BSDH variety to cohomology module of the relative tangent bundle on BSDH variety. In section 7, we prove Theorem 1.1 using the results from the previous sections. In section 8, we prove Theorem 1.2.

2. Preliminaries

In this section, we set up some notations and preliminaries. We refer to [3,7,8,12] for preliminaries in algebraic groups and Lie algebras.

Let G be a simple algebraic group of adjoint type over $\mathbb{C}$ and T be a maximal torus of G. Let $W = N_G(T)/T$ denote the Weyl group of G with respect to T and we denote the set of roots of G with respect to T by R. Let B^+ be a Borel subgroup of G containing T. Let B be the Borel subgroup of G opposite to B^+ determined by T, i.e., $B = n_0 B^+ n_0^{-1}$, where n_0 is a representative in $N_G(T)$ of the longest element w_0 of W. Let $R^+ \subset R$ be the set of positive roots of G with respect to the Borel subgroup B^+. Note that the set of roots of B is equal to the set $R^- := -R^+$ of negative roots.

Let $S = \{\alpha_1, \ldots, \alpha_n\}$ denote the set of simple roots in R^+. For $\beta \in R^+$, we also use the notation $\beta > 0$. The simple reflection in W corresponding to α_i is denoted by s_i.

Let $\mathfrak{g}$ be the Lie algebra of G. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of T and $\mathfrak{b} \subset \mathfrak{g}$ be the Lie algebra of B. Let $X(T)$ denote the group of all characters of T. We have $X(T) \otimes \mathbb{R} = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$, the dual of the real form of $\mathfrak{h}$. The positive definite W-invariant form on $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$ induced by the Killing form of $\mathfrak{g}$ is denoted by $(\, , \,)$. We use the notation $\langle \, , \, \rangle$ to denote $\langle \mu, \alpha \rangle = \frac{2(\mu,\alpha)}{(\alpha,\alpha)}$, for every $\mu \in X(T) \otimes \mathbb{R}$ and $\alpha \in R$. We denote by $X(T)^+$ the set of dominant characters of T with respect to B^+. Let ρ denote the half sum of all the positive roots of G with respect to T and B^+. For any simple root α, we denote the fundamental weight corresponding to α by ω_α. For $1 \leq i \leq n$, let $h(\alpha_i) \in \mathfrak{h}$ be the fundamental co-weight corresponding to α_i, i.e., $\alpha_i(h(\alpha_j)) = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

For a simple root $\alpha \in S$, we denote by n_α, a representative of s_α in $N_G(T)$, and P_α the minimal parabolic subgroup of G containing B and n_α. We recall that the BSDH variety corresponds to a reduced expression $\underline{i}$ of $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ defined by

$$Z(w, \underline{i}) = \frac{P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}}}{B \times B \times \cdots \times B},$$

where the action of $B \times B \times \cdots \times B$ on $P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}}$ is given by $(p_1, p_2, \ldots, p_r)(b_1, b_2, \ldots, b_r) = (p_1 \cdot b_1, b_1^{-1} \cdot p_2 \cdot b_2, \ldots, b_{r-1}^{-1} \cdot p_r \cdot b_r)$, $p_j \in P_{\alpha_{i_j}}$, $b_j \in B$ for $1 \leq j \leq r$, and $\underline{i} = (i_1, i_2, \ldots, i_r)$ (see [4, Definition 1, p. 73], [3, Definition 2.2.1, p. 64]).

We note that for each reduced expression $\underline{i}$ of w, $Z(w, \underline{i})$ is a smooth projective variety. We denote by ϕ_w, the natural birational surjective morphism from $Z(w, \underline{i})$ to $X(w)$.

Let $f_r : Z(w, \underline{i}) \longrightarrow Z(ws_{i_r}, \underline{i'})$ denote the map induced by the projection $P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}} \longrightarrow P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_{r-1}}}$, where $\underline{i'} = (i_1, i_2, \ldots, i_{r-1})$. Then we observe that f_r is a $P_{\alpha_{i_r}}/B \simeq \mathbb{P}^1$-fibration.

For a B-module V, let $\mathcal{L}(w, V)$ denote the restriction of the associated homogeneous vector bundle on G/B to $X(w)$. By abuse of notation, we denote the pull back of $\mathcal{L}(w, V)$ via ϕ_w to $Z(w, \underline{i})$ also by $\mathcal{L}(w, V)$, when there is no confusion. Since for any B-module V the vector bundle $\mathcal{L}(w, V)$ on $Z(w, \underline{i})$ is the pull back of the homogeneous vector bundle from $X(w)$, we conclude that the cohomology modules

$$H^j(Z(w, \underline{i}), \mathcal{L}(w, V)) \simeq H^j(X(w), \mathcal{L}(w, V))$$

for all $j \geq 0$ (see [3, Theorem 3.3.4(b)]), are independent of the choice of the reduced expression $\underline{i}$. Hence we denote $H^j(Z(w, \underline{i}), \mathcal{L}(w, V))$ by $H^j(w, V)$. In particular, if λ is a character of B, then we denote the cohomology modules $H^j(Z(w, \underline{i}), \mathcal{L}_\lambda)$ by $H^j(w, \lambda)$.

We recall the following short exact sequence of B-modules from [13], which we call it SES.

If $l(w) = l(s_\gamma w) + 1$, $\gamma \in S$, then we have

(1) $H^0(w, V) \simeq H^0(s_\gamma, H^0(s_\gamma w, V))$.
(2) $0 \to H^1(s_\gamma, H^0(s_\gamma w, V)) \to H^1(w, V) \to H^0(s_\gamma, H^1(s_\gamma w, V)) \to 0$.

Let α be a simple root and $\lambda \in X(T)$ be such that $\langle \lambda, \alpha \rangle \geq 0$. Let $\mathbb{C}_\lambda$ denote the one-dimensional B-module associated to λ. Here, we recall the following result due to Demazure [5, p. 271] on the short exact sequence of B-modules.

Lemma 2.1. *Let α be a simple root and $\lambda \in X(T)$ be such that $\langle \lambda, \alpha \rangle \geq 0$. Let ev : $H^0(s_\alpha, \lambda) \longrightarrow \mathbb{C}_\lambda$ be the evaluation map. Then we have*

(1) *If $\langle \lambda, \alpha \rangle = 0$, then $H^0(s_\alpha, \lambda) \simeq \mathbb{C}_\lambda$.*
(2) *If $\langle \lambda, \alpha \rangle \geq 1$, then $\mathbb{C}_{s_\alpha(\lambda)} \hookrightarrow H^0(s_\alpha, \lambda)$, and there is a short exact sequene of B-modules*:
$$0 \to H^0(s_\alpha, \lambda - \alpha) \longrightarrow H^0(s_\alpha, \lambda)/\mathbb{C}_{s_\alpha(\lambda)} \longrightarrow \mathbb{C}_\lambda \to 0.$$
Furthermore, $H^0(s_\alpha, \lambda - \alpha) = 0$ when $\langle \lambda, \alpha \rangle = 1$.
(3) *Let $n = \langle \lambda, \alpha \rangle$. As a B-module, $H^0(s_\alpha, \lambda)$ has a composition series*
$$0 \subseteq V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_0 = H^0(s_\alpha, \lambda)$$
such that $V_i/V_{i+1} \simeq \mathbb{C}_{\lambda - i\alpha}$ for $i = 0, 1, \ldots, n-1$ and $V_n = \mathbb{C}_{s_\alpha(\lambda)}$.

We define the dot action by $w \cdot \lambda = w(\lambda + \rho) - \rho$, where ρ is the half sum of positive roots. As a consequence of exact sequences of Lemma 2.1, we can prove the following.

Let $w \in W$, α be a simple root and set $v = ws_\alpha$.

Lemma 2.2. *If $l(w) = l(v) + 1$, then we have*

(1) *If $\langle \lambda, \alpha \rangle \geq 0$, then $H^j(w, \lambda) = H^j(v, H^0(s_\alpha, \lambda))$ for all $j \geq 0$.*
(2) *If $\langle \lambda, \alpha \rangle \geq 0$, then $H^j(w, \lambda) = H^{j+1}(w, s_\alpha \cdot \lambda)$ for all $j \geq 0$.*
(3) *If $\langle \lambda, \alpha \rangle \leq -2$, then $H^{j+1}(w, \lambda) = H^j(w, s_\alpha \cdot \lambda)$ for all $j \geq 0$.*
(4) *If $\langle \lambda, \alpha \rangle = -1$, then $H^j(w, \lambda)$ vanishes for every $j \geq 0$.*

The following consequence of Lemma 2.2 will be used to compute the cohomology modules in this paper. Now onwards, we will denote the Levi subgroup of $P_\alpha (\alpha \in S)$ containing T by L_α and the subgroup $L_\alpha \cap B$ by B_α. Let $\pi : \tilde{G} \longrightarrow G$ be the universal cover. Let $\tilde{L}_\alpha$ (respectively, $\tilde{B}_\alpha$) be the inverse image of L_α (respectively, B_α).

Lemma 2.3. *Let V be an irreducible L_α-module. Let λ be a character of B_α. Then we have*

(1) *As L_α-modules, $H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) \simeq V \otimes H^j(L_\alpha/B_\alpha, \mathbb{C}_\lambda)$.*
(2) *If $\langle \lambda, \alpha \rangle \geq 0$, then $H^0(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda)$ is isomorphic as an L_α-module to the tensor product of V and $H^0(L_\alpha/B_\alpha, \mathbb{C}_\lambda)$. Further, we have $H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0$ for every $j \geq 1$.*
(3) *If $\langle \lambda, \alpha \rangle \leq -2$, then $H^0(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0$, and $H^1(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda)$ is isomorphic to the tensor product of V and $H^0(L_\alpha/B_\alpha, \mathbb{C}_{s_\alpha \cdot \lambda})$.*

(4) *If* $\langle \lambda, \alpha \rangle = -1$, *then* $H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0$ *for every* $j \geq 0$.

Proof. Let us prove (1). By [12, Proposition 4.8, p. 53, I] and [12, Proposition 5.12, p. 77, I], for all $j \geq 0$, we have the following isomorphism of L_α-modules:

$$H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) \simeq V \otimes H^j(L_\alpha/B_\alpha, \mathbb{C}_\lambda).$$

The proofs of (2), (3) and (4) follow from Lemma 2.2 by taking $w = s_\alpha$ and the fact that $L_\alpha/B_\alpha \simeq P_\alpha/B$. □

Recall the structure of indecomposable B_α-modules and $\tilde{B}_\alpha$-modules (see [1, Corollary 9.1, p. 130]).

Lemma 2.4.

(1) *Any finite dimensional indecomposable* $\tilde{B}_\alpha$*-module* V *is isomorphic to* $V' \otimes \mathbb{C}_\lambda$ *for some irreducible representation* V' *of* $\tilde{L}_\alpha$ *and for some character* λ *of* $\tilde{B}_\alpha$.
(2) *Any finite dimensional indecomposable* B_α*-module* V *is isomorphic to* $V' \otimes \mathbb{C}_\lambda$ *for some irreducible representation* V' *of* $\tilde{L}_\alpha$ *and for some character* λ *of* $\tilde{B}_\alpha$.

3. Reduced expressions

Now onwards, we will assume that G is of type F_4. Note that the longest element w_0 of the Weyl group W of G is equal to $-id$. We recall the following Proposition from [17, Proposition 1.3, p. 858].

PROPOSITION 3.1

Let $c \in W$ *be a Coxeter element,* ω_i *be the fundamental weight corresponding to the simple root* α_i. *Then there exists a least positive integer* $h(i, c)$ *such that* $c^{h(i,c)}(\omega_i) = w_0(\omega_i)$.

Now we can deduce the following:

Lemma 3.2. *Let* $c \in W$ *be a Coxeter element. Then we have*

(1) $w_0 = c^6$.
(2) *For any sequence* $\underline{i} = (\underline{i}^1, \underline{i}^2, \ldots, \underline{i}^6)$ *of reduced expressions of* c; *the sequence* $\underline{i} = (\underline{i}^1, \underline{i}^2, \ldots, \underline{i}^6)$ *is a reduced expression of* w_0.

Proof. Let us prove (1). Let $\eta : S \longrightarrow S$ be the involution of S defined by $i \to i^*$, where i^* is given by $\omega_{i^*} = -w_0(\omega_i)$. Since G is of type F_4, $w_0 = -id$, and hence $\omega_{i^*} = \omega_i$ for every i. Therefore, we have $i = i^*$ for every i. Let h be the Coxeter number. By [17, Proposition 1.7], we have $h(i, c) + h(i^*, c) = h$. Since $h = 2|R^+|/4$ (see [9, Proposition 3.18]) and $i = i^*$, we have $h(i, c) = h/2 = 6$ as $|R^+| = 24$. By Proposition 3.1, we have $c^6(\omega_i) = -\omega_i$ for all $1 \leq i \leq 4$. Since $\{\omega_i : 1 \leq i \leq 4\}$ forms an $\mathbb{R}$-basis of $X(T) \otimes \mathbb{R}$, it follows that $c^6 = -id$. Hence, we have $w_0 = c^6$. The assertion (2) follows from the fact that $l(c) = 4$ and $l(w_0) = |R^+| = 24$ (see [7, p. 66, Table 1]). □

Lemma 3.3. *Let* $v \in W$ *and* $\alpha \in S$. *Then* $H^1(s_j, H^0(v, \alpha)) = 0$ *for* $j = 1, 2$.

Proof. If $H^1(s_j, H^0(v, \alpha))_\mu \neq 0$, then there exists an indecomposable $\tilde{L}_{\alpha_j}$-summand V of $H^0(v, \alpha)$ such that $H^1(s_j, V)_\mu \neq 0$. By Lemma 2.4, we have $V \simeq V' \otimes \mathbb{C}_\lambda$ for some character λ of $\tilde{B}_{\alpha_j}$ and for some irreducible $\tilde{L}_{\alpha_j}$-module V'. Since $H^1(s_j, V)_\mu \neq 0$, from Lemma 2.3(3), we have $\langle \lambda, \alpha_j \rangle \leq -2$. If α is a short root, then $H^1(w, \alpha) = 0$ for all $w \in W$ (see [15, Corollary 5.6, p. 778]). Hence we may assume that α is a long root. Then there exists $w \in W$ such that $w(\alpha) = \alpha_0$. Thus $H^0(v, \alpha) \subseteq H^0(vw, \alpha_0)$. Again, since α_0 is the highest long root, $H^0(w_0, \alpha_0) = \mathfrak{g} \longrightarrow H^0(vw, \alpha_0)$ is surjective. Let μ' be the lowest weight of V. Then by the above argument, μ' is a root. Therefore we have $\mu' = \mu_1 + \lambda$, where μ_1 is the lowest weight of V'. Hence, we have $\langle \mu', \alpha_j \rangle \leq -2$. Since α_j is a long root and μ' is a root, we have $\langle \mu', \alpha_j \rangle = -1, 0, 1$. This is a contradiction. Thus we have $H^1(s_j, H^0(v, \alpha))_\mu = 0$. □

4. Cohomology modules $H^0(w, \alpha_i)$

Let $w_r = (s_1s_2s_3s_4)^r s_1s_2$ for $1 \leq r \leq 5$. In this section, we compute various cohomology modules $H^0(w, \alpha_i)$ for some elements $w \in W$ and $i = 2, 3$.

Lemma 4.1.

(1) $H^0(w_3, \alpha_2) = 0$.
(2) $H^0(w_r, \alpha_2) = 0$ *for* $r = 4, 5$.

Proof. We have $w_3 = [1, 4]^3 12$. By using SES, we have

$$H^0(s_1s_2, \alpha_2) = \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2)}.$$

Since $\langle \alpha_2, \alpha_4 \rangle = 0$, by using SES, we have

$$H^0(s_4s_1s_2, \alpha_2) = H^0(s_1s_2, \alpha_2).$$

Since $\langle -\alpha_2, \alpha_3 \rangle = 2$ and $\langle -(\alpha_1 + \alpha_2), \alpha_3 \rangle = 2$, by using SES, we have

$$\begin{aligned} H^0(s_3s_4s_1s_2, \alpha_2) = \mathbb{C}h(\alpha_2) &\oplus (\mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3)}) \\ &\oplus (\mathbb{C}_{-(\alpha_1+\alpha_2)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}). \quad (4.1.1) \end{aligned}$$

Since $\mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2}$ is an indecomposable two-dimensional $\tilde{B}_{\alpha_2}$-module, by Lemma 2.4, we have $\mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} = V \otimes \mathbb{C}_{-\omega_2}$, where V is the standard two-dimensional irreducible $\tilde{L}_{\alpha_2}$-module.

Thus by Lemma 2.3(4), we have $H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2}) = 0$.

Since $\langle -(\alpha_2 + \alpha_3), \alpha_2 \rangle = -1$, $\langle -(\alpha_1 + \alpha_2), \alpha_2 \rangle = -1$, by Lemma 2.2(4), we have $H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_2+\alpha_3)}) = 0$ and $H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_1+\alpha_2)}) = 0$.

Since $\langle -(\alpha_2 + 2\alpha_3), \alpha_2 \rangle = 0$, $\langle -(\alpha_1 + \alpha_2 + \alpha_3), \alpha_2 \rangle = 0$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_2+2\alpha_3)}) = \mathbb{C}_{-(\alpha_2+2\alpha_3)}$$

and

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Since $\langle -(\alpha_1 + \alpha_2 + 2\alpha_3), \alpha_2 \rangle = 1$, we have

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}.$$

Thus we have

$$H^0(s_2s_3s_4s_1s_2, \alpha_2) = \mathbb{C}_{-(\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}. \tag{4.1.2}$$

Since $\langle -(\alpha_1 + \alpha_2 + \alpha_3), \alpha_1 \rangle = -1$, by using Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = 0$$

and

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}) = 0.$$

Since $\mathbb{C}_{-(\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}$ the standard two-dimensional irreducible $\tilde{L}_{\alpha_1}$-module, by using Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}.$$

Since $\langle -(\alpha_2 + 2\alpha_3), \alpha_1 \rangle = 1$, by using Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\alpha_2+2\alpha_3)}) = \mathbb{C}_{-(\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}.$$

Therefore, we have

$$H^0(w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}. \tag{4.1.3}$$

By using SES, we have

$$H^0(w_3, \alpha_2) = H^0([1, 4]^2, H^0(w_1, \alpha_2)).$$

Note that the computations of the module $H^0([1, 4]^2, H^0(w_1, \alpha_2))$ is independent of the choice of a reduced expression of $[1, 4]^2$. We consider the reduced expression $s_2s_1s_2s_3s_2s_3s_4s_3$, of $[1, 4]^2$ to compute $H^0([1, 4]^2, H^0(w_1, \alpha_2))$.

Since $\langle -(\alpha_2 + 2\alpha_3), \alpha_3 \rangle = -2$, $\langle -(\alpha_1 + \alpha_2 + 2\alpha_3), \alpha_3 \rangle = -2$, by using Lemma 2.3(3), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_2+2\alpha_3)}) = 0$$

and

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}) = 0.$$

Since $\langle -(\alpha_1+2\alpha_2+2\alpha_3), \alpha_3\rangle = 0$, by using Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}.$$

Thus, from the above discussion, we have

$$H^0(s_3w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}.$$

Since $\langle -(\alpha_1+2\alpha_2+2\alpha_3), \alpha_4\rangle = 2$, by using SES and Lemma 2.3(2), we have

$$H^0(s_4s_3w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)}.$$

Since $\langle -(\alpha_1+2\alpha_2+2\alpha_3), \alpha_3\rangle = 0$, $\langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_3\rangle = 1$, $\langle -(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4), \alpha_3\rangle = 2$, by using Lemma 2.3(2), we have

$$\begin{aligned} H^0(s_3s_4s_3w_1, \alpha_2) = {} & \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}. \end{aligned}$$

Since $\langle -(\alpha_1+2\alpha_2+2\alpha_3), \alpha_2\rangle = -1$, $\langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_2\rangle = -1$, $\langle -(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4), \alpha_2\rangle = -1$, $\langle -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), \alpha_2\rangle = 0$, $\langle -(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4), \alpha_2\rangle = 0$, $\langle -(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4), \alpha_2\rangle = 1$, by using Lemma 2.3(2) and Lemma 2.3(4), we have

$$\begin{aligned} H^0(s_2s_3s_4s_3w_1, \alpha_2) = {} & \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}. \end{aligned}$$

Since $\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}$ is a two-dimensional indecomposable $\tilde{B}_{\alpha_3}$-module, by Lemma 2.4(1), we have $\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} = V \otimes \mathbb{C}_{-\omega_3}$, where V is the standard two-dimensional irreducible $\tilde{L}_{\alpha_3}$-module.

Thus by Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}) = 0.$$

Since $\langle -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), \alpha_3\rangle = -1$ and $\langle -(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4), \alpha_3\rangle = 0$, by Lemma 2.3(2) and Lemma 2.3(4), we have $H^0(s_3s_2s_3s_4s_3w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}$.

Since $\langle -(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4), \alpha_2\rangle = -1$, by Lemma 2.3(4), we have $H^0(s_2s_3s_2s_3s_4s_3w_1, \alpha_2) = 0$.

Thus, by using SES and Lemma 2.3(2), we have $H^0(s_1s_2s_3s_2s_3s_4s_3w_1, \alpha_2) = 0$. Again, by using SES and Lemma 2.3(2), we have $H^0(w_3, \alpha_2) = H^0([1,4]^2, H^0(w_1, \alpha_2)) = 0$. Proof of (2) follows from (1). □

Recall that $\omega_4 = \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$. From now onwards, we replace $\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$ by ω_4.

Lemma 4.2.

(1) $H^0(w_2s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4}$.

(2) $H^0(w_3s_3, \alpha_3) = \mathbb{C}_{-\omega_4}$.

Proof.

Proof of (1). Using SES, we have $H^0(s_3, \alpha_3) = \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{\alpha_3}$. Since $\langle \alpha_3, \alpha_2 \rangle = -1$, by using SES and Lemma 2.3, we have

$$H^0(s_2s_3, \alpha_3) = \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}_{-(\alpha_3+\alpha_2)}.$$

Further, since $\langle \alpha_3, \alpha_1 \rangle = 0$ and $\langle -(\alpha_3 + \alpha_2), \alpha_1 \rangle = 1$, by using SES and Lemma 2.3, we have

$$H^0(s_1s_2s_3, \alpha_3) = \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Note that the computations of the module $H^0([1,4]^2, H^0(s_1s_2s_3, \alpha_3))$ is independent of the choice of a reduced expression of $[1,4]^2$. We consider the reduced expression $s_1s_2s_1s_3s_2s_3s_4s_3$ of $[1,4]^2$ to compute $H^0([1,4]^2, H^0(s_1s_2s_3, \alpha_3))$.

Since $\mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3}$ is a two-dimensional $\tilde{B}_{\alpha_3}$-module, by Lemma 2.4(1), we have

$$\mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} = V \otimes \mathbb{C}_{-\omega_3},$$

where V is the standard two-dimensional irreducible $\tilde{L}_{\alpha_3}$-module.

Thus by using Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3}) = 0.$$

Since $\langle -(\alpha_2 + \alpha_3), \alpha_3 \rangle = 0$, $\langle -(\alpha_1 + \alpha_2 + \alpha_3), \alpha_3 \rangle = 0$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_2+\alpha_3)}$$

and

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Thus, from the above discussion, we have

$$H^0(s_3s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Since $\langle -(\alpha_2 + \alpha_3), \alpha_4 \rangle = 1$ and $\langle -(\alpha_1 + \alpha_2 + \alpha_3), \alpha_4 \rangle = 1$, by using Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}$$

and

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.$$

Thus, from the above discussion, we have

$$H^0(s_4s_3s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_2+\alpha_3), \alpha_3\rangle = 0$ and $\langle -(\alpha_1+\alpha_2+\alpha_3), \alpha_3\rangle = 0$, by using Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_2+\alpha_3)}$$

and

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Since $\langle -(\alpha_2+\alpha_3+\alpha_4), \alpha_3\rangle = 1$ and $\langle -(\alpha_1+\alpha_2+\alpha_3+\alpha_4), \alpha_3\rangle = 1$, by using Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}$$

and

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.$$

Thus, from the above discussion, we have

$$\begin{aligned} H^0(s_3s_4s_3s_1s_2s_3, \alpha_3) = {} & \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}. \end{aligned}$$

Since $\langle -(\alpha_2+\alpha_3), \alpha_2\rangle = -1$, $\langle -(\alpha_2+\alpha_3+\alpha_4), \alpha_2\rangle = -1$, by using Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_2+\alpha_3)}) = 0$$

and

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}) = 0.$$

Since $\langle -(\alpha_2+2\alpha_3+\alpha_4), \alpha_2\rangle = 0$, $\langle -(\alpha_1+\alpha_2+\alpha_3), \alpha_2\rangle = 0$, $\langle -(\alpha_1+\alpha_2+\alpha_3+\alpha_4), \alpha_2\rangle = 0$ and $\langle -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4), \alpha_2\rangle = 1$, by using Lemma 2.3(2), we have

$$\begin{aligned} H^0(s_2s_3s_4s_3s_1s_2s_3, \alpha_3) = {} & \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}. \end{aligned}$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$ is the standard two-dimensional irreducible $\tilde{L}_{\alpha_3}$-module, $\langle -(\alpha_1+\alpha_2+\alpha_3), \alpha_3\rangle = 0$, $\langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_3\rangle = 1$ and $\langle -(\alpha_2+2\alpha_3+\alpha_4), \alpha_3\rangle = -1$, by using similar arguments as above and using Lemma 2.3(2) and Lemma 2.3(4), we have

$$\begin{aligned} H^0(s_3s_2s_3s_4s_3s_1s_2s_3, \alpha_3) = {} & \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}. \end{aligned}$$

Since $\langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_1\rangle = 0$, $\langle -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), \alpha_1\rangle = 0$, $\langle -(\alpha_1+\alpha_2+\alpha_3), \alpha_1\rangle = -1$, $\langle -(\alpha_1+\alpha_2+\alpha_3+\alpha_4), \alpha_1\rangle = -1$, $\langle -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4), \alpha_1\rangle = -1$, by using similar arguments as above and using Lemma 2.3(2) and Lemma 2.3(4), we have

$$H^0(s_1s_3s_2s_3s_4s_3s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), \alpha_2\rangle = 0$, $\langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_2\rangle = -1$, by using Lemma 2.3(2) and Lemma 2.3(4), we have

$$H^0(s_2s_1s_3s_2s_3s_4s_3s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), \alpha_1\rangle = 0$, by using Lemma 2.3(2), we have

$$H^0(s_1s_2s_1s_3s_2s_3s_4s_3s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Thus we have

$$H^0(w_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} = \mathbb{C}_{-\omega_4+\alpha_4}.$$

Proof of (2). By the proof of (1), we have

$$H^0(w_2s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4}.$$

Since $\langle -\omega_4+\alpha_4, \alpha_4\rangle = 1$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-\omega_4+\alpha_4}) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.$$

Therefore, we have

$$H^0(s_4w_2s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.$$

Since $\langle -\omega_4+\alpha_4, \alpha_3\rangle = -1$, by Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-\omega_4+\alpha_4}) = 0.$$

Since $\langle -\omega_4, \alpha_3\rangle = 0$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-\omega_4}) = \mathbb{C}_{-\omega_4}.$$

Thus from the above discussion, we have

$$H^0(s_3s_4w_2s_3, \alpha_3) = \mathbb{C}_{-\omega_4}.$$

Since α_1, α_2 are othogonal to ω_4, by Lemma 2.3(2), we have

$$H^0(w_3s_3, \alpha_3) = \mathbb{C}_{-\omega_4}.$$

□

COROLLARY 4.3

(1) $H^0(s_4w_1s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}$.
(2) $H^0(s_4w_2s_3, \alpha_3) = \mathbb{C}_{-\omega_4} \oplus \mathbb{C}_{-\omega_4+\alpha_4}$.
(3) $H^0(s_4w_rs_3, \alpha_3) = 0$ *for* $r = 3, 4, 5$.

Proof.

Proof of (1). We have

$$H^0(s_1s_2s_3, \alpha_3) = \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Since $\langle -\alpha_3, \alpha_4 \rangle = 1$, $\langle -(\alpha_2+\alpha_3), \alpha_4 \rangle = 1$ and $\langle -(\alpha_1+\alpha_2+\alpha_3), \alpha_4 \rangle = 1$, by using SES and Lemma 2.3(2), we have

$$H^0(s_4s_1s_2s_3, \alpha_3) = \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}_{-(\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3)} \\ \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.$$

Since $\mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3}$ is a two-dimensional $\tilde{B}_{\alpha_3}$-module, by Lemma 2.4(1), we have

$$\mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} = V \otimes \mathbb{C}_{-\omega_3},$$

where V is the standard two-dimensional $\tilde{L}_{\alpha_3}$-module.
Thus by using Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3}) = 0.$$

Since $\langle -(\alpha_3+\alpha_4), \alpha_3 \rangle = -1$, by Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_3+\alpha_4)}) = 0.$$

Since $\langle -(\alpha_2+\alpha_3), \alpha_3 \rangle = 0$, $\langle -(\alpha_1+\alpha_2+\alpha_3), \alpha_3 \rangle = 0$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_2+\alpha_3)}$$

and

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Since $\langle -(\alpha_2+\alpha_3+\alpha_4), \alpha_3 \rangle = 1$, $\langle -(\alpha_1+\alpha_2+\alpha_3+\alpha_4), \alpha_3 \rangle = 1$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}$$

and

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.$$

Thus, combining the above discussion, we have

$$\begin{aligned} H^0(s_3s_4s_1s_2s_3, \alpha_3) = &\ \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \\ &\oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \\ &\oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}. \end{aligned} \tag{4.3.1}$$

Since $\langle -(\alpha_2+\alpha_3), \alpha_2 \rangle = -1$, $\langle -(\alpha_2+\alpha_3+\alpha_4), \alpha_2 \rangle = -1$, by Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_2+\alpha_3)}) = 0$$

and

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}) = 0.$$

Since $\langle -(\alpha_1+\alpha_2+\alpha_3), \alpha_2 \rangle = 0$, $\langle -(\alpha_2+2\alpha_3+\alpha_4), \alpha_2 \rangle = 0$ and $\langle -(\alpha_1+\alpha_2+\alpha_3+\alpha_4), \alpha_2 \rangle = 0$, by Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)},$$
$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}$$

and

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4), \alpha_2 \rangle = 1$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Thus, combining the above discussion, we have

$$\begin{aligned} H^0(s_2s_3s_4s_1s_2s_3, \alpha_3) = &\ \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \\ &\oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \\ &\oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}. \end{aligned}$$

Since $\langle -(\alpha_1+\alpha_2+\alpha_3), \alpha_1 \rangle = -1$ and $\langle -(\alpha_1+\alpha_2+\alpha_3+\alpha_4), \alpha_1 \rangle = -1$, by Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = 0$$

and

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}) = 0.$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}$ is the standard two-dimensional irreducible $\tilde{L}_{\alpha_1}$-module, by using Lemma 2.3(2), we have

$$\begin{aligned} &H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}) \\ &\quad = \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}. \end{aligned}$$

Since $\langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_1\rangle = 0$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Thus, combining the above discussion, we have

$$H^0(w_1 s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_2+2\alpha_3+\alpha_4), \alpha_4\rangle = 0$, $\langle -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4), \alpha_4\rangle = 0$ and $\langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_4\rangle = 0$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)},$$
$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$$

and

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Therefore, we have

$$H^0(s_4 w_1 s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Proof of (2). By Lemma 4.2(1), we have

$$H^0(w_2 s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4}.$$

Since $\langle -\omega_4+\alpha_4, \alpha_4\rangle = 1$, by using SES and Lemma 2.3(2), we have

$$H^0(s_4 w_2 s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.$$

Proof of (3). By the Lemma 4.2(3), we have

$$H^0(w_3 s_3, \alpha_3) = \mathbb{C}_{-\omega_4}.$$

□

Since $\langle -\omega_4, \alpha_4\rangle = -1$, by Lemma 2.3(4), we have

$$H^0(s_4 w_3 s_3, \alpha_3) = 0.$$

By using SES repeatedly, we have

$$H^0(s_4 w_r s_3, \alpha_3) = 0 \quad \text{for } r = 4, 5.$$

COROLLARY 4.4

(1) $H^0(s_3 s_4 s_1 s_2 s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$.
(2) $H^0(s_3 s_4 w_1 s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4}$.
(3) $H^0(s_3 s_4 w_2 s_3, \alpha_3) = \mathbb{C}_{-\omega_4}$.

Proof.

Proof of (1). Proof follows from (4.3.1).
Proof of (2). Proof follows from Corollary 4.3(1).
Proof of (3). Proof follows from Corollary 4.3(2). □

COROLLARY 4.5

(1) $H^0(s_2s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}$.
(2) $H^0(s_2s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4}$.
(3) $H^1(s_2s_3s_4w_2s_3, \alpha_3) = \mathbb{C}_{-\omega_4}$.

Proof.

Proof of (1). Proof follows from Corollary 4.4(1).
Proof of (2). Proof follows from Corollary 4.4(2).
Proof of (3). Proof follows from Corollary 4.4(3). □

COROLLARY 4.6

(1) $H^0(s_4s_3s_4s_1s_2s_3, \alpha_3)=\mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$.
(2) $H^0(s_4s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}$.

Proof.

Proof of (1). By Corollary 4.4(1), we have

$$\begin{aligned} H^0(s_3s_4s_1s_2s_3, \alpha_3) = {} & \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}. \end{aligned}$$

Since $\mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}$ is the standard two-dimensional irreducible $\tilde{L}_{\alpha_4}$-module, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}.$$

Also, since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$ is the standard two-dimensional irreducible $\tilde{L}_{\alpha_4}$-module, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_2+2\alpha_3+\alpha_4), \alpha_4\rangle = 0$ and $\langle -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4), \alpha_4\rangle = 0$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}$$

and

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.$$

Thus, combining the above discussion, we have

$$\begin{aligned} H^0(s_4s_3s_4s_1s_2s_3, \alpha_3) = {} & \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}. \end{aligned}$$

Proof of (2). By Corollary 4.4(2), we have

$$H^0(s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_4 \rangle = 0$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Further, since $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_4 \rangle = 1$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}.$$

Thus, combining the above discussion, we have

$$H^0(s_4s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4},$$

since $\omega_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$. □

COROLLARY 4.7

(1) $H^0(s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}$.
(2) $H^0(s_4s_2s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}$.

Proof.

Proof of (1). By Corollary 4.5(1), we have

$$\begin{aligned} H^0(s_2s_3s_4s_1s_2s_3, \alpha_3) = {} & \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}. \end{aligned}$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$ is the standard two-dimensional irreducible $\tilde{L}_{\alpha_4}$-module, by Lemma 2.3(2), we have

$$\begin{aligned} & H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}) \\ & \quad = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}. \end{aligned}$$

Moreover, since $\langle -(\alpha_2+2\alpha_3+\alpha_4), \alpha_4\rangle = 0$, $\langle -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4), \alpha_4\rangle = 0$ and $\langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_4\rangle = 0$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}$$
$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$$

and

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Thus, combining the above discussion, we have

$$\begin{aligned} H^0(s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = {} & \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}. \end{aligned}$$

Proof of (2). By Corollary 4.5(2), we have

$$H^0(s_2s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4}.$$

Since $\langle -\omega_4+\alpha_4, \alpha_4\rangle = 1$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-\omega_4+\alpha_4}) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.$$

Thus we have

$$H^0(s_4s_2s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.$$

□

COROLLARY 4.8

(1) $H^0(s_3s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4}$.
(2) $H^0(s_3s_4s_2s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-\omega_4}$.

Proof.

Proof of (1). By Corollary 4.7(1), we have

$$\begin{aligned} H^0(s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = {} & \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}. \end{aligned}$$

Since $\langle -(\alpha_2+2\alpha_3+\alpha_4), \alpha_3\rangle = -1$, by Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}) = 0.$$

Since $\langle -(\alpha_1+\alpha_2+\alpha_3), \alpha_3\rangle = 0$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$ is the standard two-dimensional irreducible $\tilde{L}_{\alpha_3}$-module, by Lemma 2.3(2), we have

$$\begin{aligned} &H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}) \\ &\quad = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}. \end{aligned}$$

Since $\langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_3\rangle = 1$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Thus, combining the above discussion, we have

$$\begin{aligned} H^0(s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = {} & \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}. \end{aligned}$$

Proof of (2). By Corollary 4.7(2), we have

$$H^0(s_4s_2s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.$$

Since $\langle -\omega_4+\alpha_4, \alpha_3\rangle = -1$, by Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-\omega_4+\alpha_4}) = 0.$$

Further, since $\langle -\omega_4, \alpha_3\rangle = 0$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-\omega_4}) = \mathbb{C}_{-\omega_4}.$$

Thus, from the above discussion, we have

$$H^0(s_3s_4s_2s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-\omega_4}.$$

□

COROLLARY 4.9

(1) $H^0(s_4s_3s_4s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}$.
(2) $H^0(s_4s_3s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}$.

Proof.

Proof of (1). It is easy to see that

$$H^0(s_3, \alpha_3) = \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{\alpha_3}.$$

Since $\langle -\alpha_3, \alpha_2\rangle = 1$, by using Lemma 2.3(2) and Lemma 2.3(4), we have

$$H^0(s_2s_3, \alpha_3) = \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3)}.$$

Since $\langle -\alpha_3, \alpha_4\rangle = 1$, by using Lemma 2.3(2), we have

$$H^0(s_4s_2s_3, \alpha_3) = \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}_{-(\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}.$$

Since $\mathbb{C}h(\alpha_3)\oplus\mathbb{C}_{-\alpha_3}$ is the two-dimensionl indecomposable $\tilde{B}_{\alpha_3}$-module, by Lemma 2.4(1), we have

$$\mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} = V \otimes \mathbb{C}_{-\omega_3}$$

where V is the standard two-dimensional irreducible $\tilde{L}_{\alpha_3}$-module. Thus by Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_3, \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3}) = 0.$$

Also, since $\langle -(\alpha_3 + \alpha_4), \alpha_3\rangle = -1$, by Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_3, \mathbb{C}_{-(\alpha_3+\alpha_4)}) = 0.$$

Since $\langle -(\alpha_2 + \alpha_3), \alpha_3\rangle = 0$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_3, \mathbb{C}_{-(\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_2+\alpha_3)}.$$

Since $\langle -(\alpha_2 + \alpha_3 + \alpha_4), \alpha_3\rangle = 1$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_3, \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}.$$

Thus, combining the above discussion, we have

$$H^0(s_3s_4s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}.$$

Since $\mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}$ is the standard two-dimensional irreducible $\tilde{L}_{\alpha_4}$-module, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_4, \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}.$$

Further, since $\langle -(\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_4\rangle = 0$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_4, \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}.$$

Therefore, we have

$$H^0(s_4s_3s_4s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}.$$

Proof of (2). By Corollary 4.8(1), we have

$$H^0(s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$ and $\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}$ are the two-dimensional irreducible $\tilde{L}_{\alpha_3}$-modules and $\langle -(\alpha_1+\alpha_2+\alpha_3), \alpha_3\rangle = 0$, by Lemma 2.3(2), we have

$$H^0(s_3s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$ is the standard two-dimensional irreducible $\tilde{L}_{\alpha_4}$-module by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.$$

Moreover, since $\langle -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4), \alpha_4\rangle = 0$ and $\langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_4\rangle = 0$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$$

and

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), \alpha_4\rangle = 1$, by Lemma 2.3, we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}.$$

Therefore, combining the above discussion, we have

$$H^0(s_4s_3s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4},$$

since $\omega_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$. □

5. Computions of relative tangent bundles $H^1(w, \alpha_2)$

In this section, we compute cohomology modules $H^1(w, \alpha_2)$ corresponding to some special Weyl group elements.

Lemma 5.1.

(1) $H^1(w_r, \alpha_2) = 0$ *for* $r = 1, 2, 5$.
(2) $H^1(w_3, \alpha_2) = \mathbb{C}_{-\omega_4+\alpha_4}$.
(3) $H^1(w_4, \alpha_2) = \mathbb{C}_{-\omega_4}$.

Proof. It is easy to see that $H^1(s_2, \alpha_2) = 0$. Note that we have

$$H^0(s_2, \alpha_2) = \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{\alpha_2}.$$

Since $\langle -\alpha_2, \alpha_1 \rangle = 1$, by using Lemma 2.3(2) and Lemma 2.3(4), we have

$$H^1(s_1, H^0(s_2, \alpha_2)) = 0.$$

Since $H^1(s_2, \alpha_2) = 0$, by using Lemma 2.3(1), we have

$$H^0(s_1, H^1(s_2, \alpha_2)) = 0.$$

Thus, by using SES and the above discussion, we have

$$H^1(s_1 s_2, \alpha_2) = 0.$$

By using SES and Lemma 2.3(2), we have

$$H^0(s_1 s_2, \alpha_2) = \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2)}.$$

Since $\langle \alpha_2, \alpha_4 \rangle = 0$, by using Lemma 2.3(2), we have

$$H^1(s_4, H^0(s_1 s_2, \alpha_2)) = 0$$

and

$$H^0(s_4 s_1 s_2, \alpha_2) = \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2)}. \quad (5.1.1)$$

Again, since $H^1(s_1 s_2, \alpha_2) = 0$, by using Lemma 2.3(1), we have

$$H^0(s_4, H^1(s_1 s_2, \alpha_2)) = 0.$$

Thus, by using SES and the above discussion, we have

$$H^1(s_4 s_1 s_2, \alpha_2) = 0. \quad (5.1.2)$$

Since $\langle -\alpha_2, \alpha_3 \rangle = 2$, $\langle -(\alpha_1 + \alpha_2), \alpha_3 \rangle = 2$, by using (5.1.1) and Lemma 2.3(2), we have

$$H^1(s_3, H^0(s_4 s_1 s_2, \alpha_2)) = 0.$$

Further, by (5.1.2), we have

$$H^0(s_3, H^1(s_4 s_1 s_2, \alpha_2)) = 0.$$

Thus, by using SES, we have

$$H^1(s_3s_4s_1s_2, \alpha_2) = 0. \tag{5.1.3}$$

Therefore, we have

$$H^0(s_2, H^1(s_3s_4s_1s_2, \alpha_2)) = 0.$$

By using Lemma 3.3, we have

$$H^1(s_2, H^0(s_3s_4s_1s_2, \alpha_2)) = 0.$$

Thus by SES, we have

$$H^1(s_2s_3s_4s_1s_2, \alpha_2) = 0. \tag{5.1.4}$$

Therefore, we have

$$H^0(s_1, H^1(s_2s_3s_4s_1s_2, \alpha_2)) = 0.$$

By Lemma 3.3, we have

$$H^1(s_1, H^0(s_2s_3s_4s_1s_2, \alpha_2)) = 0.$$

Therefore, by using SES, we have

$$H^1(w_1, \alpha_2) = 0.$$

Since $H^1(w_1, \alpha_2) = 0$, we have

$$H^0(s_4, H^1(w_1, \alpha_2)) = 0.$$

Recall that by (4.1.3), we have

$$H^0(w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}.$$

Since $\langle -(\alpha_2+2\alpha_3), \alpha_4\rangle = 2$, $\langle -(\alpha_1+\alpha_2+2\alpha_3), \alpha_4\rangle = 2$, $\langle -(\alpha_1+2\alpha_2+2\alpha_3), \alpha_4\rangle = 2$, by using Lemma 2.3(2), we have

$$H^1(s_4, H^0(w_1, \alpha_2)) = 0.$$

Thus, by using SES and the above discussion, we have

$$H^1(s_4w_1, \alpha_2) = 0 \tag{5.1.5}$$

and

$$\begin{aligned} H^0(s_4w_1, \alpha_2) = {} & (\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)}) \oplus (\mathbb{C}_{-(\alpha_2+2\alpha_3)} \\ & \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)}) \\ & \oplus (\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)}). \end{aligned}$$

Since $H^1(s_4w_1, \alpha_2) = 0$, we have

$$H^0(s_3, H^1(s_4w_1, \alpha_2)) = 0.$$

Since $\langle -(\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_3\rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3), \alpha_3\rangle = 0$, $\langle -(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_3\rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3\rangle = 1$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_3\rangle = 2$, $\langle -(\alpha_2 + 2\alpha_3), \alpha_3\rangle = -2$, $\langle -(\alpha_1 + \alpha_2 + 2\alpha_3), \alpha_3\rangle = -2$, $\langle -(\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3\rangle = -1$, $\langle -(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3\rangle = -1$, by using Lemma 2.3, we have

$$H^1(s_3, H^0(s_4w_1, \alpha_2)) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Thus, by using SES and the above discussion, we have

$$H^1(s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}, \tag{5.1.6}$$

and

$$\begin{aligned} H^0(s_3s_4w_1, \alpha_2) = {} & \mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}. \end{aligned} \tag{5.1.7}$$

Since $H^1(s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}$, by using Lemma 2.3, we have

$$H^0(s_2, H^1(s_3s_4w_1, \alpha_2)) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

By Lemma 3.3, we have

$$H^1(s_2, H^0(s_3s_4w_1, \alpha_2)) = 0.$$

Thus, using SES and the above discussion, we have

$$H^1(s_2s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}. \tag{5.1.8}$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)}$ is the standard two-dimensional irreducible $\tilde{L}_{\alpha_2}$-module and $\langle -(\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_2\rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_2\rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4), \alpha_2\rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_2\rangle = 1$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3), \alpha_2\rangle = -1$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_2\rangle = -1$, by using SES and Lemma 2.3, we have

$$\begin{aligned} H^0(s_2s_3s_4w_1, \alpha_2) = {} & \mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}. \end{aligned} \tag{5.1.9}$$

Since $\langle -(\alpha_1+\alpha_2+\alpha_3), \alpha_1\rangle = -1$, by using Lemma 2.3(4), we have

$$H^0(s_1, H^1(s_2s_3s_4w_1, \alpha_2)) = 0.$$

Further, by Lemma 3.3, we have

$$H^1(s_1, H^0(s_2s_3s_4w_1, \alpha_2)) = 0.$$

Thus using SES, we have

$$H^1(w_2, \alpha_2) = 0.$$

Since $\mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)}$ is the standard two-dimensional irreducible $\tilde{L}_{\alpha_1}$-module and $\langle -(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4), \alpha_1\rangle = 0$, $\langle -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), \alpha_1\rangle = 0$, $\langle -(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4), \alpha_1\rangle = 0$ and $\langle -(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4), \alpha_1\rangle = 0$, $\langle -(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4), \alpha_1\rangle = 1$, by using SES and Lemma 2.3, we have

$$\begin{aligned} H^0(w_2, \alpha_2) = {} & \mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}. \end{aligned}$$

Since $H^1(w_2, \alpha_2) = 0$, we have

$$H^0(s_4, H^1(w_2, \alpha_2)) = 0.$$

Since $\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}$ is the standard two-dimensional irreducible $\tilde{L}_{\alpha_4}$-module and $\langle -(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4), \alpha_4\rangle = 0$, $\langle -(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4), \alpha_4\rangle = 0$, $\langle -(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4), \alpha_4\rangle = 0$, $\langle -(\alpha_2+2\alpha_3+2\alpha_4), \alpha_4\rangle = -2$, $\langle -(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4), \alpha_4\rangle = -2$, $\langle -(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4), \alpha_4\rangle = -2$, by using SES and Lemma 2.3, we have

$$\begin{aligned} H^1(s_4, H^0(w_2, \alpha_2)) = {} & \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}. \end{aligned}$$

Thus, from the above discussion, we have

$$\begin{aligned} H^1(s_4w_2, \alpha_2) = {} & \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \end{aligned} \tag{5.1.10}$$

and

$$\begin{aligned} H^0(s_4w_2, \alpha_2) = {} & \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}. \end{aligned}$$

Since $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3\rangle = 1$, $\langle -(\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3\rangle = -1$ and $\langle -(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3\rangle = -1$, by using Lemma 2.3, we have

$$H^0(s_3, H^1(s_4w_2, \alpha_2)) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_3\rangle = 0$, $\langle -(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_3\rangle = 0$, $\langle -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), \alpha_3\rangle = -1$ and $\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} = V \otimes \mathbb{C}_{-\omega_3}$(where V is the standard two-dimensional irreducible $\tilde{L}_{\alpha_3}$-module), by Lemma 2.3, we have

$$H^1(s_3, H^0(s_4w_2, \alpha_2)) = 0$$

and

$$H^0(s_3s_4w_2, \alpha_2) = \mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}. \tag{5.1.11}$$

Therefore, we have

$$H^1(s_3s_4w_2, \alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}. \tag{5.1.12}$$

Since $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_2\rangle = 0$ and $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_2\rangle = -1$, by using Lemma 2.3, we have

$$H^0(s_2, H^1(s_3s_4w_2, \alpha_2)) = \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

By Lemma 3.3, we have

$$H^1(s_2, H^0(s_3s_4w_2, \alpha_2)) = 0.$$

Thus, from the above discussion, we have

$$H^1(s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} = \mathbb{C}_{-\omega_4+\alpha_4}. \tag{5.1.13}$$

Since $\langle -(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_2\rangle = 0$ and $\langle -(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_2\rangle = -1$, by using SES and Lemma 2.3, we have

$$H^0(s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}. \tag{5.1.14}$$

Since $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_1\rangle = 0$, by using Lemma 2.3(2), we have

$$H^0(s_1, H^1(s_2s_3s_4w_2, \alpha_2)) = \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

By Lemma 3.3, we have

$$H^1(s_1, H^0(s_2s_3s_4w_2, \alpha_2)) = 0.$$

Thus, from the above discussion, we have

$$H^1(w_3, \alpha_2) = \mathbb{C}_{-\omega_4+\alpha_4}$$

since $\omega_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$. This proves (2).

Since we have $H^0(w_3, \alpha_2) = 0$ (see Lemma 4.1), by using SES, we have

$$H^1(s_4w_3, \alpha_2) = H^0(s_4, H^1(w_3, \alpha_2)).$$

Since $\langle -\omega_4 + \alpha_4, \alpha_4 \rangle = 1$, by using Lemma 2.3(2), we have

$$H^1(s_4w_3, \alpha_2) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}. \tag{5.1.15}$$

Since we have $H^0(w_3, \alpha_2) = 0$ (see Lemma 4.1), by using SES, we have

$$H^1(s_3s_4w_3, \alpha_2) = H^0(s_3, H^1(s_4w_3, \alpha_2)).$$

Since $\langle -\omega_4, \alpha_3 \rangle = 0$ and $\langle -\omega_4 + \alpha_4, \alpha_3 \rangle = -1$, by using Lemma 2.3(2) and Lemma 2.3(4), we have

$$H^1(s_3s_4w_3, \alpha_2) = \mathbb{C}_{-\omega_4}. \tag{5.1.16}$$

Since we have $H^0(w_3, \alpha_2) = 0$ (see Lemma 4.1) and α_1, α_2 are orthogonal to ω_4, by Lemma 2.3(2), we have

$$H^1(w_4, \alpha_2) = \mathbb{C}_{-\omega_4}.$$

This gives the proof of (3).

Since we have $H^0(w_3, \alpha_2) = 0$ (see Lemma 4.1) and $\langle -\omega_4, \alpha_4 \rangle = -1$, by using Lemma 2.3(4), we have

$$H^1(s_4w_4, \alpha_2) = 0.$$

Since by Lemma 4.1 we have $H^0(w_3, \alpha_2) = 0$ and $H^1(s_4w_4, \alpha_2) = 0$, by using SES repeatedly we have

$$H^1(w_5, \alpha_2) = 0.$$

This completes the proof of (1). □

COROLLARY 5.2

(1) $H^1(s_4s_1s_2, \alpha_2) = 0$.
(2) $H^1(s_4w_r, \alpha_2) = 0$ *for* $r = 1, 4, 5$.
(3) $H^1(s_4w_2, \alpha_2) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}$.
(4) $H^1(s_4w_3, \alpha_2) = \mathbb{C}_{-\omega_4} \oplus \mathbb{C}_{-\omega_4+\alpha_4}$.

Proof.

Proof of (1). Follows from (5.1.2).
Proof of (2). For $r = 1$, the proof follows from (5.1.5). For $r = 4, 5$ the proof follows by using SES, Lemma 5.1 and Lemma 4.1.
Proof of (3). Follows from (5.1.10).
Proof of (4). Follows from (5.1.15). □

COROLLARY 5.3

(1) $H^1(s_3s_4s_1s_2, \alpha_2) = 0$.
(2) $H^1(s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}$.
(3) $H^1(s_3s_4w_2, \alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4}$.
(4) $H^1(s_3s_4w_3, \alpha_2) = \mathbb{C}_{-\omega_4}$.
(5) $H^1(s_3s_4w_r, \alpha_2) = 0$ *for* $r = 4, 5$.

Proof.

Proof of (1). Follows from (5.1.3).
Proof of (2). Follows from (5.1.6).
Proof of (3). Follows from (5.1.12).
Proof of (4). Follows from (5.1.16).
Proof of (5). By Lemma 4.1, we have $H^0(w_r, \alpha_2) = 0$ for $r = 4, 5$. Therefore, $H^0(s_4w_r, \alpha_2) = 0$ for $r = 4, 5$. Hence we have $H^1(s_3, H^0(s_4w_r, \alpha_2)) = 0$ for $r = 4, 5$. On the other hand, by Corollary 5.2(2), we have $H^1(s_4w_r, \alpha_2) = 0$ for $r = 4, 5$. Therefore, $H^0(s_3, H^1(s_4w_r, \alpha_2)) = 0$ for $r = 4, 5$. Thus by SES, we have $H^1(s_3s_4w_r, \alpha_2) = 0$ for $r = 4, 5$. □

COROLLARY 5.4

(1) $H^1(s_2s_3s_4s_1s_2, \alpha_2) = 0$.
(2) $H^1(s_2s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}$.
(3) $H^1(s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-\omega_4+\alpha_4}$.
(4) $H^1(s_2s_3s_4w_3, \alpha_2) = \mathbb{C}_{-\omega_4}$.
(5) $H^1(s_2s_3s_4w_4, \alpha_2) = 0$.

Proof.

Proof of (1). Follows from (5.1.4).
Proof of (2). Follows from (5.1.8).
Proof of (3). Follows from (5.1.13).
Proof of (4). By Lemma 4.1, we have $H^0(w_3, \alpha_2) = 0$. Therefore, $H^0(s_3s_4w_3, \alpha_2) = 0$. Hence we have $H^1(s_2, H^0(s_3s_4w_3, \alpha_2)) = 0$. On the other hand, by Corollary 5.3(4), we have $H^1(s_3s_4w_3, \alpha_2) = \mathbb{C}_{-\omega_4}$. Since ω_4 is orthogonal to α_2, by Lemma 2.3(2), we have $H^0(s_2, H^1(s_3s_4w_3, \alpha_2)) = \mathbb{C}_{-\omega_4}$. Thus by SES, we have $H^1(s_2s_3s_4w_3, \alpha_2) = \mathbb{C}_{-\omega_4}$.
Proof of (5). By Lemma 4.1(2), we have $H^0(w_4, \alpha_2) = 0$. Therefore, $H^0(s_3s_4w_4, \alpha_2) = 0$. Hence we have $H^1(s_2, H^0(s_3s_4w_4, \alpha_2)) = 0$. On the other hand, by Corollary 5.3(5), we have $H^1(s_3s_4w_4, \alpha_2) = 0$. Therefore, $H^0(s_2, H^1(s_3s_4w_4, \alpha_2)) = 0$. Thus by SES, we have $H^1(s_2s_3s_4w_4, \alpha_2) = 0$.

□

COROLLARY 5.5

(1) $H^1(s_4s_3s_4s_1s_2, \alpha_2) = 0$.
(2) $H^1(s_4s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$.
(3) $H^1(s_4s_3s_4w_2, \alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}$.

(4) $H^1(s_4s_3s_4w_r, \alpha_2) = 0$ *for* $r = 3, 4$.

Proof.

Proof of (1). By (4.1.1), if $H^0(s_3s_4s_1s_2, \alpha_2)_\mu \neq 0$, then we have $\langle \mu, \alpha_4 \rangle \geq 0$. Thus using Lemma 2.3(3), we have

$$H^1(s_4, H^0(s_3s_4s_1s_2, \alpha_2)) = 0.$$

On the other hand, by using Corollary 5.3(1), we have

$$H^0(s_4, H^1(s_3s_4s_1s_2, \alpha_2)) = 0.$$

Hence we have $H^1(s_4s_3s_4s_1s_2, \alpha_2) = 0$.

Proof of (2). By (5.1.7), the $\tilde{B}_{\alpha_4}$-indecomposable summands V of $H^0(s_3s_4w_1, \alpha_2)$ for which $H^1(s_4, V) \neq 0$ are $\mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)}$ and $\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)}$. Thus using Lemma 2.3(3), we have

$$H^1(s_4, H^0(s_3s_4w_1, \alpha_2)) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.$$

On the other hand, by using Corollary 5.3(2) and Lemma 2.3(2), we have

$$\begin{aligned} H^0(s_4, H^1(s_3s_4w_1, \alpha_2)) = \; & \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}. \end{aligned}$$

Hence we have

$$\begin{aligned} H^1(s_4s_3s_4w_1, \alpha_2) = \; & \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}. \end{aligned}$$

Proof of (3). By (5.1.11), we have if $H^0(s_3s_4w_2, \alpha_2)_\mu \neq 0$, then $\langle \mu, \alpha_4 \rangle = 0$. Thus using Lemma 2.3(3), we have

$$H^1(s_4, H^0(s_3s_4w_2, \alpha_2)) = 0.$$

On the other hand, by using Corollary 5.3(3) and Lemma 2.3(2), we have

$$\begin{aligned} H^0(s_4, H^1(s_3s_4w_2, \alpha_2)) = \; & \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}. \end{aligned}$$

Hence we have

$$H^1(s_4s_3s_4w_2, \alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.$$

Proof of (4). By Lemma 4.1, we have $H^0(w_r, \alpha_2) = 0$ for $r = 3, 4$.

Therefore, $H^0(s_3s_4w_r, \alpha_2) = 0$ for $r = 3, 4$. Hence we have $H^1(s_4, H^0(s_3s_4w_r, \alpha_2)) = 0$ for $r = 3, 4$.

On the other hand, by Corollary 5.3(4) and Corollary 5.3(5), we have $H^0(s_4, H^1(s_3s_4w_r, \alpha_2)) = 0$ for $r = 3, 4$. Thus, by using SES, we have $H^1(s_4s_3s_4w_r, \alpha_2) = 0$ for $r = 3, 4$. □

COROLLARY 5.6

(1) $H^1(s_4s_2s_3s_4s_1s_2, \alpha_2) = 0$.
(2) $H^1(s_4s_2s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}$.
(3) $H^1(s_4s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}$.
(4) $H^1(s_4s_2s_3s_4w_r, \alpha_2) = 0$ *for* $r = 3, 4$.

Proof.

Proof of (1). By Lemma 3.3, we have

$$H^1(s_2, H^0(s_4s_3s_4s_1s_2, \alpha_2)) = 0.$$

On the other hand, by using Corollary 5.5(1), we have

$$H^0(s_2, H^1(s_4s_3s_4s_1s_2, \alpha_2)) = 0.$$

Hence we have $H^1(s_4s_2s_3s_4s_1s_2, \alpha_2) = H^1(s_2s_4s_3s_4s_1s_2, \alpha_2) = 0$.

Proof of (2). By Corollary 5.5(2), we have

$$\begin{aligned} H^0(s_2, H^1(s_4s_3s_4w_1, \alpha_2)) = {} & \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}. \end{aligned}$$

Now the proof of (2) follows from Lemma 3.3 and SES.

Proof of (3). By Corollary 5.5(3), using SES and Lemma 2.3, we have

$$H^0(s_2, H^1(s_4s_3s_4w_2, \alpha_2)) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.$$

Now the proof of (3) follows from Lemma 3.3 and SES.

Proof of (4). By Lemma 3.3, we have $H^1(s_2, H^0(s_4s_3s_4w_r, \alpha_2)) = 0$ for $r = 3, 4$. On the other hand, by Corollary 5.5(4), we have $H^0(s_2, H^1(s_4s_3s_4w_r, \alpha_2)) = 0$ for $r = 3, 4$. Thus by using SES, we have $H^1(s_4s_2s_3s_4w_r, \alpha_2) = 0$ for $r = 3, 4$. □

Lemma 5.7.

(1) $H^1(s_3s_4s_2s_3s_4s_1s_2, \alpha_2) = \mathbb{C}_{-(\alpha_2+\alpha_3)}$.
(2) $H^1(s_3s_4s_2s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4}$.
(3) $H^1(s_3s_4s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-\omega_4}$.
(4) $H^1(s_3s_4s_2s_3s_4w_r, \alpha_2) = 0$ *for* $r = 3, 4$.

Proof.

Proof of (1). Recall from (4.1.2) that

$$H^0(s_2s_3s_4s_1s_2, \alpha_2) = \mathbb{C}_{-(\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}.$$

Since $\langle -(\alpha_2+2\alpha_3), \alpha_4\rangle = 2$, $\langle -(\alpha_1+\alpha_2+\alpha_3), \alpha_4\rangle = 1$, $\langle -(\alpha_1+\alpha_2+2\alpha_3), \alpha_4\rangle = 2$ and $\langle -(\alpha_1+2\alpha_2+2\alpha_3), \alpha_4\rangle = 2$, by using SES and Lemma 2.3(2), we have

$$\begin{aligned} H^0(s_4s_2s_3s_4s_1s_2, \alpha_2) = {} & \mathbb{C}_{-(\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)}. \end{aligned} \tag{5.7.1}$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}$ is the indecomposable $\tilde{B}_{\alpha_3}$-module, by using Lemma 2.4(1), we have $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)} = V \otimes \mathbb{C}_{-\omega_3}$ (where V is the standard two-dimensional irreducible $\tilde{L}_{\alpha_3}$-module) and $\langle -(\alpha_2+2\alpha_3), \alpha_3\rangle = -2$, by using SES and Lemma 2.3(3), we have

$$H^1(s_3, H^0(s_4s_2s_3s_4s_1s_2, \alpha_2)) = \mathbb{C}_{-(\alpha_2+\alpha_3)}.$$

By using SES and Corollary 5.6(1), we have

$$H^0(s_3, H^1(s_4s_2s_3s_4s_1s_2, \alpha_2)) = 0.$$

Thus we have

$$H^1(s_3s_4s_2s_3s_4s_1s_2, \alpha_2) = \mathbb{C}_{-(\alpha_2+\alpha_3)}.$$

Proof of (2). Recall from (5.1.9) that

$$\begin{aligned} H^0(s_2s_3s_4w_1, \alpha_2) = {} & \mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}. \end{aligned}$$

Since $\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}$ is the standard two-dimensional irreducible $\tilde{L}_{\alpha_4}$-module, $\langle -(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4), \alpha_4\rangle = 0$, $\langle -(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4), \alpha_4\rangle = 0$, $\langle -(\alpha_2+2\alpha_3+2\alpha_4), \alpha_4\rangle = -2$, $\langle -(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4), \alpha_4\rangle = -2$, $\langle -(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4), \alpha_4\rangle = -2$, by using Lemma 2.3, we have

$$\begin{aligned} H^0(s_4s_2s_3s_4w_1, \alpha_2) = {} & \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}. \end{aligned} \tag{5.7.2}$$

By Corollary 5.6(2), we have

$$\begin{aligned} H^1(s_4s_2s_3s_4w_1, \alpha_2) = {} & \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}. \end{aligned}$$

Since $\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}$ is an indecomposable $\tilde{B}_{\alpha_3}$-module, by Lemma 2.4(1), we have

$$\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} = V \otimes \mathbb{C}_{-\omega_3}$$

where V is the standard two-dimensional irreducible $\tilde{L}_{\alpha_3}$-module.

Further, since $\langle -(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4), \alpha_3\rangle = 0$, $\langle -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), \alpha_3\rangle = -1$, by using Lemma 2.3, we have

$$H^1(s_3, H^0(s_4s_2s_3s_4w_1, \alpha_2)) = 0.$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$ is the standard two-dimensional irreduible $\tilde{L}_{\alpha_3}$-module, $\langle -(\alpha_1+\alpha_2+\alpha_3), \alpha_3\rangle = 0$, $\langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_3\rangle = 1$, $\langle -(\alpha_2+2\alpha_3+\alpha_4), \alpha_3\rangle = -1$, by using Lemma 2.3(2), we have

$$\begin{aligned} H^0(s_3, H^1(s_4s_2s_3s_4w_1, \alpha_2)) = {} & \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}. \end{aligned}$$

Thus we have

$$\begin{aligned} H^1(s_3s_4s_2s_3s_4w_1, \alpha_2) = {} & \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}. \end{aligned}$$

Proof of (3). Recall from (5.1.14) that

$$H^0(s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)} = \mathbb{C}_{-\omega_1}.$$

Since α_4 is orthogonal to ω_1, by using Lemma 2.3(2), we have

$$H^0(s_4s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)} = \mathbb{C}_{-\omega_1}.$$

Since α_3 is orthogonal to ω_1, by using Lemma 2.3(2), we have

$$H^1(s_3, H^0(s_4s_2s_3s_4w_2, \alpha_2)) = 0.$$

On the other hand, by Corollary 5.6(3), we have

$$H^1(s_4s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.$$

Since $\langle -\omega_4, \alpha_3\rangle = 0$ and $\langle -\omega_4+\alpha_4, \alpha_3\rangle = -1$, by using Lemma 2.3, we have

$$H^0(s_3, H^1(s_4s_2s_3s_4w_2, \alpha_2)) = \mathbb{C}_{-\omega_4}.$$

Thus we have

$$H^1(s_3s_4s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-\omega_4}.$$

Proof of (4). By Lemma 4.1, we have $H^0(w_r, \alpha_2) = 0$ for $r = 3, 4$. Therefore, we have $H^0(s_4s_2s_3s_4w_r, \alpha_2) = 0$ for $r = 3, 4$. Hence we have $H^1(s_3, H^0(s_4s_2s_3s_4w_r, \alpha_2)) = 0$ for $r = 3, 4$. On the other hand, by Corollary 5.6(4), we have $H^0(s_3, H^1(s_4s_2s_3s_4w_r, \alpha_2)) = 0$ for $r = 3, 4$. Thus by using SES, we have $H^1(s_3s_4s_2s_3s_4w_r, \alpha_2) = 0$ for $r = 3, 4$. □

Lemma 5.8.

(1) $H^1(s_4s_3s_4s_2, \alpha_2) = 0$.
(2) $H^1(s_4s_3s_4s_2s_3s_4s_1s_2, \alpha_2) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}$.
(3) $H^1(s_4s_3s_4s_2s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}$.
(4) $H^1(s_4s_3s_4s_2s_3s_4w_r, \alpha_2) = 0$ *for* $r = 2, 3$.

Proof.

Proof of (1). By using SES, it is easy to see that

$$H^0(s_3s_4s_2, \alpha_2) = \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3)}$$

and

$$H^1(s_3s_4s_2, \alpha_2) = \mathbb{C}_{\alpha_2+\alpha_3}.$$

Since $H^0(s_3s_4s_2, \alpha_2)_\mu \neq 0$ implies $\langle \mu, \alpha_4 \rangle \geq 0$, by using Lemma 2.3(2), we have

$$H^1(s_4, H^0(s_3s_4s_2, \alpha_2)) = 0.$$

Since $\langle \alpha_2 + \alpha_3, \alpha_4 \rangle = -1$, by using Lemma 2.3(4), we have $H^0(s_4, H^1(s_3s_4s_2, \alpha_2)) = 0$. Therefore, by using SES, we have $H^1(s_4s_3s_4s_2, \alpha_2) = 0$.

Proof of (2). By the Corollary 5.7(1), we have

$$H^1(s_3s_4s_2s_3s_4s_1s_2, \alpha_2) = \mathbb{C}_{-(\alpha_2+\alpha_3)}.$$

Since $\langle -(\alpha_2 + \alpha_3), \alpha_4 \rangle = 1$, by using Lemma 2.3(2), we have

$$H^0(s_4, H^1(s_3s_4s_2s_3s_4s_1s_2, \alpha_2)) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}.$$

Recall from (5.7.1) that

$$\begin{aligned} H^0(s_4s_2s_3s_4s_1s_2, \alpha_2) = {} & \mathbb{C}_{-(\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)}. \end{aligned}$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$ is the standard two-dimensional irreducible $\tilde{L}_{\alpha_3}$-module, $\langle -(\alpha_2+2\alpha_3+2\alpha_4), \alpha_3\rangle = 0$, $\langle -(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4), \alpha_3\rangle = 0$, $\langle -(\alpha_1+2\alpha_2+2\alpha_3), \alpha_3\rangle = 0$, $\langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_3\rangle = 1$, $\langle -(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4), \alpha_3\rangle = 2$, $\langle -(\alpha_2+2\alpha_3), \alpha_3\rangle = -2$, $\langle -(\alpha_2+2\alpha_3+\alpha_4), \alpha_3\rangle = -1$ and $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)} = V \otimes \mathbb{C}_{-\omega_3}$ (where V is the standard two-dimensional irreducible $\tilde{L}_{\alpha_3}$), by using SES and Lemma 2.3, we have

$$\begin{aligned} H^0(s_3s_4s_2s_3s_4s_1s_2, \alpha_2) = {} & \mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}. \end{aligned} \tag{5.8.1}$$

The $\tilde{B}_{\alpha_4}$-indecomposable summands V of $H^0(s_3s_4s_2s_3s_4s_1s_2, \alpha_2)$ for which $H^1(s_4, V) \neq 0$ is $\mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)}$. Thus by using SES and Lemma 2.3(3), we have

$$H^1(s_4, H^0(s_3s_4s_2s_3s_4s_1s_2, \alpha_2)) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}.$$

Therefore, by using SES, we have

$$H^1(s_4s_3s_4s_2s_3s_4s_1s_2, \alpha_2) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}.$$

Proof of (3). Recall that from Corollary 5.7(2), we have

$$\begin{aligned} H^1(s_3s_4s_2s_3s_4w_1, \alpha_2) = {} & \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}. \end{aligned}$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$ is the standard two-dimensional irreducible $\tilde{L}_{\alpha_4}$-module, $\langle -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4), \alpha_4\rangle = 0$, $\langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_4\rangle = 0$ and $\langle -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), \alpha_4\rangle = 1$, by using SES and Lemma 2.3, we have

$$\begin{aligned} H^0(s_4, H^1(s_3s_4s_2s_3s_4w_1, \alpha_2)) = {} & \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}. \end{aligned}$$

On the other hand, from (5.7.2), we have

$$\begin{aligned} H^0(s_4s_2s_3s_4w_1, \alpha_2) = {} & \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}. \end{aligned}$$

Since $\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} = V \otimes \mathbb{C}_{-\omega_3}$, where V is the standard two-dimensional irreducible $\tilde{L}_{\alpha_3}$-module, $\langle -(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4), \alpha_3\rangle = 0$ and $\langle -(\alpha_1+$

$2\alpha_2+3\alpha_3+\alpha_4), \alpha_3\rangle = -1$, by using SES and Lemma 2.3, we have

$$H^0(s_3s_4s_2s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}.$$

Since $\langle -(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4), \alpha_4\rangle = 0$, by using SES and Lemma 2.3, we have

$$H^1(s_4, H^0(s_3s_4s_2s_3s_4w_1, \alpha_2)) = 0.$$

Therefore by SES, we have

$$\begin{aligned} H^1(s_4s_3s_4s_2s_3s_4w_1, \alpha_2) = {} & \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \\ & \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}. \end{aligned}$$

Proof of (4). For $r=2$, we recall that from (5.1.14) that $H^0(s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-\omega_1}$. Since α_4, α_3 are orthogonal to ω_1, by using SES, we have $H^0(s_3s_4s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-\omega_1}$. Further, using the orthogonality of α_4 and ω_1, we have $H^1(s_4, H^0(s_3s_4s_2s_3s_4w_2, \alpha_2)) = 0$. On the other hand, by Corollary 5.7(3), we have $H^0(s_4, H^1(s_3s_4s_2s_3s_4w_2, \alpha_2)) = 0$. Thus we have $H^1(s_4s_3s_4s_2s_3s_4w_2, \alpha_2) = 0$. For $r=3$, by Lemma 4.1, we have $H^0(w_3, \alpha_2) = 0$. Therefore, $H^0(s_3s_4s_2s_3s_4w_3, \alpha_2) = 0$. Hence we have $H^1(s_4, H^0(s_3s_4s_2s_3s_4w_3, \alpha_2)) = 0$. On the other hand, by Corollary 5.7(4), we have $H^0(s_4, H^1(s_3s_4s_2s_3s_4w_3, \alpha_2)) = 0$. Thus by using SES, we have $H^1(s_4s_3s_4s_2s_3s_4w_r, \alpha_2) = 0$. □

We denote $v_r = [1,4]^r$ for $1 \le r \le 6$ and $\tau_r = [1,4]^r 1$ for $1 \le r \le 5$.

Lemma 5.9. *We have*

(1) $H^i(\tau_r, \alpha_1) = 0$ *for all* $i \ge 0, 1 \le r \le 5$.
(2) $H^i(v_r, \alpha_4) = 0$ *for all* $i \ge 0, 2 \le r \le 6$.

Proof.

Proof of (1). By [15, Corollary 6.4, p. 780], we have

$$H^i(\tau_r, \alpha_1) = 0 \text{ for all } i \ge 2, r \ge 1.$$

Note that $H^i(s_1s_2s_3s_4s_1, \alpha_1) = H^i(s_1s_2s_1, \alpha_1) = H^i(s_2s_1s_2, \alpha_1) = 0$ for $i = 0, 1$ (see Lemma 2.3(4)). Now by using SES repeatedly, we have the required result.

Proof of (2). By [15, Corollary 6.4, p. 780], we have

$$H^i(v_r, \alpha_4) = 0 \quad \text{for all } i \ge 2, r \ge 1.$$

We note that

$$H^i(s_4s_1s_2s_3s_4, \alpha_4) = H^i(s_1s_2s_4s_3s_4, \alpha_4) = H^i(s_1s_2s_3s_4s_3, \alpha_4) = 0 \qquad (5.9.1)$$

for $i = 0, 1$ (see Lemma 2.3(4)).

Since $2 \le r \le 6$, we have $v_r = us_4s_1s_2s_3s_4$ for some $u \in W$ such that $l(v_r) = l(u) + 5$. Thus by using SES repeatedly, we have the required result. □

COROLLARY 5.10

We have the following:

(1) $H^i(s_4\tau_r, \alpha_1) = 0$ *for* $i \geq 0, 1 \leq r \leq 5$.
$H^i(s_4 v_r, \alpha_4) = 0$ *for* $i \geq 0, 1 \leq r \leq 5$.
(2) $H^i(s_3s_4\tau_r, \alpha_1) = 0$ *for* $i \geq 0, 1 \leq r \leq 5$.
$H^i(s_3s_4 v_r, \alpha_4) = 0$ *for* $i \geq 0, 1 \leq r \leq 5$.
(3) $H^i(s_2s_3s_4\tau_r, \alpha_1) = 0$ *for* $i \geq 0, 1 \leq r \leq 5$.
$H^i(s_2s_3s_4 v_r, \alpha_4) = 0$ *for* $i \geq 0, 1 \leq r \leq 5$.
(4) $H^i(s_4s_3s_4\tau_r, \alpha_1) = 0$ *for* $i \geq 0, 1 \leq r \leq 4$.
$H^i(s_4s_3s_4 v_r, \alpha_4) = 0$ *for* $i \geq 0, 1 \leq r \leq 4$.
(5) $H^i(s_4s_2s_3s_4\tau_r, \alpha_1) = 0$ *for* $i \geq 0, 1 \leq r \leq 4$.
$H^i(s_4s_2s_3s_4 v_r, \alpha_4) = 0$ *for* $i \geq 0, 1 \leq r \leq 4$.
(6) $H^i(s_4s_3s_4s_2s_3s_4\tau_r, \alpha_1) = 0$ *for* $i \geq 0, 1 \leq r \leq 3$.
$H^i(s_4s_3s_4s_2s_3s_4 v_r, \alpha_4) = 0$ *for* $i \geq 0, 1 \leq r \leq 3$.

Proof.

Proof of (1). By using SES and Lemma 5.9(1), we have $H^i(s_4\tau_r, \alpha_1) = 0$ for all $1 \leq r \leq 5$, $i \geq 0$.

By (5.9.1), we have $H^i(s_4 v_1, \alpha_4) = 0$ for $i \geq 0$. On the other hand, by using SES and Lemma 5.9(2), we have $H^i(s_4 v_r, \alpha_4) = 0$ for $i \geq 0, 2 \leq r \leq 5$. Thus, by combining, we have $H^i(s_4 v_r, \alpha_4) = 0$ for $i \geq 0, 1 \leq r \leq 5$.
Proofs of (2), (3), (4), (5), and (6). Follow by using SES and (1). □

6. Surjectivity of some maps

Let $w \in W$ and let $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ be a reduced expression for w and let $\underline{i} = (i_1, i_2, \ldots, i_r)$. Let $\tau = s_{i_1}s_{i_2}\cdots s_{i_{r-1}}$ and $\underline{i'} = (i_1, i_2, \ldots, i_{r-1})$.

Recall the following long exact sequence of B-modules from [13] (see [13, Proposition 3.1, p. 673]):

$$\begin{aligned}&0 \to H^0(w, \alpha_{i_r}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^0(Z(\tau, \underline{i'}), T_{(\tau,\underline{i'})}) \longrightarrow\\ &H^1(w, \alpha_{i_r}) \longrightarrow H^1(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^1(Z(\tau, \underline{i'}),\\ &T_{(\tau,\underline{i'})}) \longrightarrow H^2(w, \alpha_{i_r}) \longrightarrow\\ &H^2(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^2(Z(\tau, \underline{i'}), T_{(\tau,\underline{i'})}) \longrightarrow H^3(w, \alpha_{i_r}) \longrightarrow \cdots.\end{aligned}$$

By [15, Corollary 6.4, p. 780], we have $H^j(w, \alpha_{i_r}) = 0$ for every $j \geq 2$. Thus we have the following exact sequence of B-modules:

$$\begin{aligned}&0 \to H^0(w, \alpha_{i_r}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^0(Z(\tau, \underline{i'}), T_{(\tau,\underline{i'})}) \longrightarrow\\ &H^1(w, \alpha_{i_r}) \longrightarrow H^1(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^1(Z(\tau, \underline{i'}), T_{(\tau,\underline{i'})}) \longrightarrow 0.\end{aligned}$$

From now onwards, we call this exact sequence by LES.

Let $w_0 = s_{j_1}s_{j_2}\cdots s_{j_N}$ be a reduced expression of w_0. Let $w = s_{j_1}s_{j_2}\cdots s_{j_r}$, $\underline{i} = (j_1, j_2, \ldots, j_r)$, and $\underline{j} = (j_1, j_2, \ldots, j_N)$.

Lemma 6.1. *The natural homomorphism*

$$f : H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$$

of B-modules is injective if and only if $w^{-1}(\alpha_0) < 0$.

Proof. Suppose $w^{-1}(\alpha_0) < 0$. By [13, Lemma 6.2, p. 667], we have $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_{-\alpha_0} \neq 0$. By [13, Theorem 7.1], $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is a parabolic subalgebra of $\mathfrak{g}$ and hence there is a unique B-stable line in $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$, namely $\mathfrak{g}_{-\alpha_0}$. Therefore, we conclude that the natural homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \to H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$$

is injective.

Conversely, suppose the natural homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \to H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$$

is injective. Then by [13, Lemma 6.2, p. 667], we have $w^{-1}(\alpha_0) < 0$. □

Lemma 6.2. *The natural homomorphism*

$$f : H^1(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^1(Z(w, \underline{i}), T_{(w,\underline{i})})$$

of B-modules is surjective.

Proof. See [14, *Lemma* 7.1, *p*. 459]. For $1 \leq r \leq 5$, let τ_r be the reduced expression of $\tau_r = [1, 4]^r s_1$, $\underline{i}_r = (\underline{j}_r, 2)$ be the reduced expression of $w_r = [1, 4]^r s_1 s_2$ and $\underline{l}_r = (\underline{i}_r, 3)$ be the reduced expression of $w_r s_3 = [1, 4]^r s_1 s_2 s_3$. □

Lemma 6.3.

(1) *We have* $\dim H^0(Z(\tau_4, \underline{j}_4), T_{(\tau_4,\underline{j}_4)})_{-\omega_4} = 2$. *Further, the natural map* $H^0(Z(\tau_4, \underline{j}_4), T_{(\tau_4,\underline{j}_4)}) \longrightarrow H^1(w_4, \alpha_2)$ *is surjective.*
(2) *We have* $\dim H^0(Z(\tau_3, \underline{j}_3), T_{(\tau_3,\underline{j}_3)})_{-\omega_4+\alpha_4} = 2$. *Further, the natural map* $H^0(Z(\tau_3, \underline{j}_3), T_{(\tau_3,\underline{j}_3)}) \longrightarrow H^1(w_3, \alpha_2)$ *is surjective.*

Proof.

Proof of (1). Since $w_4^{-1}(\alpha_0) < 0$, by Lemma 6.1, we conclude that the natural homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \to H^0(Z(w_4, \underline{i_4}), T_{(w_4,\underline{i_4})})$$

is injective.

Since α_3 is a short simple root, by [15, Corollary 5.6, p. 778] we have $H^1(w_r s_3, \alpha_3) = 0$ for $r = 4, 5$. On the other hand, by Lemma 5.1, we have $H^1(w_5, \alpha_2) = 0$ and by Lemma 5.9, $H^1(v_r, \alpha_4) = 0$ and $H^1(\tau_r, \alpha_1) = 0$ for $r = 4, 5$.

Thus from above observations and using LES the natural map

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \to H^0(Z(w_4, \underline{i_4}), T_{(w_4,\underline{i_4})}) \tag{6.3.1}$$

is surjective, hence an isomorphism.

By [13, Theorem 7.1], $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is a parabolic subalgebra of $\mathfrak{g}$. Hence for any $\mu \in X(T)\setminus\{0\}$, we have

$$\dim H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})_\mu \leq 1.$$

By using LES repeatedly and using Lemma 5.9, we have

$$H^0(Z(\tau_4, \underline{j_4}), T_{(\tau_4,\underline{j_4})}) = H^0(Z(w_3 s_3, \underline{l_3}), T_{(l_3,\underline{l_3})}). \tag{6.3.2}$$

By using LES and [15, Corollary 5.6, p. 778] we have an exact sequence

$$\begin{aligned} 0 &\to H^0(w_3 s_3, \alpha_3) \to H^0(Z(w_3 s_3, \underline{l_3}), T_{(w_3 s_3,\underline{l_3})}) \\ &\to H^0(Z(w_3, \underline{i_3}), T_{(w_3,\underline{i_3})}) \to 0 \end{aligned} \tag{6.3.3}$$

of B-modules.

Since $w_3^{-1}(\alpha_0) < 0$, by using Lemma 6.1, we conclude that the natural homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \to H^0(Z(w_3, \underline{i_3}), T_{(w_3,\underline{i_3})}) \tag{6.3.4}$$

is injective.

Thus by (6.3.4), we have $\dim H^0(Z(w_3, \underline{i_3}), T_{(w_3,\underline{i_3})})_{-\omega_4} \geq 1$. Hence by Lemma 4.2(2) and (6.3.3), we have

$$dim H^0(Z(w_3 s_3, \underline{l_3}), T_{(w_3 s_3,\underline{l_3})})_{-\omega_4} \geq 2. \tag{6.3.5}$$

By (6.3.1), we have $\dim H^0(Z(w_4, \underline{i_4}), T_{(w_4,\underline{i_4})})_{-\omega_4} \leq 1$. Therefore, by using LES, we see that $\dim H^0(Z(\tau_4, \underline{j}_4), T_{(\tau_4,\underline{j}_4)})_{-\omega_4} \leq 2$.

Thus by (6.3.2) and (6.3.5), we have $\dim H^0(Z(\tau_4, \underline{j}_4), T_{(\tau_4,\underline{j}_4)})_{-\omega_4} = 2$. Therefore, by LES, the natural map $H^0(Z(\tau_4, \underline{j}_4), T_{(\tau_4,\underline{j}_4)})_{-\omega_4} \longrightarrow H^1(w_4, \alpha_2)_{-\omega_4}$ is surjective. Hence by Lemma 5.1(3), the natural map $H^0(Z(\tau_4, \underline{j}_4), T_{(\tau_4,\underline{j}_4)}) \longrightarrow H^1(w_4, \alpha_2)$ is surjective.

Proof of (2). By using LES repeatedly and using Lemma 5.9, we have

$$H^0(Z(\tau_3, \underline{j_3}), T_{(\tau_3,\underline{j_3})}) = H^0(Z(w_2 s_3, \underline{l_2}), T_{(w_2 s_3,\underline{l_2})}). \tag{6.3.6}$$

By using LES and [15, Corollary 5.6, p. 778], we have an exact sequence

$$\begin{aligned} 0 &\to H^0(w_2 s_3, \alpha_3) \to H^0(Z(w_2 s_3, \underline{l_2}), T_{(w_2 s_3,\underline{l_2})}) \\ &\to H^0(Z(w_2, \underline{i_2}), T_{(w_2,\underline{i_2})}) \to 0 \end{aligned} \tag{6.3.7}$$

of B-modules.

Since $w_2^{-1}(\alpha_0) < 0$, by using Lemma 6.1, we conclude that the natural homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \to H^0(Z(w_2, \underline{i_2}), T_{(w_2,\underline{i_2})}) \tag{6.3.8}$$

is injective.

Thus by (6.3.8), we have $\dim H^0(Z(w_2, \underline{i_2}), T_{(w_2,\underline{i_2})})_{-\omega_4+\alpha_4} \geq 1$. Hence by Lemma 4.2(1) and (6.3.7), we have

$$\dim H^0(Z(w_2s_3, \underline{l_2}), T_{(w_2s_3,\underline{l_2})})_{-\omega_4+\alpha_4} \geq 2. \tag{6.3.9}$$

By Lemma 5.1, we have $H^1(w_4, \alpha_2)_{-\omega_4+\alpha_4} = 0$. Since α_3 is a short simple root, by [15, Corollary 5.6, p. 778], we have $H^1(w_3s_3, \alpha_3) = 0$. On the other hand, by Lemma 5.9, we have $H^1(v_4, \alpha_4) = 0$ and $H^1(\tau_4, \alpha_1) = 0$. Thus by using LES and from the above discussion we have the natural map

$$H^0(Z(w_4, \underline{i_4}), T_{(w_4,\underline{i_4})})_{-\omega_4+\alpha_4} \longrightarrow H^0(Z(w_3, \underline{i_3}), T_{(w_3,\underline{i_3})})_{-\omega_4+\alpha_4}$$

is surjective.

Thus by using (6.3.1) and the above surjectivity we have $\dim H^0(Z(w_3, \underline{i_3}), T_{(w_3,\underline{i_3})})_{-\omega_4+\alpha_4} \leq 1$. Therefore, by using LES, we see that $\dim H^0(Z(\tau_3, \underline{j_3}), T_{(\tau_3,\underline{j_3})})_{-\omega_4+\alpha_4} \leq 2$. Thus by (6.3.6) and (6.3.9), we have $\dim H^0(Z(\tau_3, \underline{j}_3), T_{(\tau_3,\underline{j}_3)})_{-\omega_4+\alpha_4} = 2$. Therefore, by LES, the natural map $H^0(Z(\tau_3, \underline{j}_3), T_{(\tau_3,\underline{j}_3)})_{-\omega_4+\alpha_4} \longrightarrow H^1(w_3, \alpha_2)_{-\omega_4+\alpha_4}$ is surjective. Hence by Lemma 5.1(2), the natural map $H^0(Z(\tau_3, \underline{j_3}), T_{(\tau_3,\underline{j_3})}) \longrightarrow H^1(w_3, \alpha_2)$ is surjective. □

Lemma 6.4.

(1) *Let* $\mu = -\omega_4, -\omega_4 + \alpha_4$. *Then we have* $\dim H^0(Z(s_4\tau_3, (4, \underline{j}_3)), T_{(s_4\tau_3,(4,\underline{j}_3))})_\mu = 2$. *Further, the natural map* $H^0(Z(s_4\tau_3, (4, \underline{j}_3)), T_{(s_4\tau_3,(4,\underline{j}_3))}) \longrightarrow H^1(s_4w_3, \alpha_2)$ *is surjective.*

(2) *Let* $\mu = -(\alpha_2 + 2\alpha_3 + \alpha_4), -(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)$. *Then we have* $\dim H^0(Z(s_4\tau_2, (4, \underline{j}_2)), T_{(s_4\tau_2,(4,\underline{j}_2))})_\mu = 2$. *Further, the natural map* $H^0(Z(s_4\tau_2, (4, \underline{j}_2)), T_{(s_4\tau_2,(4,\underline{j}_2))}) \longrightarrow H^1(s_4w_2, \alpha_2)$ *is surjective.*

Proof. Since $(s_4w_3)^{-1}(\alpha_0) < 0$, by Lemma 6.1, we conclude that the natural homomorphism

$$H^0(Z(w_0, (4, \underline{l_5})), T_{(w_0,(4,\underline{l_5})}) \to H^0(Z(s_4w_3, (4, \underline{i_3})), T_{(s_4w_3,(4,\underline{i_3}))})$$

is injective.

Since α_3 is a short simple root, by [15, Corollary 5.6, p. 778] we have $H^1(s_4w_rs_3, \alpha_3) = 0$ for $r = 3, 4, 5$. On the other hand, by Corollary 5.2, we have $H^1(s_4w_r, \alpha_2) = 0$ for $r = 4, 5$, and by Corollary 5.10(1), we have $H^1(s_4v_r, \alpha_4) = 0$ and $H^1(s_4\tau_r, \alpha_1) = 0$ for $r = 4, 5$.

Thus from the above observations and using LES, the natural map

$$H^0(Z(w_0, (4, \underline{l_5})), T_{(w_0,(4,\underline{l_5}))}) \to H^0(Z(s_4w_3, (4, \underline{i_3})), T_{(s_4w_3,(4,\underline{i_3}))}) \tag{6.4.1}$$

is surjective, hence an isomorphism.

Proof of (1). By using LES repeatedly and using Corollary 5.10(1), we have

$$H^0(Z(s_4\tau_3, (4, \underline{j_3})), T_{(s_4\tau_3,(4,\underline{j_3}))}) = H^0(Z(s_4w_2s_3, (4, \underline{l_2})), T_{(s_4w_2s_3,(4,\underline{l_2}))}). \tag{6.4.2}$$

By using LES and [15, Corollary 5.6, p. 778], we have an exact sequence

$$\begin{aligned} 0 \to H^0(s_4w_2s_3, \alpha_3) &\to H^0(Z(s_4w_2s_3, (4, \underline{l_2})), T_{(s_4w_2s_3,(4,\underline{l_2}))}) \\ &\to H^0(Z(s_4w_2, (4, \underline{i_2})), T_{(s_4w_2,(4,\underline{i_2}))}) \to 0 \end{aligned} \tag{6.4.3}$$

of B-modules. On the other hand, since $(s_4w_2)^{-1}(\alpha_0) < 0$, by using Lemma 6.1, we conclude that the natural homomorphism

$$H^0(Z(w_0, (4, \underline{l_5})), T_{(w_0,(4,\underline{l_5}))}) \to H^0(Z(s_4w_2, (4, \underline{i_2})), T_{(s_4w_2,(4,\underline{i_2}))}) \tag{6.4.4}$$

is injective.

Let $\mu = -\omega_4, -\omega_4 + \alpha_4$. Thus by (6.4.4), we have $\dim H^0(Z(s_4w_2, (4, \underline{i_2})), T_{(s_4w_2,(4,\underline{i_2}))})_\mu \geq 1$. Hence by (6.4.2) and by Corollary 4.3(2), we have

$$\dim H^0(Z(s_4w_2s_3, (4, \underline{l_2})), T_{(s_4w_2s_3,(4,\underline{l_2}))})_\mu \geq 2. \tag{6.4.5}$$

By (6.4.1), $\dim H^0(Z(s_4w_3, (4, \underline{i_3})), T_{(s_4w_3,(4,\underline{i_3}))})_\mu \leq 1$.

By using LES, we have $\dim H^0(Z(s_4\tau_3, (4, \underline{j_3})), T_{(s_4\tau_3,(4,\underline{j_3}))})_\mu \leq 2$. Thus by (6.4.2) and (6.4.5), we have $\dim H^0(Z(s_4\tau_3, (4, \underline{j}_3)), T_{(s_4\tau_3,(4,\underline{j}_3))})_\mu = 2$. Therefore, by LES, the natural map $H^0(Z(s_4\tau_3, (4, \underline{j}_3)), T_{(s_4\tau_3,(4,\underline{j}_3))})_\mu \longrightarrow H^1(s_4w_3, \alpha_2)_\mu$ is surjective. Hence by Corollary 5.2(4), the natural map $H^0(Z(s_4\tau_3, (4, \underline{j_3})), T_{(s_4\tau_3,(4,\underline{j_3}))}) \longrightarrow H^1(s_4w_3, \alpha_2)$ is surjective.

Proof of (2). By using LES repeatedly and using Corollary 5.10(1), we have

$$H^0(Z(s_4\tau_2, (4, \underline{j_2})), T_{(s_4\tau_2,(4,\underline{j_2}))}) = H^0(Z(s_4w_1s_3, (4, \underline{l_1})), T_{(s_4w_1s_3,(4,\underline{l_1}))}). \tag{6.4.6}$$

By using LES and [15, Corollary 5.6, p. 778], we have an exact sequence

$$\begin{aligned} 0 \to H^0(s_4w_1s_3, \alpha_3) &\to H^0(Z(s_4w_1s_3, (4, \underline{l_1})), T_{(s_4w_1s_3,(4,\underline{l_1}))}) \\ &\to H^0(Z(s_4w_1, (4, \underline{i_1})), T_{(s_4w_1,(4,\underline{i_1}))}) \to 0 \end{aligned} \tag{6.4.7}$$

of B-modules.

Let $\mu = -(\alpha_2 + 2\alpha_3 + \alpha_4), -(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)$. Since $H^1(s_4w_2, \alpha_2)_\mu \neq 0$, by Corollary 5.2 and (5.1.5), the same weight appears in $H^0(s_4w_1, \alpha_2)$, i.e. $H^0(s_4w_1, \alpha_2)_\mu \neq 0$. This implies $H^0(Z(s_4w_1, (4, \underline{i_1})), T_{(s_4w_1,(4,\underline{i_1}))})_\mu \neq 0$.

Thus by (6.4.7) and Corollary 4.3(1), we have

$$\dim H^0(Z(s_4w_1s_3, (4, \underline{l_1})), T_{(s_4w_1s_3,(4,\underline{l_1}))})_\mu \geq 2. \tag{6.4.8}$$

Since $H^1(s_4w_2, \alpha_2)_\mu \neq 0$, by Corollary 5.2, we have $H^1(s_4w_3, \alpha_2)_\mu = 0$. Since α_3 is a short simple root, by [15, Corollary 5.6, p. 778], we have $H^1(s_4w_2s_3, \alpha_3) = 0$. On the other hand, by using Corollary 5.10(1), we have $H^1(s_4v_3, \alpha_4) = 0$ and $H^1(s_4\tau_3, \alpha_1) = 0$. Thus by using LES and from the above discussion, we have the natural map

$$H^0(Z(s_4w_3, (4, \underline{i_3})), T_{(s_4w_3,(4,\underline{i_3}))})_\mu \longrightarrow H^0(Z(s_4w_2, (4, \underline{i_2})), T_{(s_4w_2,(4,\underline{i_2}))})_\mu$$

is surjective.

By (6.4.1) and the above surjectivity, we have $\dim H^0(Z(s_4w_2, (4, \underline{i_2})), T_{(s_4w_2,(4,\underline{i_2}))})_\mu \leq 1$.

By using LES, we see that $\dim H^0(Z(s_4\tau_2, (4, \underline{j_2})), T_{(s_4\tau_2,(4,\underline{j_2}))})_\mu \leq 2$. Thus by (6.4.6) and (6.4.8), we have $\dim H^0(Z(s_4\tau_2, (4, \underline{j}_2)), T_{(s_4\tau_2,(4,\underline{j}_2))})_\mu = 2$. Therefore, by LES, the natural map $H^0(Z(s_4\tau_2, (4, \underline{j}_2)), T_{(s_4\tau_2,(4,\underline{j}_2))})_\mu \longrightarrow H^1(s_4w_2, \alpha_2)_\mu$ is surjective. Hence by Corollary 5.2(3), the natural map $H^0(Z(s_4\tau_2, (4, \underline{j_2})), T_{(s_4\tau_2,(4,\underline{j_2}))}) \longrightarrow H^1(s_4w_2, \alpha_2)$ is surjective. □

Lemma 6.5.

(1) *We have* $\dim H^0(Z(s_3s_4\tau_3, (3, 4, \underline{j}_3)), T_{(s_3s_4\tau_3,(3,4,\underline{j}_3))})_{-\omega_4} = 2$. *Further, the natural map* $H^0(Z(s_3s_4\tau_3, (3, 4, \underline{j}_3)), T_{(s_3s_4\tau_3,(3,4,\underline{j}_3))}) \longrightarrow H^1(s_3s_4w_3, \alpha_2)$ *is surjective.*

(2) *Let* $\mu = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)$. *Then we have* $\dim H^0(Z(s_3s_4\tau_2, (3, 4, \underline{j}_2)), T_{(s_3s_4\tau_2,(3,4,\underline{j}_2))})_\mu = 2$. *Further, the natural map* $H^0(Z(s_3s_4\tau_2, (3, 4, \underline{j}_2)), T_{(s_3s_4\tau_2,(3,4,\underline{j}_2))}) \longrightarrow H^1(s_3s_4w_2, \alpha_2)$ *is surjective.*

(3) *Let* $\mu = -(\alpha_2 + \alpha_3), -(\alpha_1 + \alpha_2 + \alpha_3)$. *Then we have* $\dim H^0(Z(s_3s_4\tau_1, (3, 4, \underline{j}_1)), T_{(s_3s_4\tau_1,(3,4,\underline{j}_1))})_\mu = 2$. *Further, the natural map* $H^0(Z(s_3s_4\tau_1, (3, 4, \underline{j}_1)), T_{(s_3s_4\tau_1,(3,4,\underline{j}_1))}) \longrightarrow H^1(s_3s_4w_1, \alpha_2)$ *is surjective.*

Proof. The proofs of Lemma 6.5(1), Lemma 6.5(2) and Lemma 6.5(3) are similar to that of Lemma 6.4 by using [15, Corollary 5.6, p. 778], Corollary 5.3 and Corollary 5.10(2) appropriately. □

Lemma 6.6.

(1) *We have* $\dim H^0(Z(s_2s_3s_4\tau_3, (2, 3, 4, \underline{j_3})), T_{(s_2s_3s_4\tau_3,(2,3,4,\underline{j_3}))})_{-\omega_4} = 2$. *Further, the natural map* $H^0(Z(s_2s_3s_4\tau_3, (2, 3, 4, \underline{j_3})), T_{(s_2s_3s_4\tau_3,(2,3,4,\underline{j_3}))}) \longrightarrow H^1(s_2s_3s_4w_3, \alpha_2)$ *is surjective.*

(2) *We have* $\dim H^0(Z(s_2s_3s_4\tau_2, (2, 3, 4, \underline{j_2})), T_{(s_2s_3s_4\tau_2,(2,3,4,\underline{j_2}))})_{-\omega_4+\alpha_4} = 2$. *Further, the natural map* $H^0(Z(s_2s_3s_4\tau_2, (2, 3, 4, \underline{j_2})), T_{(s_2s_3s_4\tau_2,(2,3,4,\underline{j_2}))}) \longrightarrow H^1(s_2s_3s_4w_2, \alpha_2)$ *is surjective.*

(3) *We have* $\dim H^0(Z(s_2s_3s_4\tau_1, (2, 3, 4, \underline{j_1})), T_{(s_2s_3s_4\tau_1,(2,3,4,\underline{j_1}))})_{-(\alpha_1+\alpha_2+\alpha_3)} = 2$. *Further, the natural map* $H^0(Z(s_2s_3s_4\tau_1, (2, 3, 4, \underline{j_1})), T_{(s_2s_3s_4\tau_1,(2,3,4,\underline{j_1}))}) \longrightarrow H^1(s_2s_3s_4w_1, \alpha_2)$ *is surjective.*

Proof. The proofs of Lemma 6.6(1), Lemma 6.6(2) and Lemma 6.6(3) are similar to that of Lemma 6.4 by using [15, Corollary 5.6, p. 778], Corollary 5.4 and Corollary 5.10(3) appropriately. □

Lemma 6.7.

(1) *Let* $\mu = -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), -\omega_4+\alpha_4, -\omega_4$. *Then we have* $\dim H^0(Z(s_4s_3s_4\tau_2, (4,3,4,\underline{j}_2)), T_{(s_4s_3s_4\tau_2,(4,3,4,\underline{j}_2))})_\mu = 2$. *Further, the natural map* $H^0(Z(s_4s_3s_4\tau_2, (4,3,4,\underline{j}_2)), T_{(s_4s_3s_4\tau_2,(4,3,4,\underline{j}_2))}) \longrightarrow H^1(s_4s_3s_4w_2, \alpha_2)$ *is surjective.*
(2) *Let* $\mu = -(\alpha_2+\alpha_3), -(\alpha_2+\alpha_3+\alpha_4), -(\alpha_1+\alpha_2+\alpha_3), -(\alpha_1+\alpha_2+\alpha_3+\alpha_4), -(\alpha_2+2\alpha_3+\alpha_4), -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)$. *Then we have* $\dim H^0(Z(s_4s_3s_4\tau_1, (4,3,4,\underline{j}_1)), T_{(s_4s_3s_4\tau_1,(4,3,4,\underline{j}_1))})_\mu = 2$. *Further, the natural map* $H^0(Z(s_4s_3s_4\tau_1, (4,3,4,\underline{j}_1)), T_{(s_4s_3s_4\tau_1,(4,3,4,\underline{j}_1))}) \longrightarrow H^1(s_4s_3s_4w_1, \alpha_2)$ *is surjective.*

Proof. The proofs of Lemma 6.7(1) and Lemma 6.7(2) are similar to that of Lemma 6.4 by using [15, Corollary 5.6, p. 778], Corollary 5.5 and Corollary 5.10(4) appropriately. □

Lemma 6.8.

(1) *Let* $\mu = -\omega_4+\alpha_4, -\omega_4$. *Then we have* $\dim H^0(Z(s_4s_2s_3s_4\tau_2, (4,2,3,4,\underline{j}_2)), T_{(s_4s_2s_3s_4\tau_2,(4,2,3,4,\underline{j}_2))})_\mu = 2$. *Further, the natural map* $H^0(Z(s_4s_2s_3s_4\tau_2, (4,2,3,4,\underline{j}_2)), T_{(s_4s_2s_3s_4\tau_2,(4,2,3,4,\underline{j}_2))}) \longrightarrow H^1(s_4s_2s_3s_4w_2, \alpha_2)$ *is surjective.*
(2) *Let* $\mu = -(\alpha_1+\alpha_2+\alpha_3), -(\alpha_1+\alpha_2+\alpha_3+\alpha_4), -(\alpha_2+2\alpha_3+\alpha_4), -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4), -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)$. *Then we have* $\dim H^0(Z(s_4s_2s_3s_4\tau_1, (4,3,4,\underline{j}_1)), T_{(s_4s_2s_3s_4\tau_1,(4,2,3,4,\underline{j}_1))})_\mu = 2$. *Further, the natural map* $H^0(Z(s_4s_2s_3s_4\tau_1, (4,2,3,4,\underline{j}_1)), T_{(s_4s_2s_3s_4\tau_1,(4,2,3,4,\underline{j}_1))}) \longrightarrow H^1(s_4s_2s_3s_4w_1, \alpha_2)$ *is surjective.*

Proof. Proofs of Lemma 6.8(1) and Lemma 6.8(2) are similar to that of Lemma 6.4 by using [15, Corollary 5.6, p. 778], Corollary 5.6 and Corollary 5.10(5) appropriately. □

Lemma 6.9. *Let* $\underline{j}'_1 = (4,3,4,2,3,4,\underline{j}_1)$ *and* $\underline{j}' = (4,3,4,2,3,4,1)$.

(1) *Let* $\Lambda = \{-(\alpha_1+\alpha_2+\alpha_3), -(\alpha_1+\alpha_2+\alpha_3+\alpha_4), -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4), -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), -(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)\}$. *Then we have* $\dim H^0(Z(s_4s_3s_4s_2s_3s_4\tau_1, \underline{j}'_1), T_{(s_4s_3s_4s_2s_3s_4\tau_1,\underline{j}'_1)})_\mu = 2$ *for all* $\mu \in \Lambda$. *Further, the natural map* $H^0(Z(s_4s_3s_4s_2s_3s_4\tau_1, \underline{j}'_1), T_{(s_4s_3s_4s_2s_3s_4\tau_1,\underline{j}'_1)}) \longrightarrow H^1(s_4s_3s_4s_2s_3s_4w_1, \alpha_2)$ *is surjective.*
(2) *Let* $\Pi = \{-(\alpha_2+\alpha_3), -(\alpha_2+\alpha_3+\alpha_4), -(\alpha_2+2\alpha_3+\alpha_4)\}$. *Then we have* $\dim H^0(Z(s_4s_3s_4s_2s_3s_4s_1, \underline{j}'), T_{(s_4s_3s_4s_2s_3s_4s_1,\underline{j}')})_\mu = 2$ *for all* $\mu \in \Pi$. *Further, the natural map* $H^0(Z(s_4s_3s_4s_2s_3s_4s_1, \underline{j}'), T_{(s_4s_3s_4s_2s_3s_4s_1,\underline{j}')}) \longrightarrow H^1(s_4s_3s_4s_2s_3s_4s_1s_2, \alpha_2)$ *is surjective.*

Proof. Let $u_1 = s_4s_3s_4s_2s_3s_4\tau_1$ and $u = s_4s_3s_4s_2s_3s_4s_1$. Note that $w_0 = s_4s_3s_4s_2s_3s_4w_3s_3s_1s_2$. Let $\underline{i}'$ be this reduced expression of w_0. By Lemma 4.1(2) and Corollary 5.2(2), we have $H^i(s_4w_4, \alpha_2) = 0$ for $i \geq 0$. Since s_4 commutes with s_1, s_2, we have $H^i(s_4w_4, \alpha_2) = H^i(s_4w_3s_3s_4s_1s_2, \alpha_2) = H^i(s_4w_3s_3s_1s_2, \alpha_2)$ for $i \geq 0$. Thus we have $H^i(s_4w_3s_3s_1s_2, \alpha_2) = 0$ for $i \geq 0$.

Therefore by using SES, we have $H^i(s_4s_3s_4s_2s_3s_4w_3s_3s_1s_2, \alpha_2) = 0$ for $i \geq 0$. Since s_3 commutes with s_1, we have $H^i(s_4s_3s_4s_2s_3s_4w_3s_3s_1, \alpha_1) = H^i(s_4s_3s_4s_2s_3s_4w_3s_1, \alpha_1)$

for $i \geq 0$. $H^i(s_4s_3s_4s_2s_3s_4w_3s_1, \alpha_1) = H^i(s_4s_3s_4s_2s_3s_4[1,4]^3s_2s_1s_2, \alpha_1) = 0$ for $i \geq 0$ (see Lemma 2.3(4)). Thus we have $H^i(s_4s_3s_4s_2s_3s_4w_3s_3s_1, \alpha_1) = 0$ for $i \geq 0$. By Lemma 5.8(4), we have $H^1(s_4s_3s_4s_2s_3s_4w_r, \alpha_2) = 0$ for $r = 2, 3$. Since α_3 is a short simple root, by [15, Corollary 5.6, p. 778], we have $H^1(s_4s_3s_4s_2s_3s_4w_rs_3, \alpha_3) = 0$ for $r = 1, 2, 3$. On the other hand, by using Corollary 5.10(6), we have $H^1(s_4s_3s_4s_2s_3s_4v_r, \alpha_4) = 0$ and $H^1(s_4s_3s_4s_2s_3s_4\tau_r, \alpha_1) = 0$ for $r = 2, 3$.

Thus by using LES and the above discussion, we have the natural map

$$\begin{aligned} &H^0(Z(w_0, \underline{i'}), T_{(w_0,\underline{i'})}) \\ &\quad\to H^0(Z(s_4s_3s_4s_2s_3s_4w_1, (\underline{j_1'}, 2)), T_{(s_4s_3s_4s_2s_3s_4w_1,(\underline{j_1'},2))}) \end{aligned} \tag{6.9.1}$$

which is surjective.

Proof of (1). By using LES repeatedly and Corollary 5.10(6), we have

$$\begin{aligned} &H^0(Z(u_1, \underline{j_1'}), T_{(u_1,\underline{j_1'})}) \\ &\quad= H^0(Z(us_2s_3, (\underline{j'}, 2, 3)), T_{(us_2s_3,(\underline{j'},2,3))}). \end{aligned} \tag{6.9.2}$$

By using LES and [15, Corollary 5.6, p. 778], we have an exact sequence

$$\begin{aligned} &0 \to H^0(us_2s_3, \alpha_3) \to H^0(Z(us_2s_3), T_{(us_2s_3,(\underline{j'},2,3))}) \\ &\quad\to H^0(Z(us_2), T_{(us_2,(\underline{j'},2))}) \to 0 \end{aligned} \tag{6.9.3}$$

of B-modules.

Let $\Lambda_1 = \{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), -(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)\}$. Let $\Lambda_2 = \{-(\alpha_1+\alpha_2+\alpha_3), -(\alpha_1+\alpha_2+\alpha_3+\alpha_4), -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)\}$. By (5.8.1), we have

$$\begin{aligned} H^0(s_3s_4s_2s_3s_4s_1s_2, \alpha_2) = {}&\mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \\ &\oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \\ &\oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)} \\ &\oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \\ &\oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)} \\ &\oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}. \end{aligned}$$

Thus by using SES, we see that $H^0(us_2, \alpha_2)_\mu \neq 0$ for all $\mu \in \Lambda_1$. By using LES and Lemma 5.8(2), we have an exact sequence

$$\begin{aligned} &0 \to H^0(us_2, \alpha_2)_\mu \to H^0(Z(us_2, (\underline{j'}, 2)), T_{(us_2,(\underline{j'},2))})_\mu \\ &\quad\to H^0(Z(u, \underline{j'}), T_{(u,(\underline{j'}))})_\mu \to 0 \end{aligned}$$

for all $\mu \in \Lambda$.

Note that $H^0(u, \alpha_1) = H^0(s_4s_3s_2s_1, \alpha_1)$. Now it is easy to see that $H^0(s_4s_3s_2s_1, \alpha_1)_\mu \neq 0$ for $\mu \in \Lambda_2$. Therefore, we have $H^0(Z(u, \underline{j'}), T_{(u,\underline{j'})})_\mu \neq 0$ for all $\mu \in \Lambda_2$. Thus combining the above discussion, we have

$$H^0(Z(us_2, (\underline{j'}, 2)), T_{(us_2,(\underline{j'},2))})_\mu \neq 0 \tag{6.9.4}$$

for all $\mu \in \Lambda$. Therefore, by using (6.9.3), (6.9.4) and Corollary 4.9(2), we have

$$\dim H^0(Z(us_2s_3, (\underline{j'}, 2, 3)), T_{(us_2s_3,(\underline{j'},2,3))})_\mu \geq 2 \tag{6.9.5}$$

for all $\mu \in \Lambda$. By (6.9.1), we have $H^0(Z(s_4s_3s_4s_2s_3s_4w_1, (\underline{j_1'}, 2)), T_{(s_4s_3s_4s_2s_3s_4w_1,(\underline{j_1'},2))})_\mu \leq 1$ for all $\mu \in \Lambda$.

Therefore by using LES and Lemma 5.8(3), we have $\dim H^0(Z(u_1, \underline{j_1'}), T_{(u_1,\underline{j_1'})})_\mu \leq 2$ for all $\mu \in \Lambda$.

Thus by (6.9.2) and (6.9.5), we have $\dim H^0(Z(u_1, \underline{j_1'}), T_{(u_1,\underline{j_1'})})_\mu = 2$ for all $\mu \in \Lambda$.

By using LES, we have $H^0(Z(u_1, \underline{j_1'}), T_{(u_1,\underline{j_1'})})_\mu \longrightarrow H^1(s_4s_3s_4s_2s_3s_4w_1, \alpha_2)_\mu$ is surjective for all $\mu \in \Lambda$. Hence by Lemma 5.8(3), the natural map $H^0(Z(u_1, \underline{j_1'}), T_{(u_1,\underline{j_1'})}) \longrightarrow H^1(s_4s_3s_4s_2s_3s_4w_1, \alpha_2)$ is surjective.

Proof of (2). It is easy to see that $H^1(u, \alpha_1) = H^1(s_4s_3s_2s_1, \alpha_1) = 0$ and $H^0(u, \alpha_1) = H^0(s_4s_3s_2s_1, \alpha_1)_\mu = 0$ for all $\mu \in \Pi$.

Further, we have $H^i(s_4s_3s_4s_2s_3s_4, \alpha_4) = H^i(s_4s_3s_2s_3s_4s_3, \alpha_3) = 0$ for all $i \geq 0$ (see Lemma 2.3(4)).

From the above disussions and using LES repeatedly, we have

$$\begin{aligned} &H^0(Z(u, \underline{j'}), T_{(u,\underline{j'})})_\mu \\ &\quad = H^0(Z(s_4s_3s_4s_2s_3, (4, 3, 4, 2, 3)), T_{(s_4s_3s_4s_2s_3,(4,3,4,2,3))})_\mu \end{aligned} \tag{6.9.6}$$

for all $\mu \in \Pi$.

By using LES and [15, Corollary 5.6, p. 778], we have an exact sequence

$$\begin{aligned} 0 &\to H^0(s_4s_3s_4s_2s_3, \alpha_3) \to H^0(Z(s_4s_3s_4s_2s_3), T_{(s_4s_3s_4s_2s_3,(4,3,4,2,3))}) \\ &\to H^0(Z(s_4s_3s_4s_2), T_{(s_4s_3s_4s_2,(4,3,4,2))}) \to 0. \end{aligned} \tag{6.9.7}$$

It is easy to see that $H^0(s_4s_3s_4s_2, \alpha_2)_\mu \neq 0$ for all $\mu \in \Pi$. Therefore, we have $H^0(Z(s_4s_3s_4s_2), T_{(s_4s_3s_4s_2,(4,3,4,2))})_\mu \neq 0$ for all $\mu \in \Pi$. Thus from (6.9.7) and Corollary 4.9(1), we have

$$\dim H^0(Z(s_4s_3s_4s_2s_3), T_{(s_4s_3s_4s_2s_3,(4,3,4,2,3))})_\mu \geq 2 \text{ for } \mu \in \Pi. \tag{6.9.8}$$

Since α_3 is a short simple root, by [15, Corollary 5.6, p. 778], we have $H^1(us_2s_3, \alpha_3) = 0$.

By using Corollary 5.10(6), we have

$H^1(s_4s_3s_4s_2s_3s_4\tau_1, \alpha_1) = 0$ and $H^1(s_4s_3s_4s_2s_3s_4v_1, \alpha_4) = 0$.

By Lemma 5.8, we have $H^1(s_4s_3s_4s_2s_3s_4w_1, \alpha_2)_\mu = 0$ for all $\mu \in \Pi$. Thus combining the above discussion, we have the natural map

$$\begin{aligned} &H^0(Z(s_4s_3s_4s_2s_3s_4w_1, (\underline{j_1'}, 2)), T_{(s_4s_3s_4s_2s_3s_4w_1,(\underline{j_1'},2))})_\mu \\ &\qquad \to H^0(Z(us_2, (\underline{j'}, 2)), T_{(us_2,(\underline{j'},2))})_\mu, \end{aligned}$$

is surjective for all $\mu \in \Pi$.

Now, using (6.9.1) and the above surjectivity, we have $H^0(Z(us_2, (\underline{j'}, 2)), T_{(us_2,(\underline{j'},2))})_\mu \leq 1$ for all $\mu \in \Pi$. Further, by Lemma 5.8(2), dim $H^1(us_2, \alpha_2)_\mu = 1$ for all $\mu \in \Pi$.

Therefore, by using LES,

$$0 \longrightarrow H^0(us_2, \alpha_2) \longrightarrow H^0(Z(us_2, (\underline{j'}, 2)), T_{(us_2,(\underline{j'},2))}) \longrightarrow H^0(Z(u, \underline{j'}), T_{(u,\underline{j'})}) \longrightarrow H^1(us_2, \alpha_2) \longrightarrow H^1(Z(us_2, (\underline{j'}, 2)), T_{(us_2,(\underline{j'},2))}) \longrightarrow H^1(Z(u, \underline{j'}), T_{(u,\underline{j'})}) \longrightarrow 0,$$

we have $H^0(Z(u, \underline{j'}), T_{(u,\underline{j'})})_\mu \leq 2$ for all $\mu \in \Pi$.

Therefore by (6.9.6) and (6.9.8), we have dim $H^0(Z(s_4s_3s_4s_2s_3), T_{(s_4s_3s_4s_2s_3,(4,3,4,2,3))})_\mu = 2$ for all $\mu \in \Pi$.

Therefore, $H^0(Z(u, \underline{j'}), T_{(u,\underline{j'})})_\mu \to H^1(us_2, \alpha_2)_\mu$ is surjective for all $\mu \in \Pi$.

Hence by Lemma 5.8(2), the natural map $H^0(Z(u, \underline{j'}), T_{(u,\underline{j'})}) \to H^1(us_2, \alpha_2)$ is surjective. □

7. Main theorem

In this section, we prove the main theorem. Let c be a Coxeter element of W. Then there exists a decreasing sequence $4 \geq a_1 > a_2 > \cdots > a_k = 1$ of positive integers such that $c = [a_1, 4][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$, where for $i \leq j$ denotes $[i, j] = s_is_{i+1} \cdots s_j$.

Theorem 7.1. *$H^j(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) = 0$ for all $j \geq 1$ if and only if $a_1 \neq 3$ or $a_2 \neq 2$.*

Proof. From [13, Proposition 3.1, p. 673], we have $H^j(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) = 0$ for all $j \geq 2$. It is enough to prove the following: $H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) = 0$ if and only if c is of the form $[a_1, 4][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$ with $a_1 \neq 3$ or $a_2 \neq 2$.

Proof of ($\Longrightarrow$): If $a_1 = 3$ and $a_2 = 2$, then $c = s_3s_4s_2s_1$. Let $u = s_3s_4s_2$. Then $c = us_1$. Let $\underline{j} = (3, 4, 2)$ be the sequence corresponding to u. Then using LES, we have

$$0 \to H^0(u, \alpha_2) \to H^0(Z(u, \underline{j}), T_{(u,\underline{j})}) \to H^0(Z(s_3s_4, (3, 4)), T_{(s_3s_4,(3,4))}) \to H^1(u, \alpha_2) \xrightarrow{f} H^1(Z(u, \underline{j}), T_{(u,\underline{j})}) \to H^1(Z(s_3s_4, (3, 4)), T_{(s_3s_4,(3,4))}) \to 0.$$

We see that $H^1(u, \alpha_2) = \mathbb{C}_{\alpha_2+\alpha_3}$, $H^0(s_3, \alpha_3)_{\alpha_2+\alpha_3} = 0$ and $H^0(s_3s_4, \alpha_4)_{\alpha_2+\alpha_3} = 0$.

Therefore by LES, we have $H^0(Z(s_3s_4, (3, 4)), T_{(s_3s_4,(3,4))})_{\alpha_2+\alpha_3} = 0$. Hence f is a non zero homomorphism. Hence $H^1(Z(u, \underline{j}), T_{(u,\underline{j})})) \neq 0$. By Lemma 6.2, the natural homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(u, \underline{j}), T_{(u,\underline{j})})$$

is surjective.

Hence we have $H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \neq 0$.

Proof of ($\Longleftarrow$): Assume that $a_1 \neq 3$ or $a_2 \neq 2$. We prove the result by studying case by case. Note that by using Lemma 2.3(4), we have $H^1(w_0, \alpha_i) = 0$ for $i = 1, 2, 3, 4$. In each of the following cases we use these appropriately.

Case 1: $c = s_1s_2s_3s_4$. Then in this case we have $w_0 = v_6 = [1, 4]^6$. By using LES and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) = H^1(Z(w_5, \underline{i_5}), T_{(w_5,\underline{i_5})}).$$

By using LES, Lemma 5.1, Lemma 5.9 and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(w_5, \underline{i_5}), T_{(w_5,\underline{i_5})}) = H^1(Z(w_4, \underline{i_4}), T_{(w_4,\underline{i_4})}).$$

By using LES and Lemma 6.3(1), we have

$$H^1(Z(w_4, \underline{i_4}), T_{(w_4,\underline{i_4})}) = H^1(Z(\tau_4, \underline{j_4}), T_{(\tau_4,\underline{j_4})}).$$

By using LES, Lemma 5.9 and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(\tau_4, \underline{j_4}), T_{(\tau_4,\underline{j_4})}) = H^1(Z(w_3, \underline{i_3}), T_{(w_3,\underline{i_3})}).$$

By using LES and Lemma 6.3(2), we have

$$H^1(Z(w_3, \underline{i_3}), T_{(w_3,\underline{i_3})}) = H^1(Z(\tau_3, \underline{j_3}), T_{(\tau_3,\underline{j_3})}).$$

By using LES, Lemma 5.9 and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(\tau_3, \underline{j_3}), T_{(\tau_3,\underline{j_3})}) = H^1(Z(w_2, \underline{i_2}), T_{(w_2,\underline{i_2})}).$$

By using LES, Lemma 5.1, Lemma 5.9 and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(w_2, \underline{i_2}), T_{(w_2,\underline{i_2})}) = H^1(Z(w_1, \underline{i_1}), T_{(w_1,\underline{i_1})}).$$

By using LES, Lemma 5.1, Lemma 5.9 and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(w_1, \underline{i_1}), T_{(w_1,\underline{i_1})}) = H^1(Z(s_1s_2, (1, 2)), T_{(s_1s_2,(1,2))}).$$

We see that $H^1(s_1, \alpha_1) = 0$, $H^1(s_1s_2, \alpha_2) = 0$. Thus by using LES, we have $H^1(Z(s_1s_2, (1, 2)), T_{(s_1s_2,(1,2))}) = 0$. Thus by combining all, we have $H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) = 0$.

Case 2: $c = s_4s_1s_2s_3$. Then in this case we have $w_0 = s_4w_5s_3$. By using LES and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(w_0, (4, \underline{l_5})), T_{(w_0,(4,\underline{l_5}))}) = H^1(Z(s_4w_5, (4, \underline{i_5})), T_{(s_4w_5,(4,\underline{i_5}))}).$$

By using LES, Corollary 5.2, Corollary 5.10(1) and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(s_4w_5, (4, \underline{i_5})), T_{(s_4w_5,(4,\underline{i_5}))}) = H^1(Z(s_4w_4, (4, \underline{i_4})), T_{(s_4w_4,(4,\underline{i_4}))}).$$

By using LES, Corollary 5.2, Corollary 5.10(1) and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(s_4w_4, (4, \underline{i_4})), T_{(s_4w_4,(4,\underline{i_4}))}) = H^1(Z(s_4w_3, (4, \underline{i_3})), T_{(s_4w_3,(4,\underline{i_3}))}).$$

By using LES and Lemma 6.4(1), we have

$$H^1(Z(s_4w_3, (4, \underline{i_3})), T_{(s_4w_3,(4,\underline{i_3}))}) = H^1(Z(s_4\tau_3, (4, \underline{j_3})), T_{(s_4\tau_3,(4,\underline{j_3}))}).$$

By using LES, Corollary 5.10(1) and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(s_4\tau_3, (4, \underline{j_3})), T_{(s_4\tau_3,(4,\underline{j_3}))}) = H^1(Z(s_4w_2, (4, \underline{i_2})), T_{(s_4w_2,(4,\underline{i_2}))}).$$

By using LES and Lemma 6.4(2), we have

$$H^1(Z(s_4w_2, (4, \underline{i_2})), T_{(s_4w_2,(4,\underline{i_2}))}) = H^1(Z(s_4\tau_2, (4, \underline{j_2})), T_{(s_4\tau_2,(4,\underline{j_2}))}).$$

By using LES, Corollary 5.10(1) and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(s_4\tau_2, (4, \underline{j_2})), T_{(s_4\tau_2,(4,\underline{j_2}))}) = H^1(Z(s_4w_1, (4, \underline{i_1})), T_{(s_4w_1,(4,\underline{i_1}))}).$$

By using LES, Corollary 5.2, Corollary 5.10(1) and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(s_4w_1, (4, \underline{i_1})), T_{(s_4w_1,(4,\underline{i_1}))}) = H^1(Z(s_4s_1s_2, (4, 1, 2)), T_{(s_4s_1s_2,(4,1,2))}).$$

We see that $H^1(s_4s_1, \alpha_1) = 0$, $H^1(s_4s_1s_2, \alpha_2) = 0$. Thus by using LES, we have

$$H^1(Z(s_4s_1s_2, (4, 1, 2)), T_{(s_4s_1s_2,(4,1,2))}) = 0.$$

Thus combining all we have $H^1(Z(w_0, (4, \underline{l_5})), T_{(w_0,(4,\underline{l_5}))}) = 0$.

Case 3: $c = s_3s_4s_1s_2$. Then we have $w_0 = s_3s_4w_5$. By using LES, Corollary 5.3, Corollary 5.10(2) and [15, Corollary 5.6, p. 778], we have

$$\begin{aligned}&H^1(Z(s_3s_4w_5, (3, 4, \underline{i_5})), T_{(s_3s_4w_5,(3,4,\underline{i_5}))})\\ &\quad = H^1(Z(s_3s_4w_4, (3, 4, \underline{i_4})), T_{(s_3s_4w_4,(3,4,\underline{i_4}))}).\end{aligned}$$

By using LES, Corollary 5.3, Corollary 5.10(2) and [15, Corollary 5.6, p. 778], we have

$$\begin{aligned}&H^1(Z(s_3s_4w_4, (3, 4, \underline{i_4})), T_{(s_3s_4w_4,(3,4,\underline{i_4}))})\\ &\quad = H^1(Z(s_3s_4w_3, (3, 4, \underline{i_3})), T_{(s_3s_4w_3,(3,4,\underline{i_3}))}).\end{aligned}$$

By using LES and Lemma 6.5(1), we have

$$\begin{aligned}&H^1(Z(s_3s_4w_3, (3, 4, \underline{i_3})), T_{(s_3s_4w_3,(3,4,\underline{i_3}))})\\ &\quad = H^1(Z(s_3s_4\tau_3, (3, 4, \underline{j_3})), T_{(s_3s_4\tau_3,(3,4,\underline{j_3}))}).\end{aligned}$$

By using LES, Corollary 5.10(2) and [15, Corollary 5.6, p. 778] we have

$$\begin{aligned}&H^1(Z(s_3s_4\tau_3, (3, 4, \underline{j_3})), T_{(s_3s_4\tau_3,(3,4,\underline{j_3}))})\\ &\quad = H^1(Z(s_3s_4w_2, (3, 4, \underline{i_2})), T_{(s_3s_4w_2,(3,4,\underline{i_2}))}).\end{aligned}$$

By using LES and Lemma 6.5(2), we have

$$\begin{aligned}&H^1(Z(s_3s_4w_2,(3,4,\underline{i_2})),T_{(s_3s_4w_2,(3,4,\underline{i_2}))})\\&\quad=H^1(Z(s_3s_4\tau_2,(3,4,\underline{j_2})),T_{(s_3s_4\tau_2,(3,4,\underline{j_2}))}).\end{aligned}$$

By using LES, Corollary 5.10(2) and [15, Corollary 5.6, p. 778], we have

$$\begin{aligned}&H^1(Z(s_3s_4\tau_2,(3,4,\underline{j_2})),T_{(s_3s_4\tau_2,(3,4,\underline{j_2}))})\\&\quad=H^1(Z(s_3s_4w_1,(3,4,\underline{i_1})),T_{(s_3s_4w_1,(3,4,\underline{i_1}))}).\end{aligned}$$

By using LES and Lemma 6.5(3), we have

$$\begin{aligned}&H^1(Z(s_3s_4w_1,(3,4,\underline{i_1})),T_{(s_3s_4w_1,(3,4,\underline{i_1}))})\\&\quad=H^1(Z(s_3s_4\tau_1,(3,4,\underline{j_1})),T_{(s_3s_4\tau_1,(3,4,\underline{j_1}))}).\end{aligned}$$

By using LES, Corollary 5.10(2) and [15, Corollary 5.6, p. 778], we have

$$\begin{aligned}&H^1(Z(s_3s_4\tau_1,(3,4,\underline{j_1})),T_{(s_3s_4\tau_1,(3,4,\underline{j_1}))})\\&\quad=H^1(Z(s_3s_4s_1s_2,(3,4,1,2)),T_{(s_3s_4s_1s_2,(3,4,1,2))}).\end{aligned}$$

We see that $H^1(s_3s_4,\alpha_4)=0$ (see [15, Corollary 5.6, p. 778]), $H^1(s_3s_4s_1,\alpha_1)=0$, $H^1(s_3s_4s_1s_2,\alpha_2)=0$. Thus by using LES, we have

$$H^1(Z(s_3s_4s_1s_2,(3,4,1,2)),T_{(s_3s_4s_1s_2,(3,4,1,2))})=0.$$

Thus combining all we have $H^1(Z(w_0,(3,4,\underline{i_5})),T_{(w_0,(3,4,\underline{i_5}))})=0$.

Case 4: $c=s_2s_3s_4s_1$. Then $w_0=s_2s_3s_4\tau_5$. Let $t_1=s_2s_3s_4$. By using LES, Corollary 5.10(3) and [15, Corollary 5.6, p. 778], we have

$$\begin{aligned}&H^1(Z(w_0,(2,3,4,\underline{j_5})),T_{(w_0,(2,3,4,\underline{j_5}))})\\&\quad=H^1(Z(t_1w_4,(2,3,4,\underline{i_4})),T_{(t_1w_4,(2,3,4,\underline{i_4}))}).\end{aligned}$$

By using LES, Corollary 5.4, Corollary 5.10(3) and [15, Corollary 5.6, p. 778], we have

$$\begin{aligned}&H^1(Z(t_1w_4,(2,3,4,\underline{i_4})),T_{(t_1w_4,(2,3,4,\underline{i_4}))})\\&\quad=H^1(Z(t_1w_3,(2,3,4,\underline{i_3})),T_{(t_1w_3,(2,3,4,\underline{i_3}))}).\end{aligned}$$

By using LES and Lemma 6.6(1), we have

$$\begin{aligned}&H^1(Z(t_1w_3,(2,3,4,\underline{i_3})),T_{(t_1w_3,(2,3,4,\underline{i_3}))})\\&\quad=H^1(Z(t_1\tau_3,(2,3,4,\underline{j_3})),T_{(t_1\tau_3,(2,3,4,\underline{j_3}))}).\end{aligned}$$

By using LES, Corollary 5.10(3) and [15, Corollary 5.6, p. 778], we have

$$\begin{aligned}&H^1(Z(t_1\tau_3,(2,3,4,\underline{j_3})),T_{(t_1\tau_3,(2,3,4,\underline{j_3}))})\\&\quad=H^1(Z(t_1w_2,(2,3,4,\underline{i_2})),T_{(t_1w_2,(2,3,4,\underline{i_2}))}).\end{aligned}$$

By using LES and Lemma 6.6(2), we have

$$H^1(Z(t_1w_2, (2,3,4,\underline{i_2})), T_{(t_1w_2,(2,3,4,\underline{i_2}))})$$
$$= H^1(Z(t_1\tau_2, (2,3,4,\underline{j_2})), T_{(t_1\tau_2,(2,3,4,\underline{j_2}))}).$$

By using LES, Corollary 5.10(3) and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(t_1\tau_2, (2,3,4,\underline{j_2})), T_{(t_1\tau_2,(2,3,4,\underline{j_2}))})$$
$$= H^1(Z(t_1w_1, (2,3,4,\underline{i_1})), T_{(t_1w_1,(2,3,4,\underline{i_1}))}).$$

By using LES and Lemma 6.6(3), we have

$$H^1(Z(t_1w_1, (2,3,4,\underline{i_1})), T_{(t_1w_1,(2,3,4,\underline{i_1}))})$$
$$= H^1(Z(t_1\tau_1, (2,3,4,\underline{j_1})), T_{(t_1\tau_1,(2,3,4,\underline{j_1}))}).$$

By using LES, Corollary 5.10(3) and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(t_1\tau_1, (2,3,4,\underline{j_1})), T_{(t_1\tau_1,(2,3,4,\underline{j_1}))})$$
$$= H^1(Z(t_1s_1s_2, (2,3,4,1,2)), T_{(t_1s_1s_2,(2,3,4,1,2))}).$$

It is easy to see that $H^1(t_1s_1, \alpha_1) = H^1(s_2s_1, \alpha_1) = 0$. We see that $H^1(s_2s_3, \alpha_3) = 0$, $H^1(t_1, \alpha_4) = 0$ by [15, Corollary 5.6, p. 778]. $H^1(t_1s_1s_2, \alpha_2) = 0$ by Corollay 5.4.

Thus by using LES, we have

$$H^1(Z(t_1s_1s_2, (2,3,4,1,2)), T_{(t_1s_1s_2,(2,3,4,1,2))}) = 0.$$

Thus combining all we have $H^1(Z(w_0, (2,3,4,\underline{j_5})), T_{(w_0,(2,3,4,\underline{j_5}))}) = 0$.

Case 5: $c = s_4s_3s_1s_2$. In this case we have $w_0 = s_4s_3s_4w_4s_3s_1s_2$. Let $t_2 = s_4s_3s_4$. Since s_3 commutes with s_1, we have $H^i(t_2w_4s_3s_1, \alpha_1) = H^i(t_2w_3s_3s_4s_2s_1s_2, \alpha_1) = 0$ for $i \geq 0$ (see Lemma 2.3(4)).

Thus by using LES, Corollary 5.10(4) and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(w_0, (4,3,4,\underline{i_4},3,1,2)), T_{(w_0,(4,3,4,\underline{i_4},3,1,2))})$$
$$= H^1(Z(t_2w_4, (4,3,4,\underline{i_4})), T_{(t_2w_4,(4,3,4,\underline{i_4}))}).$$

By using LES, Corollary 5.5, Corollary 5.10(4) and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(t_2w_4, (4,3,4,\underline{i_4})), T_{(t_2w_4,(4,3,4,\underline{i_4}))})$$
$$= H^1(Z(t_2w_3, (4,3,4,\underline{i_3})), T_{(t_2w_3,(4,3,4,\underline{i_3}))}).$$

By using LES, Corollary 5.5, Corollary 5.10(4) and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(t_2w_3, (4,3,4,\underline{i_3})), T_{(t_2w_3,(4,3,4,\underline{i_3}))})$$
$$= H^1(Z(t_2w_2, (4,3,4,\underline{i_2})), T_{(t_2w_2,(4,3,4,\underline{i_2}))}).$$

By using LES and Lemma 6.7(1), we have

$$H^1(Z(t_2w_2,(4,3,4,\underline{i_2})),T_{(t_2w_2,(4,3,4,\underline{i_2}))})$$
$$=H^1(Z(t_2\tau_2,(4,3,4,\underline{j_2})),T_{(t_2\tau_2,(4,3,4,\underline{j_2}))}).$$

By using LES, Corollary 5.10(4) and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(t_2\tau_2,(4,3,4,\underline{j_2})),T_{(t_1\tau_2,(4,3,4,\underline{j_2}))})$$
$$=H^1(Z(t_2w_1,(4,3,4,\underline{i_1})),T_{(t_2w_1,(4,3,4,\underline{i_1}))}).$$

By using LES and Lemma 6.7(2), we have

$$H^1(Z(t_2w_1,(4,3,4,\underline{i_1})),T_{(t_2w_1,(4,3,4,\underline{i_1}))})$$
$$=H^1(Z(t_2\tau_1,(4,3,4,\underline{j_1})),T_{(t_2\tau_1,(4,3,4,\underline{j_1}))}).$$

By using LES, Corollary 5.10(4) and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(t_2\tau_1,(4,3,4,\underline{j_1})),T_{(t_2\tau_1,(4,3,4,\underline{j_1}))})$$
$$=H^1(Z(t_2s_1s_2,(4,3,4,1,2)),T_{(t_2s_1s_2,(4,3,4,1,2))}).$$

We see that $H^1(s_4s_3,\alpha_3)=0$, $H^1(t_2,\alpha_4)=0$ by [15, Corollary 5.6, p. 778]. Since s_3, s_4 commutes with s_1 we have $H^1(t_2s_1,\alpha_1)=H^1(s_1,\alpha_1)=0$. By Corollary 5.5, we have $H^1(t_2s_1s_2,\alpha_2)=0$.

Thus by using LES we have $H^1(Z(t_2s_1s_2,(4,3,4,1,2)),T_{(t_2s_1s_2,(4,3,4,1,2))})=0$. Thus combining all we have $H^1(Z(w_0,(4,3,4,\underline{i_4},3,1,2)),T_{(w_0,(4,3,4,\underline{i_4},3,1,2))})=0$.

Case 6: $c=s_4s_2s_3s_1$. In this case we have $w_0=s_4s_2s_3s_4w_4s_3s_1$. Let $t_3=s_4s_2s_3s_4$. By using LES and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(w_0,(4,2,3,4,\underline{i_4},3,1)),T_{(w_0,(4,2,3,4,\underline{i_4},3,1))})$$
$$=H^1(Z(t_3w_4,(4,2,3,4,\underline{i_4})),T_{(t_3w_4,(4,2,3,4,\underline{i_4}))}).$$

By using LES, Corollary 5.6, Corollary 5.10(5) and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(t_3w_4,(4,2,3,4,\underline{i_4})),T_{(t_3w_4,(4,2,3,4,\underline{i_4}))})$$
$$=H^1(Z(t_3w_3,(4,2,3,4,\underline{i_3})),T_{(t_3w_3,(4,2,3,4,\underline{i_3}))}).$$

By using LES, Corollary 5.6, Corollary 5.10(5) and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(t_3w_3,(4,2,3,4,\underline{i_3})),T_{(t_3w_3,(4,2,3,4,\underline{i_3})})$$
$$=H^1(Z(t_3w_2,(4,2,3,4,\underline{i_2})),T_{(t_3w_2,(4,2,3,4,\underline{i_2}))}).$$

By using LES and Lemma 6.8(1), we have

$$H^1(Z(t_3w_2,(4,2,3,4,\underline{i_2})),T_{(t_3w_2,(4,2,3,4,\underline{i_2}))})$$
$$=H^1(Z(t_3\tau_2,(4,2,3,4,\underline{j_2})),T_{(t_3\tau_2,(4,2,3,4,\underline{j_2}))}).$$

By using Corollary 5.10(5) and [15, Corollary 5.6, p. 778], we have

$$\begin{aligned}&H^1(Z(t_3\tau_2, (4, 2, 3, 4, \underline{j_2})), T_{(t_3\tau_2,(4,2,3,4,\underline{j_2}))})\\&\quad = H^1(Z(t_3w_1, (4, 2, 3, 4, \underline{i_1})), T_{(t_3w_1,(4,2,3,4,\underline{i_1}))}).\end{aligned}$$

By using LES and Lemma 6.8(2), we have

$$\begin{aligned}&H^1(Z(t_3w_1, (4, 2, 3, 4, \underline{i_1})), T_{(t_3w_1,(4,2,3,4,\underline{i_1}))})\\&\quad = H^1(Z(t_3\tau_1, (4, 2, 3, 4, \underline{j_1})), T_{(t_3\tau_1,(4,2,3,4,\underline{j_1}))}).\end{aligned}$$

By using LES, Corollary 5.10(5) and [15, Corollary 5.6, p. 778], we have

$$\begin{aligned}&H^1(Z(t_3\tau_1, (4, 2, 3, 4, \underline{j_1})), T_{(t_3\tau_1,(4,2,3,4,\underline{j_1}))})\\&\quad = H^1(Z(t_3s_1s_2, (4, 2, 3, 4, 1, 2)), T_{(t_3s_1s_2,(4,2,3,4,1,2))}).\end{aligned}$$

We see that $H^1(s_4s_2, \alpha_2) = 0$, $H^1(t_3s_1, \alpha_1) = 0$. Further, by using [15, Corollary 5.6, p. 778], we have $H^1(s_4s_2s_3, \alpha_3) = 0$, $H^1(t_3, \alpha_4) = 0$. By Corollary 5.6, we have $H^1(t_3s_1s_2, \alpha_2) = 0$.

Therefore by using LES we have $H^1(Z(t_3s_1s_2, (4, 2, 3, 4, 1, 2)), T_{(t_3s_1s_2,(4,2,3,4,1,2))}) = 0$. Thus combining all we have $H^1(Z(w_0, (4, 2, 3, 4, \underline{i_4}, 3, 1)), T_{(w_0,(4,2,3,4,\underline{i_4},3,1))}) = 0$.

Case 7: $c = s_4s_3s_2s_1$. In this case we have $w_0 = s_4s_3s_4s_2s_3s_4w_3s_3s_1s_2s_1$. Let $t_4 = s_4s_3s_4s_2s_3s_4$. Let $\underline{i'} = (4, 3, 4, 2, 3, 4, \underline{l_3}, 1, 2, 1)$. Recall that $l_r = (i_r, 3)$. Let $\underline{i'_r} = (4, 3, 4, 2, 3, 4, \underline{i_r})$ be the reduced expressions of t_4w_r for $r = 1, 2, 3$. Let $\underline{j'_r} = (4, 3, 4, 2, 3, 4, \underline{j_r})$ be the reduced expressions of $t_4\tau_r$ for $r = 1, 2, 3$. Let $\underline{j'} = (4, 3, 4, 2, 3, 4, 1)$ be the reduced expression of t_4s_1.

By Lemma 4.1(2) and Corollary 5.2(2), we have $H^i(s_4w_4, \alpha_2) = 0$ for $i \geq 0$. Since s_4 commutes with s_1, s_2, we have $H^i(s_4w_4, \alpha_4) = H^i(s_4w_3s_3s_4s_1s_2, \alpha_2) = H^i(s_4w_3s_3s_1s_2, \alpha_2)$ for $i \geq 0$. Thus we have $H^i(s_4w_3s_3s_1s_2, \alpha_2) = 0$ for $i \geq 0$. Therefore, by using SES, we have $H^i(t_4w_3s_3s_1s_2, \alpha_2) = 0$ for $i \geq 0$. Since s_3 commutes with s_1 we have $H^i(t_4w_3s_3s_1, \alpha_1) = H^i(t_4w_3s_1, \alpha_1)$ for $i \geq 0$. $H^i(t_4w_3s_1, \alpha_1) = H^i(t_4[1, 4]^3s_2s_1s_2, \alpha_1) = 0$ for $i \geq 0$ (see Lemma 2.3(4)). Thus we have $H^i(t_4w_3s_3s_1, \alpha_1) = 0$ for $i \geq 0$. Thus by using LES, above discussion, and [15, Corollary 5.6, p.778] we have

$$\begin{aligned}&H^1(Z(w_0, \underline{i'}), T_{(w_0,\underline{i'})})\\&\quad = H^1(Z(t_4w_3, \underline{i'_3}), T_{(t_4w_3,\underline{i'_3})}).\end{aligned}$$

By using LES, Lemma 5.8(4), Corollary 5.10(6) and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(t_4w_3, \underline{i'_3}), T_{(t_4w_3,\underline{i'_3})}) = H^1(Z(t_4w_2, \underline{i'_2}), T_{(t_4w_2,\underline{i'_2})}).$$

By using LES, Lemma 5.8(4), Corollary 5.10(6) and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(t_4w_2, \underline{i'_2}), T_{(t_4w_2,\underline{i'_2})}) = H^1(Z(t_4w_1, \underline{i'_1}), T_{(t_4w_1,\underline{i'_1})}).$$

By using LES and Lemma 6.9(1), we have

$$H^1(Z(t_4w_1, \underline{i_1'}), T_{(t_1w_1, \underline{i_1'})}) = H^1(Z(t_4\tau_1, \underline{j_1'}), T_{(t_4\tau_1, \underline{j_1'})}).$$

By using LES, Corollary 5.10(6) and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(t_4\tau_1, \underline{j_1'}), T_{(t_4\tau_1, \underline{j_1'})}) = H^1(Z(t_4s_1s_2, (\underline{j'}, 2)), T_{(t_4s_1s_2, (\underline{j'}, 2))}).$$

By using LES and Lemma 6.9(2), we have

$$H^1(Z(t_4s_1s_2, (\underline{j'}, 2)), T_{(t_4s_1s_2, (\underline{j'}, 2))}) = H^1(Z(t_4s_1, \underline{j'}), T_{(t_4s_1, \underline{j'})}).$$

By [15, Corollary 5.6, p. 778], we see that $H^1(s_4s_3, \alpha_3) = 0$, $H^1(s_4s_3s_4, \alpha_4) = 0$, $H^1(s_4s_3s_4s_2s_3, \alpha_3) = 0$ and $H^1(t_4, \alpha_4) = 0$. By Lemma 5.8(1), we have $H^1(s_4s_3s_4s_2, \alpha_2) = 0$.

Since s_3, s_4 commute with s_1, we have $H^1(t_4s_1, \alpha_1) = H^1(s_4s_3s_2s_1, \alpha_1)$. It is easy to see by using SES that $H^1(s_4s_3s_2s_1, \alpha_1) = 0$. Thus we have $H^1(t_4s_1, \alpha_1) = 0$. Therefore, by using LES, we have $H^1(Z(t_4s_1, \underline{j'}), T_{(t_4s_1, \underline{j'})}) = 0$. Thus combining all we have $H^1(Z(w_0, \underline{i'}), T_{(w_0, \underline{i'})}) = 0$. □

COROLLARY 7.2

Let c be a Coxeter element such that c is of the form $[a_1, 4][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$ with $a_1 \neq 3$ or $a_2 \neq 2$ and $a_k = 1$. Let $(w_0, \underline{i})$ be a reduced expression of w_0 in terms of c as in Theorem 7.1. Then, $Z(w_0, \underline{i})$ has no deformations.

Proof. By Theorem 7.1 and [13, Proposition 3.1, p. 673], we have $H^i(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) = 0$ for all $i > 0$. Hence, by [11, Proposition 6.2.10, p. 272], we see that $Z(w_0, \underline{i})$ has no deformations. □

8. Non rigidity for G_2

Now onwards, we will assume that G is of type G_2. Note that the longest element w_0 of the Weyl group W of G is equal to $-id$. We recall the following proposition from [17, Proposition 1.3, p. 858]. We use Proposition 3.1 and the notation as in [17] to deduce the following.

Lemma 8.1. *Let $c \in W$ be a Coxeter element. Then we have*

(1) $w_0 = c^3$.
(2) *For any sequence $\underline{i} = (\underline{i}^1, \underline{i}^2, \underline{i}^3)$ of reduced expressions of c; the sequence $\underline{i} = (\underline{i}^1, \underline{i}^2, \underline{i}^3)$ is a reduced expression of w_0.*

Proof.

Proof of (1). Let $\eta : S \longrightarrow S$ be the involution of S defined by $i \to i^*$, where i^* is given by $\omega_{i^*} = -w_0(\omega_i)$. Since G is of type G_2, $w_0 = -id$, therefore, we have $i = i^*$ for every

i. Let h be the Coxeter number. By [17, Proposition 1.7], we have $h(i, c) + h(i^*, c) = h$. Since $h = 2|R^+|/2$ (see [9, Proposition 3.18]) and $i = i^*$, we have $h(i, c) = h/2 = 3$, as $|R^+| = 6$. By Proposition 3.1, we have $c^6(\omega_i) = -\omega_i$ for all $i = 1, 2$. Since $\{\omega_i : i = 1, 2\}$ forms an $\mathbb{R}$-basis of $X(T) \otimes \mathbb{R}$, it follows that $c^3 = -id$. Hence, we have $w_0 = c^3$. The assertion (2) follows from the fact that $l(c) = 2$ and $l(w_0) = |R^+| = 6$ (see [7, p. 66, Table 1]). □

Let c be a Coxeter element of W. Then $c = s_1s_2$ or $c = s_2s_1$. Then from Lemma 8.1, we have $w_0 = s_1s_2s_1s_2s_1s_2$ or $w_0 = s_2s_1s_2s_1s_2s_1$ according as $c = s_1s_2$ or $c = s_2s_1$.

Let $\underline{i_1}$ (repectively, $\underline{i_2}$) be the the reduced expression of $w_0 = s_1s_2s_1s_2s_1s_2$ (respectively, $w_0 = s_2s_1s_2s_1s_2s_1$). Then we have as follows.

Theorem 8.2. *$H^1(Z(w_0, \underline{i_r}), T_{(w_0, \underline{i_r})}) \neq 0$ for $r = 1, 2$.*

Proof. Let $c = s_1s_2$. Let $\underline{i} = (1, 2)$ be the sequence corresponding to c. Then using LES, we have

$$0 \longrightarrow H^0(c, \alpha_2) \longrightarrow H^0(Z(c, \underline{i}), T_{(c,\underline{i})}) \longrightarrow H^0(s_1, \alpha_1) \longrightarrow$$
$$H^1(c, \alpha_2) \xrightarrow{g} H^1(Z(c, \underline{i}), T_{(c,\underline{i})}) \longrightarrow 0.$$

By using SES, we see that $H^1(s_1s_2, \alpha_2) = \mathbb{C}_{\alpha_2+\alpha_1} \oplus \mathbb{C}_{\alpha_2+2\alpha_1}$. Now $H^0(s_1, \alpha_1)_{\alpha_2+\alpha_1} = 0$. Hence g is a non zero homomorphism. Hence $H^1(Z(c, \underline{i}), T_{(c,\underline{i})})) \neq 0$. By Lemma 6.2, the natural homomorphism

$$H^1(Z(w_0, \underline{i_1}), T_{(w_0,\underline{i_1})}) \longrightarrow H^1(Z(c, \underline{i}), T_{(c,\underline{i})})$$

is surjective.

Hence we have $H^1(Z(w_0, \underline{i_1}), T_{(w_0,\underline{i_1})}) \neq 0$.

Let $c = s_2s_1$, $u = s_2s_1s_2$. Let $\underline{j} = (2, 1, 2)$ be the sequence corresponding to u. Then using LES, we have

$$0 \longrightarrow H^0(u, \alpha_2) \longrightarrow H^0(Z(u, \underline{j}), T_{(u,\underline{j})})$$
$$\longrightarrow H^0(Z(s_2s_1, (2, 1)), T_{(s_2s_1,(2,1))}) \longrightarrow$$
$$H^1(u, \alpha_2) \xrightarrow{h} H^1(Z(u, \underline{j}), T_{(u,\underline{j})}) \longrightarrow H^1(Z(s_2s_1, (2, 1)), T_{(s_2s_1,(2,1))}) \to 0.$$

We see that $H^1(u, \alpha_2) = \mathbb{C}_{\alpha_1} \oplus \mathbb{C}_{\alpha_2+\alpha_1} \oplus \mathbb{C}_{\alpha_2+2\alpha_1}$, $H^0(s_1, \alpha_1)_{\alpha_2+\alpha_1} = 0$ and $H^0(s_2s_1, \alpha_1)_{\alpha_2+\alpha_1} = 0$.

Therefore, by LES, we have $H^0(Z(s_2s_1, (2, 1)), T_{(s_2s_1,(2,1))})_{\alpha_2+\alpha_1} = 0$. Hence h is a non zero homomorphism. Hence $H^1(Z(u, \underline{j}), T_{(u,\underline{j})})) \neq 0$. By Lemma 6.2, the natural homomorphism

$$H^1(Z(w_0, \underline{i_2}), T_{(w_0,\underline{i_2})}) \longrightarrow H^1(Z(u, \underline{j}), T_{(u,\underline{j})})$$

is surjective.

Hence we have $H^1(Z(w_0, \underline{i_2}), T_{(w_0,\underline{i_2})}) \neq 0$. □

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