Rigidity of Bott–Samelson–Demazure–Hansen variety for \( F_4 \) and \( G_2 \)

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Abstract. Let \( G \) be a simple algebraic group of adjoint type over \( \mathbb{C} \), whose root system is of type \( F_4 \). Let \( T \) be a maximal torus of \( G \) and \( B \) be a Borel subgroup of \( G \) containing \( T \). Let \( w \) be an element of the Weyl group \( W \) and \( X(w) \) be the Schubert variety in the flag variety \( G/B \) corresponding to \( w \). Let \( Z(w, i) \) be the Bott–Samelson–Demazure–Hansen variety (the desingularization of \( X(w) \)) corresponding to a reduced expression \( i \) of \( w \). In this article, we study the cohomology modules of the tangent bundle on \( Z(w_0, i) \), where \( w_0 \) is the longest element of the Weyl group \( W \). We describe all the reduced expressions of \( w_0 \) in terms of a Coxeter element such that \( Z(w_0, i) \) is rigid (see Theorem 7.1). Further, if \( G \) is of type \( G_2 \), there is no reduced expression \( i \) of \( w_0 \) for which \( Z(w_0, i) \) is rigid (see Theorem 8.2).

Keywords. Bott–Samelson–Demazure–Hansen variety; coexeter element; tangent bundle.

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1. Introduction

Let \( G \) be a simple algebraic group of adjoint type over the field \( \mathbb{C} \) of complex numbers. We fix a maximal torus \( T \) of \( G \) and let \( W = N_G(T)/T \) denote the Weyl group of \( G \) with respect to \( T \). We denote by \( R \) the set of roots of \( G \) with respect to \( T \) and by \( R^+ \subset R \) a set of positive roots. Let \( B^+ \) be the Borel subgroup of \( G \) containing \( T \) with respect to \( R^+ \). Let \( w_0 \) denote the longest element of the Weyl group \( W \). Let \( B \) be the Borel subgroup of \( G \) opposite to \( B^+ \) determined by \( T \), i.e. \( B = n_{w_0}B^+n_{w_0}^{-1} \), where \( n_{w_0} \) is a representative of \( w_0 \) in \( N_G(T) \). Note that the roots of \( B \) is the set \( R^- := -R^+ \) of negative roots. We use the notation \( \beta < 0 \) for \( \beta \in R^- \). Let \( S = \{\alpha_1, \ldots, \alpha_n\} \) denote the set of all simple roots in \( R^+, \) where \( n \) is the rank of \( G \). For simplicity of notation, the simple reflection \( s_{\alpha_i} \) corresponding to a simple root \( \alpha_i \) is denoted by \( s_i \). For \( w \in W \), let \( X(w) := BwB/B \) denote the Schubert variety in \( G/B \) corresponding to \( w \). Given a reduced expression \( w = s_{i_1}s_{i_2}\cdots s_{i_r} \) of \( w \), with the corresponding tuple \( i := (i_1, \ldots, i_r) \), we denote by \( Z(w, i) \) the desingularization of the Schubert variety \( X(w) \), which is now known as the Bott–Samelson–Demazure–Hansen variety. This was first introduced by Bott and Samelson in a differential geometric and
topological context (see [2]). Demazure in [4] and Hansen in [6] independently adapted the construction in algebro-geometric situation, which explains the reason for the name. For the sake of simplicity, we will denote any Bott–Samelson–Demazure–Hansen variety by BSDH variety.

The construction of the BSDH variety $Z(w, \bar{i})$ depends on the choice of the reduced expression $i$ of $w$. In [13], the automorphism groups of these varieties were studied. There, the following vanishing results of the tangent bundle $T_{Z(w, \bar{i})}$ on $Z(w, \bar{i})$ were proved (see [13, section 3]):

1. $H^j(Z(w, \bar{i}), T_{Z(w, \bar{i})}) = 0$ for all $j \geq 2$.
2. If $G$ is simply laced, then $H^j(Z(w, \bar{i}), T_{Z(w, \bar{i})}) = 0$ for all $j \geq 1$.

As a consequence, it follows that the BSDH varieties are rigid for simply laced groups and their deformations are unobstructed, in general (see [5, section 3]). The above vanishing result is independent of the choice of the reduced expression $i$ of $w$. While computing the first cohomology module $H^1(Z(w, \bar{i}), T_{Z(w, \bar{i})})$ for non simply laced group, we observed that this cohomology module very much depended on the choice of a reduced expression $\bar{i}$ of $w$.

It is a natural question to ask for which reduced expressions $i$ of $w$, the cohomology module $H^1(Z(w, i), T_{Z(w, i)})$ does vanish? In [14], a partial answer is given to this question for $w = w_0$ when $G = PSp(2n, \mathbb{C})$. In [16], a partial answer is given to this question for $w = w_0$ when $G = PSO(2n + 1, \mathbb{C})$. In this article, we give partial answers to this question for $w = w_0$ when $G$ is of type $F_4, G_2$.

Recall that a Coxeter element is an element of the Weyl group having a reduced expression of the form $s_i s_i s_j \cdots s_n$ such that $i_j \neq i_l$ whenever $j \neq l$ (see [10, p. 56, section 4.4]). Note that for any Coxeter element $c \in W$, the Weyl group corresponding to the root system of type $F_4$ (respectively, $G_2$), there is a decreasing sequence of integers $4 \geq a_1 > a_2 > \cdots > a_k = 1$ (respectively, $2 \geq a_1 > \cdots > a_k = 1$) such that $c = \prod_{j=1}^{k} [a_j, a_{j-1} - 1]$, where $a_0 := 5$ (respectively, $a_0 := 3$), $[i, j] := s_i s_{i+1} \cdots s_j$ for $i \leq j$.

In this paper, we prove the following theorems.

**Theorem 1.1.** Assume that $G$ is of type $F_4$. Then $H^j(Z(w_0, i), T_{Z(w_0, i)}) = 0$ for all $j \geq 1$ if and only if $a_1 \neq 3$ or $a_2 \neq 2$.

**Theorem 1.2.** Assume that $G$ is of type $G_2$. Then $H^1(Z(w_0, \bar{i}, T_{Z(w_0, \bar{i})}) = 0$ for $r = 1, 2$.

By the above results, we conclude that if $G$ is of type $F_4$ (respectively, $G_2$) and $\bar{i} = (i^1, i^2, i^3, i^4, i^5, i^6)$ (respectively, $\bar{i} = (\bar{i}^1, \bar{i}^2)$) is a reduced expression of $w_0$ as above, then the BSDH variety $Z(w_0, i)$ is rigid (respectively, non rigid).

The organization of the paper is as follows: In section 2, we recall some preliminaries on BSDH varieties. We deal with $G$ which is of type $F_4$ in sections 3, 4, 5, 6 and 7. In section 3, we prove $H^1(w, \alpha_j) = 0$ for $j = 1, 2$ and $w \in W$. In section 4 (respectively, section 5), we compute the weight spaces of $H^0$ (respectively, $H^1$) of the relative tangent bundle of BSDH varieties associated to some elements of the Weyl group. In section 6, we prove surjectivity results of some maps from the cohomology module of the tangent bundle on BSDH variety to cohomology module of the relative tangent bundle on BSDH variety. In section 7, we prove Theorem 1.1 using the results from the previous sections. In section 8, we prove Theorem 1.2.
2. Preliminaries

In this section, we set up some notations and preliminaries. We refer to [3, 7, 8, 12] for preliminaries in algebraic groups and Lie algebras.

Let $G$ be a simple algebraic group of adjoint type over $\mathbb{C}$ and $T$ be a maximal torus of $G$. Let $W = N_G(T)/T$ denote the Weyl group of $G$ with respect to $T$ and we denote the set of roots of $G$ with respect to $T$ by $\Phi$. Let $B^+$ be a Borel subgroup of $G$ containing $T$. Let $B$ be the Borel subgroup of $G$ opposite to $B^+$ determined by $T$, i.e., $B = n_0B^+n_0^{-1}$, where $n_0$ is a representative in $N_G(T)$ of the longest element $w_0$ of $W$. Let $R^+ \subset R$ be the set of positive roots of $G$ with respect to the Borel subgroup $B^+$. Note that the set of roots of $B$ is equal to the set $R^- := -R^+$ of negative roots.

Let $S = \{\alpha_1, \ldots, \alpha_n\}$ denote the set of simple roots in $R^+$ for $\beta \in R^+$, we also use the notation $\beta > 0$. The simple reflection in $W$ corresponding to $\alpha_i$ is denoted by $s_i$.

Let $\mathfrak{g}$ be the Lie algebra of $G$. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of $T$ and $\mathfrak{b} \subset \mathfrak{g}$ be the Lie algebra of $B$. Let $X(T)$ denote the group of all characters of $T$. We have $X(T) \otimes \mathbb{R} = \text{Hom}_{\mathbb{R}}(\mathfrak{h}, \mathbb{R})$, the dual of the real form of $\mathfrak{h}$. The positive definite $W$-invariant form on $\text{Hom}_{\mathbb{R}}(\mathfrak{h}, \mathbb{R})$ induced by the Killing form of $\mathfrak{g}$ is denoted by $\langle , \rangle$. We use the notation $\langle , \rangle$ to denote $\langle \mu, \alpha \rangle = 2\frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle}$, for every $\mu \in X(T) \otimes \mathbb{R}$ and $\alpha \in R$. We denote by $X(T)^+$ the set of dominant characters of $T$ with respect to $B^+$. Let $\rho$ denote the half sum of all the positive roots of $G$ with respect to $T$ and $B^+$. For any simple root $\alpha$, we denote the fundamental weight corresponding to $\alpha$ by $\omega_\alpha$. For $1 \leq i \leq n$, let $h(\alpha_i) \in \mathfrak{h}$ be the fundamental co-weight corresponding to $\alpha_i$, i.e., $\alpha_i(h(\alpha_j)) = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta.

For a simple root $\alpha \in S$, we denote by $n_\alpha$, a representative of $s_\alpha$ in $N_G(T)$, and $P_\alpha$ the minimal parabolic subgroup of $G$ containing $B$ and $n_\alpha$. We recall that the BSDH variety corresponds to a reduced expression $i$ of $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ defined by

$$Z(w, i) = \frac{P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}}}{B \times B \times \cdots \times B},$$

where the action of $B \times B \times \cdots \times B$ on $P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}}$ is given by $(p_1, p_2, \ldots, p_r)(b_1, b_2, \ldots, b_r) = (p_1 \cdot b_1, b_1^{-1} \cdot p_2 \cdot b_2, \ldots, b_{r-1}^{-1} \cdot p_r \cdot b_r)$, $p_j \in P_{\alpha_{i_j}}$, $b_j \in B$ for $1 \leq j \leq r$, and $i = (i_1, i_2, \ldots, i_r)$ (see [4, Definition 1, p. 73], [3, Definition 2.2.1, p. 64]).

We note that for each reduced expression $i$ of $w$, $Z(w, i)$ is a smooth projective variety. We denote by $\phi_w$, the natural birational surjective morphism from $Z(w, i)$ to $X(w)$.

Let $f_r : Z(w, i) \longrightarrow Z(w_{s_{i_1}}^{-1}i')$ denote the map induced by the projection $P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}} \longrightarrow P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r-1}}$, where $i' = (i_1, i_2, \ldots, i_{r-1})$. Then we observe that $f_r$ is a $P_{\alpha_{i_r}}/B \simeq \mathbb{P}^1$-fibration.

For a $B$-module $V$, let $\mathcal{L}(w, V)$ denote the restriction of the associated homogeneous vector bundle on $G/B$ to $X(w)$. By abuse of notation, we denote the pull back of $\mathcal{L}(w, V)$ via $\phi_w$ to $Z(w, i)$ also by $\mathcal{L}(w, V)$, when there is no confusion. Since for any $B$-module $V$ the vector bundle $\mathcal{L}(w, V)$ on $Z(w, i)$ is the pull back of the homogeneous vector bundle from $X(w)$, we conclude that the cohomology modules

$$H^j(Z(w, i), \mathcal{L}(w, V)) \simeq H^j(X(w), \mathcal{L}(w, V))$$

for all $j \geq 0$ (see [3, Theorem 3.3.4(b)]), are independent of the choice of the reduced expression $i$. Hence we denote $H^j(Z(w, i), \mathcal{L}(w, V))$ by $H^j(w, V)$. In particular, if $\lambda$ is a character of $B$, then we denote the cohomology modules $H^j(Z(w, i), \mathcal{L}_\lambda)$ by $H^j(w, \lambda)$. 
We recall the following short exact sequence of $B$-modules from [13], which we call it SES.

If $l(w) = l(s_\gamma w) + 1$, $\gamma \in S$, then we have

1. $H^0(w, V) \simeq H^0(s_\gamma, H^0(s_\gamma w, V))$.
2. $0 \to H^1(s_\gamma, H^0(s_\gamma w, V)) \to H^1(w, V) \to H^0(s_\gamma, H^1(s_\gamma w, V)) \to 0$.

Let $\alpha$ be a simple root and $\lambda \in X(T)$ be such that $\langle \lambda, \alpha \rangle \geq 0$. Let $\mathbb{C}_\lambda$ denote the one-dimensional $B$-module associated to $\lambda$. Here, we recall the following result due to Demazure [5, p. 271] on the short exact sequence of $B$-modules.

Lemma 2.1. Let $\alpha$ be a simple root and $\lambda \in X(T)$ be such that $\langle \lambda, \alpha \rangle \geq 0$. Let $ev : H^0(s_\alpha, \lambda) \to \mathbb{C}_\lambda$ be the evaluation map. Then we have

1. If $\langle \lambda, \alpha \rangle = 0$, then $H^0(s_\alpha, \lambda) \simeq \mathbb{C}_\lambda$.
2. If $\langle \lambda, \alpha \rangle \geq 1$, then $\mathbb{C}_{s_\alpha(\lambda)} \hookrightarrow H^0(s_\alpha, \lambda)$, and there is a short exact sequence of $B$-modules:

$$0 \to H^0(s_\alpha, \lambda - \alpha) \to H^0(s_\alpha, \lambda)/\mathbb{C}_{s_\alpha(\lambda)} \to \mathbb{C}_\lambda \to 0.$$  

Furthermore, $H^0(s_\alpha, \lambda - \alpha) = 0$ when $\langle \lambda, \alpha \rangle = 1$.

3. Let $n = \langle \lambda, \alpha \rangle$. As a $B$-module, $H^0(s_\alpha, \lambda)$ has a composition series $0 \subseteq V_0 \subseteq V_{n-1} \subseteq \cdots \subseteq V_0 = H^0(s_\alpha, \lambda)$ such that $V_i/V_{i+1} \simeq \mathbb{C}_{\lambda-i\alpha}$ for $i = 0, 1, \ldots, n-1$ and $V_n = \mathbb{C}_{s_\alpha(\lambda)}$.

We define the dot action by $w \cdot \lambda = w(\lambda + \rho) - \rho$, where $\rho$ is the half sum of positive roots. As a consequence of exact sequences of Lemma 2.1, we can prove the following.

Let $w \in W$, $\alpha$ be a simple root and set $v = ws_\alpha$.

Lemma 2.2. If $l(w) = l(v) + 1$, then we have

1. If $\langle \lambda, \alpha \rangle \geq 0$, then $H^j(w, \lambda) = H^j(v, H^0(s_\alpha, \lambda))$ for all $j \geq 0$.
2. If $\langle \lambda, \alpha \rangle \geq 0$, then $H^j(w, \lambda) = H^{j+1}(w, s_\alpha \cdot \lambda)$ for all $j \geq 0$.
3. If $\langle \lambda, \alpha \rangle \leq -2$, then $H^{j+1}(w, \lambda) = H^j(w, s_\alpha \cdot \lambda)$ for all $j \geq 0$.
4. If $\langle \lambda, \alpha \rangle = -1$, then $H^j(w, \lambda)$ vanishes for every $j \geq 0$.

The following consequence of Lemma 2.2 will be used to compute the cohomology modules in this paper. Now onwards, we will denote the Levi subgroup of $P_\alpha (\alpha \in S)$ containing $T$ by $L_\alpha$ and the subgroup $L_\alpha \cap B$ by $B_\alpha$. Let $\pi : \tilde{G} \to G$ be the universal cover. Let $\tilde{L}_\alpha$ (respectively, $\tilde{B}_\alpha$) be the inverse image of $L_\alpha$ (respectively, $B_\alpha$).

Lemma 2.3. Let $V$ be an irreducible $L_\alpha$-module. Let $\lambda$ be a character of $B_\alpha$. Then we have

1. As $L_\alpha$-modules, $H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) \simeq V \otimes H^j(L_\alpha/B_\alpha, \mathbb{C}_\lambda)$.
2. If $\langle \lambda, \alpha \rangle \geq 0$, then $H^0(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda)$ is isomorphic as an $L_\alpha$-module to the tensor product of $V$ and $H^0(L_\alpha/B_\alpha, \mathbb{C}_\lambda)$. Further, we have $H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0$ for every $j \geq 1$.
3. If $\langle \lambda, \alpha \rangle \leq -2$, then $H^0(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0$, and $H^1(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda)$ is isomorphic to the tensor product of $V$ and $H^0(L_\alpha/B_\alpha, \mathbb{C}_{s_\alpha \cdot \lambda})$. 

(4) If $\langle \lambda, \alpha \rangle = -1$, then $H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0$ for every $j \geq 0$.

**Proof.** Let us prove (1). By [12, Proposition 4.8, p. 53, I] and [12, Proposition 5.12, p. 77, I], for all $j \geq 0$, we have the following isomorphism of $L_\alpha$-modules:

$$H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) \cong V \otimes H^j(L_\alpha/B_\alpha, \mathbb{C}_\lambda).$$

The proofs of (2), (3) and (4) follow from Lemma 2.2 by taking $w = s_\alpha$ and the fact that $L_\alpha/B_\alpha \cong P_\alpha/B_\alpha$.

Recall the structure of indecomposable $B_\alpha$-modules and $\tilde{B}_\alpha$-modules (see [1, Corollary 9.1, p. 130]).

**Lemma 2.4.**

(1) Any finite dimensional indecomposable $\tilde{B}_\alpha$-module $V$ is isomorphic to $V' \otimes \mathbb{C}_\lambda$ for some irreducible representation $V'$ of $\tilde{L}_\alpha$ and for some character $\lambda$ of $\tilde{B}_\alpha$.  

(2) Any finite dimensional indecomposable $B_\alpha$-module $V$ is isomorphic to $V' \otimes \mathbb{C}_\lambda$ for some irreducible representation $V'$ of $\tilde{L}_\alpha$ and for some character $\lambda$ of $\tilde{B}_\alpha$.

### 3. Reduced expressions

Now onwards, we will assume that $G$ is of type $F_4$. Note that the longest element $w_0$ of the Weyl group $W$ of $G$ is equal to $-id$. We recall the following Proposition from [17, Proposition 1.3, p. 858].

**PROPOSITION 3.1**

Let $c \in W$ be a Coxeter element, $\omega_i$ be the fundamental weight corresponding to the simple root $\alpha_i$. Then there exists a least positive integer $h(i, c)$ such that $c^{h(i, c)}(\omega_i) = w_0(\omega_i)$.

Now we can deduce the following:

**Lemma 3.2.** Let $c \in W$ be a Coxeter element. Then we have

1. $w_0 = c^6$.
2. For any sequence $\underline{i}_c = (i^1, i^2, \ldots, i^6)$ of reduced expressions of $c$, the sequence $\underline{i} = (i^1, i^2, \ldots, i^6)$ is a reduced expression of $w_0$.

**Proof.** Let us prove (1). Let $\eta : S \longrightarrow S$ be the involution of $S$ defined by $i \rightarrow i^*$, where $i^*$ is given by $\omega_{i^*} = -w_0(\omega_i)$. Since $G$ is of type $F_4$, $w_0 = -id$, and hence $\omega_{i^*} = \omega_i$ for every $i$. Therefore, we have $i = i^*$ for every $i$. Let $h$ be the Coxeter number. By [17, Proposition 1.7], we have $h(i, c) + h(i^*, c) = h$. Since $h = 2|R^+/4$ (see [9, Proposition 3.18]) and $i = i^*$, we have $h(i, c) = h/2 = 6$ as $|R^+| = 24$. By Proposition 3.1, we have $c^6(\omega_i) = -\omega_i$ for all $1 \leq i \leq 4$. Since $\{\omega_i : 1 \leq i \leq 4\}$ forms an $\mathbb{R}$-basis of $X(T) \otimes \mathbb{R}$, it follows that $c^6 = -id$. Hence, we have $w_0 = c^6$. The assertion (2) follows from the fact that $l(c) = 4$ and $l(w_0) = |R^+| = 24$ (see [7, p. 66, Table 1]).
Lemma 3.3. Let \( v \in W \) and \( \alpha \in S \). Then \( H^1(s_j, H^0(v, \alpha)) = 0 \) for \( j = 1, 2 \).

Proof. If \( H^1(s_j, H^0(v, \alpha)) \neq 0 \), then there exists an indecomposable \( \tilde{L}_{\alpha_j} \)-summand \( V \) of \( H^0(v, \alpha) \) such that \( H^1(s_j, V) \neq 0 \). By Lemma 2.4, we have \( V \cong V' \otimes \mathbb{C}_{\lambda} \) for some character \( \lambda \) of \( \tilde{B}_{\alpha_j} \) and for some irreducible \( \tilde{L}_{\alpha_j} \)-module \( V' \). Since \( H^1(s_j, V) \neq 0 \), from Lemma 2.3(3), we have \( \langle \lambda, \alpha_j \rangle \leq -2 \). If \( \alpha \) is a short root, then \( H^1(w, \alpha) = 0 \) for all \( w \in W \) (see [15, Corollary 5.6, p. 778]). Hence we may assume that \( \alpha \) is a long root. Then there exists \( w \in W \) such that \( w(\alpha) = \alpha_0 \). Thus \( H^0(v, \alpha) \subseteq H^0(vw, \alpha_0) \). Again, since \( \alpha_0 \) is the highest long root, \( H^0(w_0, \alpha_0) = g \twoheadrightarrow H^0(vw, \alpha_0) \) is surjective. Let \( \mu' \) be the lowest weight of \( V \). Then by the above argument, \( \mu' \) is a root. Therefore we have \( \mu' = \mu_1 + \lambda \), where \( \mu_1 \) is the lowest weight of \( V' \). Hence, we have \( \langle \mu', \alpha_j \rangle \leq -2 \). Since \( \alpha_j \) is a long root and \( \mu' \) is a root, we have \( \langle \mu', \alpha_j \rangle = -1, 0, 1 \). This is a contradiction. Thus we have \( H^1(s_j, H^0(v, \alpha)) \mu = 0 \). \( \square \)

4. Cohomology modules \( H^0(w, \alpha_i) \)

Let \( w_r = (s_1 s_2 s_3 s_4)^r s_1 s_2 \) for \( 1 \leq r \leq 5 \). In this section, we compute various cohomology modules \( H^0(w, \alpha_i) \) for some elements \( w \in W \) and \( i = 2, 3 \).

Lemma 4.1.

1. \( H^0(w_3, \alpha_2) = 0 \).
2. \( H^0(w_r, \alpha_2) = 0 \) for \( r = 4, 5 \).

Proof. We have \( w_3 = [1, 4]^3 \). By using SES, we have

\[
H^0(s_1 s_2, \alpha_2) = \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{-(\alpha_1 + \alpha_2)}.
\]

Since \( \langle \alpha_2, \alpha_4 \rangle = 0 \), by using SES, we have

\[
H^0(s_4 s_1 s_2, \alpha_2) = H^0(s_1 s_2, \alpha_2).
\]

Since \( \langle -\alpha_2, \alpha_3 \rangle = 2 \) and \( \langle -(\alpha_1 + \alpha_2), \alpha_3 \rangle = 2 \), by using SES, we have

\[
H^0(s_3 s_4 s_1 s_2, \alpha_2) = \mathbb{C}h(\alpha_2) \oplus (\mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{-(\alpha_2 + \alpha_3)} \oplus \mathbb{C}_{-(\alpha_2 + 2\alpha_3)})
\]

\[
\quad \oplus (\mathbb{C}_{-(\alpha_1 + \alpha_2)} \oplus \mathbb{C}_{-(\alpha_1 + 2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1 + \alpha_2 + 2\alpha_3)}).
\]

Since \( \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} \) is an indecomposable two-dimensional \( \tilde{B}_{\alpha_2} \)-module, by Lemma 2.4, we have \( \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} = V \otimes \mathbb{C}_{-\alpha_2} \), where \( V \) is the standard two-dimensional irreducible \( \tilde{L}_{\alpha_2} \)-module.

Thus by Lemma 2.3(4), we have \( H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2}) = 0 \).

Since \( \langle -\alpha_2 + \alpha_3, \alpha_2 \rangle = -1 \), \( \langle -(\alpha_1 + \alpha_2), \alpha_2 \rangle = -1 \), by Lemma 2.2(4), we have \( H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_2 + \alpha_3)}) = 0 \) and \( H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_1 + \alpha_2)}) = 0 \).

Since \( \langle -\alpha_2 + 2\alpha_3, \alpha_2 \rangle = 0 \), \( \langle -(\alpha_1 + \alpha_2 + \alpha_3), \alpha_2 \rangle = 0 \), by Lemma 2.3(2), we have

\[
H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_2 + 2\alpha_3)}) = \mathbb{C}_{-(\alpha_2 + 2\alpha_3)}.
\]
and

\[ H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, C_{-(\alpha_1+\alpha_2+\alpha_3)}) = C_{-(\alpha_1+\alpha_2+\alpha_3)}. \]

Since \((-\langle \alpha_1+\alpha_2+2\alpha_3, \alpha_2 \rangle) = 1\), we have

\[ H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, C_{-(\alpha_1+\alpha_2+2\alpha_3)}) = C_{-(\alpha_1+\alpha_2+2\alpha_3)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3)}. \]

Thus we have

\[ H^0(s_2s_3s_4s_1s_2, \alpha_2) = C_{-(\alpha_2+2\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3)} \]

\[ \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3)}. \]  \hspace{1cm} (4.1.2)

Since \((-\langle \alpha_1+\alpha_2+\alpha_3, \alpha_1 \rangle) = -1\), by using Lemma 2.3(4), we have

\[ H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, C_{-(\alpha_1+\alpha_2+\alpha_3)}) = 0 \]

and

\[ H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, C_{-(\alpha_1+\alpha_2+2\alpha_3)}) = 0. \]

Since \(C_{-(\alpha_2+2\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3)}\) the standard two-dimensional irreducible \(\tilde{L}_{\alpha_1}\)-module, by using Lemma 2.3(2), we have

\[ H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, C_{-(\alpha_1+2\alpha_2+2\alpha_3)}) = C_{-(\alpha_1+2\alpha_2+2\alpha_3)}. \]

Since \((-\langle \alpha_2+2\alpha_3, \alpha_1 \rangle) = 1\), by using Lemma 2.3(2), we have

\[ H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, C_{-(\alpha_2+2\alpha_3)}) = C_{-(\alpha_2+2\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3)}. \]

Therefore, we have

\[ H^0(w_1, \alpha_2) = C_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus C_{-(\alpha_2+2\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3)}. \]  \hspace{1cm} (4.1.3)

By using SES, we have

\[ H^0(w_3, \alpha_2) = H^0([1, 4]^2, H^0(w_1, \alpha_2)). \]

Note that the computations of the module \(H^0([1, 4]^2, H^0(w_1, \alpha_2))\) is independent of the choice of a reduced expression of \([1, 4]^2\). We consider the reduced expression \(s_2s_1s_2s_3s_2s_3s_4s_1s_2\), of \([1, 4]^2\) to compute \(H^0([1, 4]^2, H^0(w_1, \alpha_2))\).

Since \(-\langle \alpha_2+2\alpha_3, \alpha_3 \rangle = -2\), \(-\langle \alpha_1+\alpha_2+2\alpha_3, \alpha_3 \rangle = -2\), by using Lemma 2.3(3), we have

\[ H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, C_{-(\alpha_2+2\alpha_3)}) = 0 \]

and

\[ H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, C_{-(\alpha_1+\alpha_2+2\alpha_3)}) = 0. \]
Since $\langle - (\alpha_1 + 2\alpha_2 + 2\alpha_3), \alpha_3 \rangle = 0$, by using Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, C_{-(\alpha_1+2\alpha_2+2\alpha_3)}) = C_{-(\alpha_1+2\alpha_2+2\alpha_3)}.$$ 

Thus, from the above discussion, we have

$$H^0(s_3w_1, \alpha_2) = C_{-(\alpha_1+2\alpha_2+2\alpha_3)}.$$ 

Since $\langle - (\alpha_1 + 2\alpha_2 + 2\alpha_3), \alpha_4 \rangle = 2$, by using SES and Lemma 2.3(2), we have

$$H^0(s_4s_3w_1, \alpha_2) = C_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)}.$$ 

Since $\langle - (\alpha_1 + 2\alpha_2 + 2\alpha_3), \alpha_3 \rangle = 0$, $\langle - (\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3 \rangle = 1$, $\langle - (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_3 \rangle = 2$, by using Lemma 2.3(2), we have

$$H^0(s_3s_4s_3w_1, \alpha_2) = C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+4\alpha_4)}.$$ 

Since $\langle - (\alpha_1 + 2\alpha_2 + 2\alpha_3), \alpha_2 \rangle = -1$, $\langle - (\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_2 \rangle = -1$, $\langle - (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_2 \rangle = -1$, $\langle - (\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_2 \rangle = 0$, $\langle - (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_2 \rangle = 0$, $\langle - (\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4), \alpha_2 \rangle = 1$, by using Lemma 2.3(2) and Lemma 2.3(4), we have

$$H^0(s_2s_3s_4s_3w_1, \alpha_2) = C_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4+\alpha_3)} \oplus C_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}.$$ 

Since $C_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}$ is a two-dimensional indecomposable \( \tilde{B}_{\alpha_3} \)-module, by Lemma 2.4(1), we have $C_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} = V \otimes C_{-\alpha_3}$, where \( V \) is the standard two-dimensional irreducible \( \tilde{L}_{\alpha_3} \)-module.

Thus by Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, C_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}) = 0.$$ 

Since $\langle - (\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_3 \rangle = -1$ and $\langle - (\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_3 \rangle = 0$, by Lemma 2.3(2) and Lemma 2.3(4), we have $H^0(s_3s_2s_3s_4s_3w_1, \alpha_2) = C_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}$. 

Since $\langle - (\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_2 \rangle = -1$, by Lemma 2.3(4), we have $H^0(s_2s_3s_2s_3s_4s_3w_1, \alpha_2) = 0$.

Thus, by using SES and Lemma 2.3(2), we have $H^0(s_1s_2s_3s_2s_3s_4s_3w_1, \alpha_2) = 0$. Again, by using SES and Lemma 2.3(2), we have $H^0(w_3, \alpha_2) = H^0(1, [4])^2, H^0(w_1, \alpha_2) = 0$.

Proof of (2) follows from (1).

Recall that \( \omega_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 \). From now onwards, we replace \( \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 \) by \( \omega_4 \).

\textbf{Lemma 4.2.}

(1) $H^0(w_2s_3, \alpha_3) = C_{-\omega_4 + \alpha_4}$.
(2) $H^0(w_3^3 , \alpha_3) = \mathbb{C}_{-\alpha_3}$.

**Proof.**

**Proof of (1).** Using SES, we have $H^0(s_3^3 , \alpha_3) = \mathbb{C}_{-\alpha_3} \oplus ch(\alpha_3) \oplus \mathbb{C}_{\alpha_3}$. Since $\langle \alpha_3 , \alpha_2 \rangle = -1$, by using SES and Lemma 2.3, we have

$$H^0(s_2^3 s_3 , \alpha_3) = ch(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}_{-(\alpha_3+\alpha_2)}.$$  

Further, since $\langle \alpha_3 , \alpha_1 \rangle$ = 0 and $\langle - (\alpha_3 + \alpha_2) , \alpha_1 \rangle = 1$, by using SES and Lemma 2.3, we have

$$H^0(s_1 s_2^3 s_3 , \alpha_3) = ch(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$  

Note that the computations of the module $H^0([1, 4]^2 , H^0(s_2^3 s_3 , \alpha_3))$ is independent of the choice of a reduced expression of $[1, 4]^2$. We consider the reduced expression $s_1 s_2^3 s_1 s_3 s_2^3 s_3 s_4 s_5$ of $[1, 4]^2$ to compute $H^0((1, 4]^2 , H^0(s_1 s_2^3 s_3 , \alpha_3))$.

Since $ch(\alpha_3) \oplus \mathbb{C}_{-\alpha_3}$ is a two-dimensional $\tilde{B}_{\alpha_3}$-module, by Lemma 2.4(1), we have

$$ch(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} = V \otimes \mathbb{C}_{-\alpha_3},$$

where $V$ is the standard two-dimensional irreducible $\tilde{L}_{\alpha_3}$-module.

Thus by using Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_3} / \tilde{B}_{\alpha_3} , ch(\alpha_3) \oplus \mathbb{C}_{-\alpha_3}) = 0.$$  

Since $\langle - (\alpha_2 + \alpha_3) , \alpha_3 \rangle = 0 , \langle - (\alpha_1 + \alpha_2 + \alpha_3) , \alpha_3 \rangle = 0$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_3} / \tilde{B}_{\alpha_3} , \mathbb{C}_{-(\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_2+\alpha_3)}$$

and

$$H^0(\tilde{L}_{\alpha_3} / \tilde{B}_{\alpha_3} , \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$  

Thus, from the above discussion, we have

$$H^0(s_3 s_1 s_2^3 s_3 , \alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$  

Since $\langle - (\alpha_2 + \alpha_3) , \alpha_4 \rangle$ = 1 and $\langle - (\alpha_1 + \alpha_2 + \alpha_3) , \alpha_4 \rangle$ = 1, by using Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4} / \tilde{B}_{\alpha_4} , \mathbb{C}_{-(\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}$$

and

$$H^0(\tilde{L}_{\alpha_4} / \tilde{B}_{\alpha_4} , \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.$$  

Thus, from the above discussion, we have

$$H^0(s_4 s_3 s_1 s_2^3 s_3 , \alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.$$
Since $(-\alpha_2 + \alpha_3), \alpha_3) = 0$ and $(-\alpha_1 + \alpha_2 + \alpha_3), \alpha_3) = 0$, by using Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, C_{-(\alpha_2+\alpha_3)}) = C_{-(\alpha_2+\alpha_3)}$$

and

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, C_{-(\alpha_1+\alpha_2+\alpha_3)}) = C_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Since $(-\alpha_2 + \alpha_3 + \alpha_4), \alpha_3) = 1$ and $(-\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4), \alpha_3) = 1$, by using Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, C_{-(\alpha_2+\alpha_3+\alpha_4)}) = C_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus C_{-(\alpha_2+2\alpha_3+\alpha_4)}$$

and

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, C_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}) = C_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.$$

Thus, from the above discussion, we have

$$H^0(s_3s_4s_3s_1s_2s_3, \alpha_3) = C_{-(\alpha_2+\alpha_3)} \oplus C_{-(\alpha_2+\alpha_3+\alpha_4)}$$

$$\oplus C_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+\alpha_3)}$$

$$\oplus C_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.$$

Since $(-\alpha_2 + \alpha_3), \alpha_2) = -1, (-\alpha_2 + \alpha_3 + \alpha_4), \alpha_2) = -1$, by using Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, C_{-(\alpha_2+\alpha_3)}) = 0$$

and

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, C_{-(\alpha_2+\alpha_3+\alpha_4)}) = 0.$$

Since $(-\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_2) = 0, (-\alpha_1 + \alpha_2 + \alpha_3), \alpha_2) = 0, (-\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4), \alpha_2) = 0$ and $(-\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), \alpha_2) = 1$, by using Lemma 2.3(2), we have

$$H^0(s_2s_3s_4s_3s_1s_2s_3, \alpha_3) = C_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+\alpha_3)}$$

$$\oplus C_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$$

$$\oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Since $C_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$ is the standard two-dimensional irreducible $\tilde{L}_{\alpha_3}$-module, $(-\alpha_1 + \alpha_2 + \alpha_3), \alpha_3) = 0, (-\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3) = 1$ and $(-\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3) = -1$, by using similar arguments as above and using Lemma 2.3(2) and Lemma 2.3(4), we have

$$H^0(s_3s_2s_3s_4s_3s_1s_2s_3, \alpha_3) = C_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$$

$$\oplus C_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$$

$$\oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$
Since $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_1 \rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_1 \rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + \alpha_3), \alpha_1 \rangle = -1$, $\langle -(\alpha_1 + \alpha_2 + 3\alpha_3 + \alpha_4), \alpha_1 \rangle = -1$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_1 \rangle = -1$, by using similar arguments as above and using Lemma 2.3(2) and Lemma 2.3(4), we have

$$H^0(s_1s_3s_2s_3s_4s_3s_1s_2s_3, \alpha_3) = C_{-(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)}.$$ 

Since $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_2 \rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_2 \rangle = -1$, by using Lemma 2.3(2) and Lemma 2.3(4), we have

$$H^0(s_2s_1s_3s_2s_3s_4s_3s_1s_2s_3, \alpha_3) = C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)}.$$ 

Since $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_1 \rangle = 0$, by using Lemma 2.3(2), we have

$$H^0(s_1s_2s_1s_3s_2s_3s_4s_3s_1s_2s_3, \alpha_3) = C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)}.$$ 

Thus we have

$$H^0(w_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)} = \mathbb{C}_{-\omega_4 + \alpha_4}.$$ 

**Proof of (2).** By the proof of (1), we have

$$H^0(w_2s_3, \alpha_3) = \mathbb{C}_{-\omega_4 + \alpha_4}.$$ 

Since $\langle -\omega_4 + \alpha_4, \alpha_4 \rangle = 1$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-\omega_4 + \alpha_4}) = \mathbb{C}_{-\omega_4 + \alpha_4} \oplus \mathbb{C}_{-\omega_4}.$$ 

Therefore, we have

$$H^0(s_4w_2s_3, \alpha_3) = \mathbb{C}_{-\omega_4 + \alpha_4} \oplus \mathbb{C}_{-\omega_4}.$$ 

Since $\langle -\omega_4 + \alpha_4, \alpha_3 \rangle = -1$, by Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-\omega_4 + \alpha_4}) = 0.$$ 

Since $\langle -\omega_4, \alpha_3 \rangle = 0$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-\omega_4}) = \mathbb{C}_{-\omega_4}.$$ 

Thus from the above discussion, we have

$$H^0(s_3s_4w_2s_3, \alpha_3) = \mathbb{C}_{-\omega_4}.$$ 

Since $\alpha_1, \alpha_2$ are orthogonal to $\omega_4$, by Lemma 2.3(2), we have

$$H^0(w_3s_3, \alpha_3) = \mathbb{C}_{-\omega_4}.$$ 

□
COROLLARY 4.3

(1) \( H^0(s_4w_1s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \).

(2) \( H^0(s_4w_2s_3, \alpha_3) = \mathbb{C}_{-\omega_4} \oplus \mathbb{C}_{-\omega_4+\alpha_4} \).

(3) \( H^0(s_4w_r s_3, \alpha_3) = 0 \) for \( r = 3, 4, 5 \).

Proof.

Proof of (1). We have

\[ H^0(s_1s_2s_3, \alpha_3) = \mathcal{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}. \]

Since \( (-\alpha_3, \alpha_4) = 1, (-\alpha_2+\alpha_3, \alpha_4) = 1 \) and \( (-\alpha_1+\alpha_2+\alpha_3, \alpha_4) = 1 \), by using SES and Lemma 2.3(2), we have

\[ H^0(s_4s_1s_2s_3, \alpha_3) = \mathcal{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}_{-(\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3)} \]

\[ \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}. \]

Since \( \mathcal{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \) is a two-dimensional \( \tilde{B}_{\alpha_3}\)-module, by Lemma 2.4(1), we have

\[ \mathcal{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} = V \otimes \mathbb{C}_{-\omega_3}, \]

where \( V \) is the standard two-dimensional \( \tilde{L}_{\alpha_3}\)-module.

Thus by using Lemma 2.3(4), we have

\[ H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathcal{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3}) = 0. \]

Since \( (-\alpha_3+\alpha_4), \alpha_3) = -1 \), by Lemma 2.3(4), we have

\[ H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_3+\alpha_4)}) = 0. \]

Since \( (-\alpha_2+\alpha_3), \alpha_3) = 0, (-\alpha_1+\alpha_2+\alpha_3), \alpha_3) = 0 \), by Lemma 2.3(2), we have

\[ H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \]

and

\[ H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}. \]

Since \( (-\alpha_2+\alpha_3+\alpha_4), \alpha_3) = 1, (-\alpha_1+\alpha_2+\alpha_3+\alpha_4), \alpha_3) = 1 \), by Lemma 2.3(2), we have

\[ H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \]

and

\[ H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}. \]
Thus, combining the above discussion, we have

\[
H^0(s_3 s_4 s_1 s_2 s_3, \alpha_3) = \mathbb{C} - (\alpha_2 + \alpha_3) \oplus \mathbb{C} - (\alpha_1 + \alpha_2 + \alpha_3) \oplus \mathbb{C} - (\alpha_2 + \alpha_3 + \alpha_4) \\
\quad \oplus \mathbb{C} - (\alpha_2 + 2\alpha_3 + \alpha_4) \\
\quad \oplus \mathbb{C} - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \oplus \mathbb{C} - (\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4). \tag{4.3.1}
\]

Since \((-\alpha_2 + \alpha_3, \alpha_2) = -1, \((-\alpha_2 + \alpha_3 + \alpha_4, \alpha_2) = -1, \) by Lemma 2.3(4), we have

\[
H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C} - (\alpha_2 + \alpha_3)) = 0
\]

and

\[
H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C} - (\alpha_2 + \alpha_3 + \alpha_4)) = 0.
\]

Since \((-\alpha_1 + \alpha_2 + \alpha_3, \alpha_2) = 0, (-\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_2 = 0 \) and \((-\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4), \alpha_2 = 0, \) by Lemma 2.3(4), we have

\[
H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C} - (\alpha_1 + \alpha_2 + \alpha_3)) = \mathbb{C} - (\alpha_1 + \alpha_2 + \alpha_3),
\]

\[
H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C} - (\alpha_2 + 2\alpha_3 + \alpha_4)) = \mathbb{C} - (\alpha_2 + 2\alpha_3 + \alpha_4)
\]

and

\[
H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C} - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)) = \mathbb{C} - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4).
\]

Since \((-\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), \alpha_2 = 1, \) by Lemma 2.3(2), we have

\[
H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C} - (\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4)) = \mathbb{C} - (\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4) \oplus \mathbb{C} - (\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4).
\]

Thus, combining the above discussion, we have

\[
H^0(s_2 s_3 s_4 s_1 s_2 s_3, \alpha_3) = \mathbb{C} - (\alpha_1 + \alpha_2 + \alpha_3) \oplus \mathbb{C} - (\alpha_2 + 2\alpha_3 + \alpha_4) \\
\quad \oplus \mathbb{C} - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \oplus \mathbb{C} - (\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4) \\
\quad \oplus \mathbb{C} - (\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4).
\]

Since \((-\alpha_1 + \alpha_2 + \alpha_3, \alpha_1) = -1\) and \((-\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1) = -1, \) by Lemma 2.3(4), we have

\[
H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C} - (\alpha_1 + \alpha_2 + \alpha_3)) = 0
\]

and

\[
H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C} - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)) = 0.
\]

Since \(\mathbb{C} - (\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4) \oplus \mathbb{C} - (\alpha_2 + 2\alpha_3 + \alpha_4)\) is the standard two-dimensional irreducible \(\tilde{L}_{\alpha_1}\)-module, by using Lemma 2.3(2), we have

\[
H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C} - (\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4) \oplus \mathbb{C} - (\alpha_2 + 2\alpha_3 + \alpha_4)) \\
\quad = \mathbb{C} - (\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4) \oplus \mathbb{C} - (\alpha_2 + 2\alpha_3 + \alpha_4).
\]
Since \((-\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_1\) = 0, by Lemma 2.3(2), we have
\[ H^0(\bar{L}_{\alpha_1}/\bar{B}_{\alpha_1}, C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}) = C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}. \]

Thus, combining the above discussion, we have
\[ H^0(w_1s_3, \alpha_3) = C_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}. \]

Since \((-\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_4\) = 0, \((-\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_4\) = 0 and \((-\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_4\) = 0, by Lemma 2.3(2), we have
\[ H^0(\bar{L}_{\alpha_4}/\bar{B}_{\alpha_4}, C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}) = C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \]

and
\[ H^0(\bar{L}_{\alpha_4}/\bar{B}_{\alpha_4}, C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}) = C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}. \]

Therefore, we have
\[ H^0(s_4w_1s_3, \alpha_3) = C_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}. \]

**Proof of (2).** By Lemma 4.2(1), we have
\[ H^0(w_2s_3, \alpha_3) = C_{-\omega_4+\alpha_4}. \]

Since \((-\omega_4 + \alpha_4, \alpha_4\) = 1, by using SES and Lemma 2.3(2), we have
\[ H^0(s_4w_2s_3, \alpha_3) = C_{-\omega_4+\alpha_4} \oplus C_{-\omega_4}. \]

**Proof of (3).** By the Lemma 4.2(3), we have
\[ H^0(w_3s_3, \alpha_3) = C_{-\omega_4}. \]

Since \((-\omega_4, \alpha_4\) = -1, by Lemma 2.3(4), we have
\[ H^0(s_4w_3s_3, \alpha_3) = 0. \]

By using SES repeatedly, we have
\[ H^0(s_4w_r s_3, \alpha_3) = 0 \quad \text{for } r = 4, 5. \]

**COROLLARY 4.4**

(1) \[ H^0(s_3s_4s_1s_2s_3, \alpha_3) = C_{-(\alpha_2+\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3)} \oplus C_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}. \]

(2) \[ H^0(s_3s_4w_1s_3, \alpha_3) = C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-\omega_4+\alpha_4}. \]

(3) \[ H^0(s_3s_4w_2s_3, \alpha_3) = C_{-\omega_4}. \]
Proof.

Proof of (1). Proof follows from (4.3.1).
Proof of (2). Proof follows from Corollary 4.3(1).
Proof of (3). Proof follows from Corollary 4.3(2).

COROLLARY 4.5

\[(1) \quad H^0(\mathfrak{s}_2\mathfrak{s}_3\mathfrak{s}_4\mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3, \alpha_3) = C_{(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{(\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)} \oplus C_{(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus C_{(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.\]
\[(2) \quad H^0(\mathfrak{s}_2\mathfrak{s}_3\mathfrak{s}_4\mathfrak{w}_1\mathfrak{s}_3, \alpha_3) = C_{-\omega_3+\omega_4}.\]
\[(3) \quad H^1(\mathfrak{s}_2\mathfrak{s}_3\mathfrak{s}_4\mathfrak{w}_2\mathfrak{s}_3, \alpha_3) = C_{-\omega_4}.\]

Proof.

Proof of (1). Proof follows from Corollary 4.4(1).
Proof of (2). Proof follows from Corollary 4.4(2).
Proof of (3). Proof follows from Corollary 4.4(3).

COROLLARY 4.6

\[(1) \quad H^0(\mathfrak{s}_4\mathfrak{s}_3\mathfrak{s}_4\mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3, \alpha_3) = C_{(\alpha_2+\alpha_3)} \oplus C_{(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{(\alpha_2+\alpha_3+\alpha_4)} \oplus C_{(\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus C_{(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.\]
\[(2) \quad H^0(\mathfrak{s}_4\mathfrak{s}_3\mathfrak{s}_4\mathfrak{w}_1\mathfrak{s}_3, \alpha_3) = C_{(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-\omega_3+\omega_4} \oplus C_{-\omega_4}.\]

Proof.

Proof of (1). By Corollary 4.4(1), we have

\[H^0(\mathfrak{s}_3\mathfrak{s}_4\mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3, \alpha_3) = C_{(\alpha_2+\alpha_3)} \oplus C_{(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{(\alpha_2+\alpha_3+\alpha_4)} \oplus C_{(\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus C_{(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.\]

Since \(C_{(\alpha_2+\alpha_3)} \oplus C_{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\) is the standard two-dimensional irreducible \(\tilde{L}_{\alpha_4}\)-module, by Lemma 2.3(2), we have

\[H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, C_{(\alpha_2+\alpha_3)} \oplus C_{(\alpha_2+\alpha_3+\alpha_4)}) = C_{(\alpha_2+\alpha_3)} \oplus C_{(\alpha_2+\alpha_3+\alpha_4)}.\]

Also, since \(C_{(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\) is the standard two-dimensional irreducible \(\tilde{L}_{\alpha_4}\)-module, by Lemma 2.3(2), we have

\[H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, C_{(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}) = C_{(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.\]

Since \((-\alpha_2+2\alpha_3+\alpha_4), \alpha_4\) = 0 and \((-\alpha_1+\alpha_2+2\alpha_3+\alpha_4), \alpha_4\) = 0, by Lemma 2.3(2), we have

\[H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, C_{(\alpha_2+2\alpha_3+\alpha_4)}) = C_{(\alpha_2+2\alpha_3+\alpha_4)}.\]
and

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}) = C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.$$  

Thus, combining the above discussion, we have

$$H^0(s_4s_3s_4s_1s_2s_3, \alpha_3) = C_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.$$  

**Proof of (2).** By Corollary 4.4(2), we have

$$H^0(s_3s_4w_1s_3, \alpha_3) = C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$  

Since $$(-\langle \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 \rangle, \alpha_4) = 0$$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}) = C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$  

Further, since $$(-\langle \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 \rangle, \alpha_4) = 1$$, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, C_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}) = C_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}.$$  

Thus, combining the above discussion, we have

$$H^0(s_4s_3s_4w_1s_3, \alpha_3) = C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-\omega_4+\alpha_4} \oplus C_{-\omega_4}.$$  

since $$\omega_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$$. □

**COROLLARY 4.7**

1. $$H^0(s_4s_3s_4s_1s_2s_3, \alpha_3) = C_{-(\alpha_1+2\alpha_2+\alpha_3)} \oplus C_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$  

2. $$H^0(s_4s_3s_4w_1s_3, \alpha_3) = C_{-\omega_4+\alpha_4} \oplus C_{-\omega_4}.$$  

**Proof.**

**Proof of (1).** By Corollary 4.5(1), we have

$$H^0(s_2s_3s_4s_1s_2s_3, \alpha_3) = C_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$  

Since $$C_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$$ is the standard two-dimensional irreducible $$\tilde{L}_{\alpha_4}$$-module, by Lemma 2.3(2), we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, C_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}) = C_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.$$
Moreover, since \((-\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3\) = \(0\), \((-\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), \alpha_4\) = \(0\) and \((-\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_4\) = \(0\), by Lemma 2.3(2), we have

\[
H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}
\]

and

\[
H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.
\]

Thus, combining the above discussion, we have

\[
H^0(s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.
\]

**Proof of (2).** By Corollary 4.5(2), we have

\[
H^0(s_2s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4}.
\]

Since \((-\omega_4 + \alpha_4, \alpha_4) = 1\), by Lemma 2.3(2), we have

\[
H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-\omega_4+\alpha_4}) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.
\]

Thus we have

\[
H^0(s_4s_2s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.
\]

\[
\square
\]

**COROLLARY 4.8**

1. \(H^0(s_3s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.
\]

2. \(H^0(s_3s_4s_2s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-\omega_4} \).

**Proof.**

**Proof of (1).** By Corollary 4.7(1), we have

\[
H^0(s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.
\]

Since \((-\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3\) = \(-1\), by Lemma 2.3(4), we have

\[
H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}) = 0.
\]
Since \((-\langle \alpha_1 + \alpha_2 + \alpha_3, \alpha_3 \rangle = 0\), by Lemma 2.3(2), we have

\[ H^0(\tilde{L}_{\alpha_3} / \tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1 + \alpha_2 + \alpha_3)}) = \mathbb{C}_{-(\alpha_1 + \alpha_2 + \alpha_3)}. \]

Since \(\mathbb{C}_{-(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)} \oplus \mathbb{C}_{-(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4)}\) is the standard two-dimensional irreducible \(\tilde{L}_{\alpha_3}\)-module, by Lemma 2.3(2), we have

\[ H^0(\tilde{L}_{\alpha_3} / \tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)} \oplus \mathbb{C}_{-(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4)}) = \mathbb{C}_{-(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4)}. \]

Since \((-\langle \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \alpha_3 \rangle = 1\), by Lemma 2.3(2), we have

\[ H^0(\tilde{L}_{\alpha_3} / \tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)}) = \mathbb{C}_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)}. \]

Thus, combining the above discussion, we have

\[ H^0(s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1 + \alpha_2 + \alpha_3)} \oplus \mathbb{C}_{-(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)} \oplus \mathbb{C}_{-(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4)} \oplus \mathbb{C}_{-(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)} \oplus \mathbb{C}_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)}. \]

**Proof of (2).** By Corollary 4.7(2), we have

\[ H^0(s_4s_2s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-\omega_4 + \alpha_4} \oplus \mathbb{C}_{-\omega_4}. \]

Since \((-\omega_4 + \alpha_4, \alpha_3) = -1\), by Lemma 2.3(4), we have

\[ H^0(\tilde{L}_{\alpha_3} / \tilde{B}_{\alpha_3}, \mathbb{C}_{-\omega_4 + \alpha_4}) = 0. \]

Further, since \((-\omega_4, \alpha_3) = 0\), by Lemma 2.3(2), we have

\[ H^0(\tilde{L}_{\alpha_3} / \tilde{B}_{\alpha_3}, \mathbb{C}_{-\omega_4}) = \mathbb{C}_{-\omega_4}. \]

Thus, from the above discussion, we have

\[ H^0(s_3s_4s_2s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-\omega_4}. \]

\[ \square \]

**COROLLARY 4.9**

1. \( H^0(s_4s_3s_4s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2 + \alpha_3)} \oplus \mathbb{C}_{-(\alpha_2 + \alpha_3 + \alpha_4)} \oplus \mathbb{C}_{-(\alpha_2 + 2\alpha_3 + \alpha_4)}. \)
2. \( H^0(s_4s_3s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1 + \alpha_2 + \alpha_3)} \oplus \mathbb{C}_{-(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)} \oplus \mathbb{C}_{-(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4)} \oplus \mathbb{C}_{-(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)} \oplus \mathbb{C}_{-\omega_4 + \alpha_4} \oplus \mathbb{C}_{-\omega_4}. \)

**Proof.**

**Proof of (1).** It is easy to see that

\[ H^0(s_3, \alpha_3) = \mathbb{C}_{-\alpha_3} \oplus \mathbb{C} h(\alpha_3) \oplus \mathbb{C}_{\alpha_3}. \]
Since \( \langle -\alpha_3, \alpha_2 \rangle = 1 \), by using Lemma 2.3(2) and Lemma 2.3(4), we have

\[
H^0(s_2s_3, \alpha_3) = \mathbb{C} h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3)}.
\]

Since \( \langle -\alpha_3, \alpha_4 \rangle = 1 \), by using Lemma 2.3(2), we have

\[
H^0(s_4s_2s_3, \alpha_3) = \mathbb{C} h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}_{-(\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}.\]

Since \( \mathbb{C} h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \) is the two-dimensional indecomposable \( \tilde{B}_{\alpha_3} \)-module, by Lemma 2.3(4), we have

\[
\mathbb{C} h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} = V \otimes \mathbb{C}_{-\alpha_3},
\]

where \( V \) is the standard two-dimensional irreducible \( \tilde{L}_{\alpha_3} \)-module. Thus by Lemma 2.3(4), we have

\[
H^0(\tilde{L}_{\alpha_3}/\tilde{B}_3, \mathbb{C} h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3}) = 0.
\]

Also, since \( \langle -\alpha_3 + \alpha_4, \alpha_3 \rangle = -1 \), by Lemma 2.3(4), we have

\[
H^0(\tilde{L}_{\alpha_3}/\tilde{B}_3, \mathbb{C}_{-(\alpha_3+\alpha_4)}) = 0.
\]

Since \( \langle -\alpha_2 + \alpha_3, \alpha_3 \rangle = 0 \), by Lemma 2.3(2), we have

\[
H^0(\tilde{L}_{\alpha_3}/\tilde{B}_3, \mathbb{C}_{-(\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_2+\alpha_3)}.
\]

Since \( \langle -\alpha_2 + \alpha_3 + \alpha_4, \alpha_3 \rangle = 1 \), by Lemma 2.3(2), we have

\[
H^0(\tilde{L}_{\alpha_3}/\tilde{B}_3, \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}.
\]

Thus, combining the above discussion, we have

\[
H^0(s_3s_4s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}.
\]

Since \( \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \) is the standard two-dimensional irreducible \( \tilde{L}_{\alpha_4} \)-module, by Lemma 2.3(2), we have

\[
H^0(\tilde{L}_{\alpha_4}/\tilde{B}_4, \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}.
\]

Further, since \( \langle -\alpha_2 + 2\alpha_3 + \alpha_4, \alpha_4 \rangle = 0 \), by Lemma 2.3(2), we have

\[
H^0(\tilde{L}_{\alpha_4}/\tilde{B}_4, \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}.
\]

Therefore, we have

\[
H^0(s_4s_3s_4s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}.
\]
Proof of (2). By Corollary 4.8(1), we have
\[ H^0(s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = C(-(\alpha_1+\alpha_2+\alpha_3) \oplus C-(\alpha_1+\alpha_2+3\alpha_3+\alpha_4) \]
\[ \oplus C-(\alpha_1+\alpha_2+2\alpha_3+3\alpha_3+\alpha_4) \oplus C-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4) \]
\[ \oplus C-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4). \]

Since \( C-(\alpha_1+\alpha_2+\alpha_3) \oplus C-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4) \) and \( C-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4) \) are the two-dimensional irreducible \( L_{\alpha_3} \)-modules and \( -(\alpha_1 + \alpha_2 + \alpha_3, \alpha_3) = 0 \) by Lemma 2.3(2), we have
\[ H^0(s_3s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = C-(\alpha_1+\alpha_2+\alpha_3) \oplus C-(\alpha_1+\alpha_2+3\alpha_3+\alpha_4) \]
\[ \oplus C-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4) \oplus C-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4) \]
\[ \oplus C-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4). \]

Since \( C-(\alpha_1+\alpha_2+\alpha_3) \oplus C-(\alpha_1+\alpha_2+3\alpha_3+\alpha_4) \) is the standard two-dimensional irreducible \( \tilde{L}_{\alpha_4} \)-module by Lemma 2.3(2), we have
\[ H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, C-(\alpha_1+\alpha_2+\alpha_3) \oplus C-(\alpha_1+\alpha_2+3\alpha_3+\alpha_4)) = C-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4) \]
\[ \oplus C-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4). \]

Moreover, since \( -(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), \alpha_4) = 0 \) and \( -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_4) = 0 \), by Lemma 2.3(2), we have
\[ H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, C-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)) = C-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4) \]
\[ \text{and} \]
\[ H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, C-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)) = C-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4). \]

Since \( -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_4) = 1 \), by Lemma 2.3, we have
\[ H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, C-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)) = C-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4) \oplus C-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4). \]

Therefore, combining the above discussion, we have
\[ H^0(s_4s_3s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = C-(\alpha_1+\alpha_2+\alpha_3) \oplus C-(\alpha_1+\alpha_2+\alpha_3+\alpha_4) \]
\[ \oplus C-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4) \oplus C-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4) \]
\[ \oplus C-(\omega_4+\alpha_4) \oplus C-\omega_4, \]

since \( \omega_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4. \)

5. Computations of relative tangent bundles \( H^1(w, \alpha_2) \)

In this section, we compute cohomology modules \( H^1(w, \alpha_2) \) corresponding to some special Weyl group elements.
Lemma 5.1.
(1) \( H^1(w_r, \alpha_2) = 0 \) for \( r = 1, 2, 5 \).
(2) \( H^1(w_3, \alpha_2) = \mathbb{C}_{-\alpha_4+\alpha_4} \).
(3) \( H^1(w_4, \alpha_2) = \mathbb{C}_{-\alpha_4} \).

Proof. It is easy to see that \( H^1(s_2, \alpha_2) = 0 \). Note that we have
\[
H^0(s_2, \alpha_2) = \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{\alpha_2}.
\]
Since \( \langle -\alpha_2, \alpha_1 \rangle = 1 \), by using Lemma 2.3(2) and Lemma 2.3(4), we have
\[
H^1(s_1, H^0(s_2, \alpha_2)) = 0.
\]
Since \( H^1(s_2, \alpha_2) = 0 \), by using Lemma 2.3(1), we have
\[
H^0(s_1, H^1(s_2, \alpha_2)) = 0.
\]
Thus, by using SES and the above discussion, we have
\[
H^1(s_1s_2, \alpha_2) = 0.
\]
By using SES and Lemma 2.3(2), we have
\[
H^0(s_1s_2, \alpha_2) = \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2)}.
\]
Since \( \langle \alpha_2, \alpha_4 \rangle = 0 \), by using Lemma 2.3(2), we have
\[
H^1(s_4, H^0(s_1s_2, \alpha_2)) = 0
\]
and
\[
H^0(s_4s_1s_2, \alpha_2) = \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2)}.
\]  \hspace{1cm} (5.1.1)
Again, since \( H^1(s_1s_2, \alpha_2) = 0 \), by using Lemma 2.3(1), we have
\[
H^0(s_4, H^1(s_1s_2, \alpha_2)) = 0.
\]
Thus, by using SES and the above discussion, we have
\[
H^1(s_4s_1s_2, \alpha_2) = 0. \hspace{1cm} (5.1.2)
\]
Since \( \langle -\alpha_2, \alpha_3 \rangle = 2 \), \( \langle -\alpha_1+\alpha_2, \alpha_3 \rangle = 2 \), by using (5.1.1) and Lemma 2.3(2), we have
\[
H^1(s_3, H^0(s_4s_1s_2, \alpha_2)) = 0.
\]
Further, by (5.1.2), we have
\[
H^0(s_3, H^1(s_4s_1s_2, \alpha_2)) = 0.
\]
Thus, by using SES, we have
\[ H^1(s_3s_4s_1s_2, \alpha_2) = 0. \] \hspace{1cm} (5.1.3)

Therefore, we have
\[ H^0(s_2, H^1(s_3s_4s_1s_2, \alpha_2)) = 0. \]

By using Lemma 3.3, we have
\[ H^1(s_2, H^0(s_3s_4s_1s_2, \alpha_2)) = 0. \]

Thus by SES, we have
\[ H^1(s_2s_3s_4s_1s_2, \alpha_2) = 0. \] \hspace{1cm} (5.1.4)

Therefore, we have
\[ H^0(s_1, H^1(s_2s_3s_4s_1s_2, \alpha_2)) = 0. \]

By Lemma 3.3, we have
\[ H^1(s_1, H^0(s_2s_3s_4s_1s_2, \alpha_2)) = 0. \]

Therefore, by using SES, we have
\[ H^1(w_1, \alpha_2) = 0. \]

Since \( H^1(w_1, \alpha_2) = 0 \), we have
\[ H^0(s_4, H^1(w_1, \alpha_2)) = 0. \]

Recall that by (4.1.3), we have
\[ H^0(w_1, \alpha_2) = C_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus C_{-(\alpha_2+2\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3)}. \]

Since \((-\alpha_2+2\alpha_3), \alpha_4) = 2, \((-\alpha_1+\alpha_2+2\alpha_3), \alpha_4) = 2, \((-\alpha_1+2\alpha_2+2\alpha_3), \alpha_4) = 2, \)
by using Lemma 2.3(2), we have
\[ H^1(s_4, H^0(w_1, \alpha_2)) = 0. \]

Thus, by using SES and the above discussion, we have
\[ H^1(s_4w_1, \alpha_2) = 0 \] \hspace{1cm} (5.1.5)

and
\[ H^0(s_4w_1, \alpha_2) = (C_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}) \oplus (C_{-(\alpha_1+2\alpha_2+2\alpha_3)}) \oplus (C_{-(\alpha_2+2\alpha_3)} \oplus C_{-(\alpha_2+2\alpha_3+\alpha_4)}) \oplus (C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}). \]
Since $H^1(s_4w_1, \alpha_2) = 0$, we have

$$H^0(s_3, H^1(s_4w_1, \alpha_2)) = 0.$$  

Since $\langle - (\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_3 \rangle = 0$, $\langle - (\alpha_1 + 2\alpha_2 + 2\alpha_3), \alpha_3 \rangle = 0$, $\langle - (\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_3 \rangle = 0$, $\langle - (\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3 \rangle = 1$, $\langle - (\alpha_1 + 2\alpha_2 + 2\alpha_3), \alpha_3 \rangle = 2$, $\langle - (\alpha_2 + 2\alpha_3), \alpha_3 \rangle = -2$, $\langle - (\alpha_1 + 2\alpha_2 + 2\alpha_3), \alpha_3 \rangle = -2$, $\langle - (\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3 \rangle = -1$, $\langle - (\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3 \rangle = -1$, by using Lemma 2.3, we have

$$H^1(s_3, H^0(s_4w_1, \alpha_2)) = C_{-(\alpha_2 + \alpha_3)} \oplus C_{-(\alpha_1 + \alpha_2 + \alpha_3)}.$$  

Thus, by using SES and the above discussion, we have

$$H^1(s_3s_4w_1, \alpha_2) = C_{-(\alpha_2 + \alpha_3)} \oplus C_{-(\alpha_1 + \alpha_2 + \alpha_3)}, \quad (5.1.6)$$  

and

$$H^0(s_3s_4w_1, \alpha_2) = C_{-(\alpha_2 + 2\alpha_3 + 2\alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 2\alpha_3)} \oplus C_{-(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_3 + 3\alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4)}. \quad (5.1.7)$$  

Since $H^1(s_3s_4w_1, \alpha_2) = C_{-(\alpha_2 + \alpha_3)} \oplus C_{-(\alpha_1 + \alpha_2 + \alpha_3)}$, by using Lemma 2.3, we have

$$H^0(s_2, H^1(s_3s_4w_1, \alpha_2)) = C_{-(\alpha_1 + \alpha_2 + \alpha_3)}.$$  

By Lemma 3.3, we have

$$H^1(s_2, H^0(s_3s_4w_1, \alpha_2)) = 0.$$  

Thus, using SES and the above discussion, we have

$$H^1(s_2s_3s_4w_1, \alpha_2) = C_{-(\alpha_1 + \alpha_2 + \alpha_3)}. \quad (5.1.8)$$  

Since $C_{-(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4)}$ is the standard two-dimensional irreducible $\widetilde{L}_{\alpha_1}$-module and $\langle - (\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_2 \rangle = 0$, $\langle - (\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_2 \rangle = 0$, $\langle - (\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4), \alpha_2 \rangle = 1$, $\langle - (\alpha_1 + 2\alpha_2 + 2\alpha_3), \alpha_2 \rangle = -1$, $\langle - (\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_2 \rangle = -1$, by using SES and Lemma 2.3, we have

$$H^0(s_2s_3s_4w_1, \alpha_2) = C_{-(\alpha_2 + 2\alpha_3 + 2\alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4)}. \quad (5.1.9)$$
Since \((-\alpha_1 + \alpha_2 + \alpha_3, \alpha_1) = -1\), by using Lemma 2.3(4), we have
\[
H^0(s_1, H^1(s_2s_3s_4w_1, \alpha_2)) = 0.
\]

Further, by Lemma 3.3, we have
\[
H^1(s_1, H^0(s_2s_3s_4w_1, \alpha_2)) = 0.
\]

Thus using SES, we have
\[
H^1(w_2, \alpha_2) = 0.
\]

Since \(C_{-(\alpha_2 + 2\alpha_3 + 2\alpha_4)} \oplus C_{-(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4)}\) is the standard two-dimensional irreducible \(L_{\alpha_1}\)-module and \(\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_1 \rangle = 0, \langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4), \alpha_1 \rangle = 0, \langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_1 \rangle = 0, \langle -(\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_1 \rangle = 0, \langle -(\alpha_1 + 3\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_1 \rangle = 1\), by using SES and Lemma 2.3, we have
\[
H^0(w_2, \alpha_2) = C_{-(\alpha_2 + 2\alpha_3 + 2\alpha_4)} \oplus C_{-(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4)} \\
\quad \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4)} \\
\quad \oplus C_{-(\alpha_1 + 3\alpha_2 + 2\alpha_3 + 2\alpha_4)} \oplus C_{-(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4)}.
\]

Since \(H^1(w_2, \alpha_2) = 0\), we have
\[
H^0(s_4, H^1(w_2, \alpha_2)) = 0.
\]

Since \(C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_1 + \alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)}\) is the standard two-dimensional irreducible \(L_{\alpha_1}\)-module and \(\langle -(\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_4 \rangle = 0, \langle -(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_4 \rangle = 0, \langle -(\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_4 \rangle = -2, \langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_4 \rangle = -2, \langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4), \alpha_4 \rangle = -2\), by using SES and Lemma 2.3, we have
\[
H^1(s_4, H^0(w_2, \alpha_2)) = C_{-(\alpha_2 + 2\alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4)} \\
\quad \oplus C_{-(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)}.
\]

Thus, from the above discussion, we have
\[
H^1(s_4w_2, \alpha_2) = C_{-(\alpha_2 + 2\alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4)} \\
\quad \oplus C_{-(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)} \quad (5.1.10)
\]

and
\[
H^0(s_4w_2, \alpha_2) = C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)} \\
\quad \oplus C_{-(\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4)} \\
\quad \oplus C_{-(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4)} \oplus C_{-(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4)}.
\]
Since \( \langle - (\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3 \rangle = 1, \langle - (\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3 \rangle = -1 \) and \( \langle - (\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3 \rangle = -1 \), by using Lemma 2.3, we have
\[
H^0(s_3, H^1(s_4 w_2, \alpha_2)) = C_{-(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)}.
\]

Since \( \langle - (\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_3 \rangle = 0, \langle - (2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_3 \rangle = 0, \langle - (\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_3 \rangle = -1 \) and \( C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4)} = V \otimes C_{-\omega_3} \) (where \( V \) is the standard two-dimensional irreducible \( \tilde{L}_{\alpha_3} \)-module), by Lemma 2.3, we have
\[
H^1(s_3, H^0(s_4 w_2, \alpha_2)) = 0
\]
and
\[
H^0(s_3 s_4 w_2, \alpha_2) = C_{-(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4)} \oplus C_{-(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4)}. \tag{5.1.11}
\]

Therefore, we have
\[
H^1(s_3 s_4 w_2, \alpha_2) = C_{-(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)}. \tag{5.1.12}
\]

Since \( \langle - (\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_2 \rangle = 0 \) and \( \langle - (\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_2 \rangle = -1 \), by using Lemma 2.3, we have
\[
H^0(s_2, H^1(s_3 s_4 w_2, \alpha_2)) = C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)}.
\]

By Lemma 3.3, we have
\[
H^1(s_2, H^0(s_3 s_4 w_2, \alpha_2)) = 0.
\]

Thus, from the above discussion, we have
\[
H^1(s_2 s_3 s_4 w_2, \alpha_2) = C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)} = C_{-\omega_4 + \alpha_4}. \tag{5.1.13}
\]

Since \( \langle - (2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_2 \rangle = 0 \) and \( \langle - (\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_2 \rangle = -1 \), by using SES and Lemma 2.3, we have
\[
H^0(s_2 s_3 s_4 w_2, \alpha_2) = C_{-(\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4)}. \tag{5.1.14}
\]

Since \( \langle - (\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_1 \rangle = 0 \), by using Lemma 2.3(2), we have
\[
H^0(s_1, H^1(s_2 s_3 s_4 w_2, \alpha_2)) = C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)}.
\]

By Lemma 3.3, we have
\[
H^1(s_1, H^0(s_2 s_3 s_4 w_2, \alpha_2)) = 0.
\]

Thus, from the above discussion, we have
\[
H^1(w_3, \alpha_2) = C_{-\omega_4 + \alpha_4}
\]
since \( \omega_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 \). This proves (2).

Since we have \( H^0(w_3, \alpha_2) = 0 \) (see Lemma 4.1), by using SES, we have
\[
H^1(s_4 w_3, \alpha_2) = H^0(s_4, H^1(w_3, \alpha_2)).
\]

Since \( \langle -\omega_4 + \alpha_4, \alpha_4 \rangle = 1 \), by using Lemma 2.3(2), we have
\[
H^1(s_4 w_3, \alpha_2) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.
\] (5.1.15)

Since we have \( H^0(w_3, \alpha_2) = 0 \) (see Lemma 4.1), by using SES, we have
\[
H^1(s_3 s_4 w_3, \alpha_2) = H^0(s_3, H^1(s_4 w_3, \alpha_2)).
\]

Since \( \langle -\omega_4, \alpha_3 \rangle = 0 \) and \( \langle -\omega_4 + \alpha_4, \alpha_3 \rangle = -1 \), by using Lemma 2.3(2) and Lemma 2.3(4), we have
\[
H^1(s_3 s_4 w_3, \alpha_2) = \mathbb{C}_{-\omega_4}.
\] (5.1.16)

Since we have \( H^0(w_3, \alpha_2) = 0 \) (see Lemma 4.1) and \( \alpha_1, \alpha_2 \) are orthogonal to \( \omega_4 \), by Lemma 2.3(2), we have
\[
H^1(w_4, \alpha_2) = \mathbb{C}_{-\omega_4}.
\]

This gives the proof of (3).

Since we have \( H^0(w_3, \alpha_2) = 0 \) (see Lemma 4.1) and \( \langle -\omega_4, \alpha_4 \rangle = -1 \), by using SES repeatedly we have
\[
H^1(w_5, \alpha_2) = 0.
\]

This completes the proof of (1). \( \square \)

COROLLARY 5.2

(1) \( H^1(s_4 s_1 s_2, \alpha_2) = 0 \).

(2) \( H^1(s_4 w_r, \alpha_2) = 0 \) for \( r = 1, 4, 5 \).

(3) \( H^1(s_4 w_2, \alpha_2) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \).

(4) \( H^1(s_4 w_3, \alpha_2) = \mathbb{C}_{-\omega_4} \oplus \mathbb{C}_{-\omega_4+\alpha_4} \).

Proof.

Proof of (1). Follows from (5.1.2).

Proof of (2). For \( r = 1 \), the proof follows from (5.1.5). For \( r = 4, 5 \) the proof follows by using SES, Lemma 5.1 and Lemma 4.1.

Proof of (3). Follows from (5.1.10).

Proof of (4). Follows from (5.1.15). \( \square \)
COROLLARY 5.3

(1) $H^1(s_3s_4s_1s_2, \alpha_2) = 0$.
(2) $H^1(s_3s_4w_1, \alpha_2) = C_{-(\alpha_2+\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+\alpha_3)}$.
(3) $H^1(s_3s_4w_2, \alpha_2) = C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-\omega_4+\alpha_4}$.
(4) $H^1(s_3s_4w_3, \alpha_2) = C_{-\omega_4}$.
(5) $H^1(s_3s_4w_r, \alpha_2) = 0$ for $r = 4, 5$.

Proof.

Proof of (1). Follows from (5.1.3).
Proof of (2). Follows from (5.1.6).
Proof of (3). Follows from (5.1.12).
Proof of (4). Follows from (5.1.16).

Proof of (5). By Lemma 4.1, we have $H^0(w_r, \alpha_2) = 0$ for $r = 4, 5$. Therefore, $H^0(s_4w_r, \alpha_2) = 0$ for $r = 4, 5$. Hence we have $H^1(s_3, H^0(s_4w_r, \alpha_2)) = 0$ for $r = 4, 5$. On the other hand, by Corollary 5.2(2), we have $H^1(s_4w_r, \alpha_2) = 0$ for $r = 4, 5$. Therefore, $H^0(s_3, H^1(s_4w_r, \alpha_2)) = 0$ for $r = 4, 5$. Thus by SES, we have $H^1(s_3s_4w_r, \alpha_2) = 0$ for $r = 4, 5$. □

COROLLARY 5.4

(1) $H^1(s_2s_3s_4s_1s_2, \alpha_2) = 0$.
(2) $H^1(s_2s_3s_4w_1, \alpha_2) = C_{-(\alpha_1+\alpha_2+\alpha_3)}$.
(3) $H^1(s_2s_3s_4w_2, \alpha_2) = C_{-\omega_4+\alpha_4}$.
(4) $H^1(s_2s_3s_4w_3, \alpha_2) = C_{-\omega_4}$.
(5) $H^1(s_2s_3s_4w_4, \alpha_2) = 0$.

Proof.

Proof of (1). Follows from (5.1.4).
Proof of (2). Follows from (5.1.8).
Proof of (3). Follows from (5.1.13).

Proof of (4). By Lemma 4.1, we have $H^0(w_3, \alpha_2) = 0$. Therefore, $H^0(s_3s_4w_3, \alpha_2) = 0$. Hence we have $H^1(s_2, H^0(s_3s_4w_3, \alpha_2)) = 0$. On the other hand, by Corollary 5.3(4), we have $H^1(s_3s_4w_3, \alpha_2) = C_{-\omega_4}$. Since $\omega_4$ is orthogonal to $\alpha_2$, by Lemma 2.3(2), we have $H^0(s_2, H^1(s_3s_4w_3, \alpha_2)) = C_{-\omega_4}$. Thus by SES, we have $H^1(s_2s_3s_4w_3, \alpha_2) = C_{-\omega_4}$.

Proof of (5). By Lemma 4.1(2), we have $H^0(w_4, \alpha_2) = 0$. Therefore, $H^0(s_3s_4w_4, \alpha_2) = 0$. Hence we have $H^1(s_2, H^0(s_3s_4w_4, \alpha_2)) = 0$. On the other hand, by Corollary 5.3(5), we have $H^1(s_3s_4w_4, \alpha_2) = 0$. Therefore, $H^0(s_2, H^1(s_3s_4w_4, \alpha_2)) = 0$. Thus by SES, we have $H^1(s_2s_3s_4w_4, \alpha_2) = 0$. □

COROLLARY 5.5

(1) $H^1(s_4s_3s_4s_1s_2, \alpha_2) = 0$.
(2) $H^1(s_4s_3s_4w_1, \alpha_2) = C_{-(\alpha_2+\alpha_3)} \oplus C_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus C_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}$.
(3) $H^1(s_4s_3s_4w_2, \alpha_2) = C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-\omega_4+\alpha_4} \oplus C_{-\omega_4}$.
On the other hand, by using Corollary 5.3(3) and Lemma 2.3(2), we have

$$H^1(s_4, H^0(s_3s_4s_3s_2, \alpha_2)) = 0.$$ 

On the other hand, by using Corollary 5.3(1), we have

$$H^0(s_4, H^1(s_3s_4s_3s_2, \alpha_2)) = 0.$$ 

Hence we have $H^1(s_4s_3s_4w_r, \alpha_2) = 0$ for $r = 3, 4$.

**Proof.**

**Proof of (1).** By (4.1.1), if $H^0(s_3s_4s_3s_2, \alpha_2) \neq 0$, then we have $\langle \mu, \alpha_4 \rangle \geq 0$. Thus using Lemma 2.3(3), we have

$$H^1(s_4, H^0(s_3s_4s_3s_2, \alpha_2)) = 0.$$ 

On the other hand, by using Corollary 5.3(1), we have

$$H^0(s_4, H^1(s_3s_4s_3s_2, \alpha_2)) = 0.$$ 

Hence we have $H^1(s_4s_3s_4s_3s_2, \alpha_2) = 0$.

**Proof of (2).** By (5.1.7), the $\tilde{B}_{\alpha_2}$-indecomposable summands $V$ of $H^0(s_3s_4w_1, \alpha_2)$ for which $H^1(s_4, V) \neq 0$ are $\mathbb{C}-(\alpha_2+2\alpha_3+2\alpha_4)$ and $\mathbb{C}-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)$. Thus using Lemma 2.3(3), we have

$$H^1(s_4, H^0(s_3s_4w_1, \alpha_2)) = \mathbb{C}-(\alpha_2+2\alpha_3+\alpha_4) \oplus \mathbb{C}-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4).$$ 

On the other hand, by using Corollary 5.3(2) and Lemma 2.3(2), we have

$$H^0(s_4, H^1(s_3s_4w_1, \alpha_2)) = \mathbb{C}-(\alpha_2+\alpha_3) \oplus \mathbb{C}-(\alpha_2+\alpha_3+\alpha_4)$$

$$\oplus \mathbb{C}-(\alpha_1+\alpha_2+\alpha_3) \oplus \mathbb{C}-(\alpha_1+\alpha_2+\alpha_3+\alpha_4).$$ 

Hence we have

$$H^1(s_4s_3s_4w_1, \alpha_2) = \mathbb{C}-(\alpha_2+\alpha_3) \oplus \mathbb{C}-(\alpha_2+\alpha_3+\alpha_4)$$

$$\oplus \mathbb{C}-(\alpha_1+\alpha_2+\alpha_3) \oplus \mathbb{C}-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$$

$$\oplus \mathbb{C}-(\alpha_2+2\alpha_3+\alpha_4) \oplus \mathbb{C}-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4).$$

**Proof of (3).** By (5.1.11), we have if $H^0(s_3s_4w_2, \alpha_2) \neq 0$, then $\langle \mu, \alpha_4 \rangle = 0$. Thus using Lemma 2.3(3), we have

$$H^1(s_4, H^0(s_3s_4w_2, \alpha_2)) = 0.$$ 

On the other hand, by using Corollary 5.3(3) and Lemma 2.3(2), we have

$$H^0(s_4, H^1(s_3s_4w_2, \alpha_2)) = \mathbb{C}-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4) \oplus \mathbb{C}-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)$$

$$\oplus \mathbb{C}-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4).$$ 

Hence we have

$$H^1(s_4s_3s_4w_2, \alpha_2) = \mathbb{C}-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4) \oplus \mathbb{C}-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)$$

$$\oplus \mathbb{C}-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4) \oplus \mathbb{C}-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4).$$

**Proof of (4).** By Lemma 4.1, we have $H^0(w_r, \alpha_2) = 0$ for $r = 3, 4$. Therefore, $H^0(s_3s_4w_r, \alpha_2) = 0$ for $r = 3, 4$. Hence we have $H^1(s_4, H^0(s_3s_4w_r, \alpha_2)) = 0$ for $r = 3, 4$.

On the other hand, by Corollary 5.3(4) and Corollary 5.3(5), we have $H^0(s_4, H^1(s_3s_4w_r, \alpha_2)) = 0$ for $r = 3, 4$. Thus, by using SES, we have $H^1(s_4s_3s_4w_r, \alpha_2) = 0$ for $r = 3, 4$. \hfill \Box
COROLLARY 5.6
(1) $H^1(s_4s_2s_3s_4s_1s_2, \alpha_2) = 0$.
(2) $H^1(s_4s_2s_3s_4w_1, \alpha_2) = C_{-(\alpha_1+\alpha_2)} \oplus C_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{-(\alpha_2+2\alpha_3+\alpha_4)}$.
(3) $H^1(s_4s_2s_3s_4w_2, \alpha_2) = C_{-\alpha_4} \oplus C_{-\alpha_4}$.
(4) $H^1(s_4s_2s_3s_4w_r, \alpha_2) = 0$ for $r = 3, 4$.

Proof.
Proof of (1). By Lemma 3.3, we have
$$H^1(s_2, H^0(s_4s_2s_3s_4s_1s_2, \alpha_2)) = 0.$$ 

On the other hand, by using Corollary 5.5(1), we have
$$H^0(s_2, H^1(s_4s_3s_4s_1s_2, \alpha_2)) = 0.$$ 

Hence we have $H^1(s_4s_2s_3s_4s_1s_2, \alpha_2) = H^1(s_2s_4s_3s_4s_1s_2, \alpha_2) = 0$.

Proof of (2). By Corollary 5.5(2), we have
$$H^0(s_2, H^1(s_4s_3s_4w_1, \alpha_2)) = C_{-(\alpha_1+\alpha_2)} \oplus C_{-(\alpha_1+\alpha_2+\alpha_3)}$$
$$\oplus C_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$$
$$\oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$ 

Now the proof of (2) follows from Lemma 3.3 and SES.

Proof of (3). By Corollary 5.5(3), using SES and Lemma 2.3, we have
$$H^0(s_2, H^1(s_4s_3s_4w_2, \alpha_2)) = C_{-\alpha_4} \oplus C_{-\alpha_4}.$$ 

Now the proof of (3) follows from Lemma 3.3 and SES.

Proof of (4). By Lemma 3.3, we have $H^1(s_2, H^0(s_4s_3s_4w_r, \alpha_2)) = 0$ for $r = 3, 4$. On the other hand, by Corollary 5.5(4), we have $H^0(s_2, H^1(s_4s_3s_4w_r, \alpha_2)) = 0$ for $r = 3, 4$. Thus by using SES, we have $H^1(s_4s_2s_3s_4w_r, \alpha_2) = 0$ for $r = 3, 4$. □

Lemma 5.7.
(1) $H^1(s_3s_4s_2s_3s_4s_1s_2, \alpha_2) = C_{-(\alpha_2+\alpha_3)}$.
(2) $H^1(s_3s_4s_2s_3s_4w_1, \alpha_2) = C_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus C_{-(\alpha_2+2\alpha_3+\alpha_4)}$.
(3) $H^1(s_3s_4s_2s_3s_4w_2, \alpha_2) = C_{-\alpha_4}$.
(4) $H^1(s_3s_4s_2s_3s_4w_r, \alpha_2) = 0$ for $r = 3, 4$.

Proof.
Proof of (1). Recall from (4.1.2) that
$$H^0(s_2s_3s_4s_1s_2, \alpha_2) = C_{-(\alpha_2+2\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3)}.$$
Since $(-\alpha_2 + 2\alpha_3, \alpha_4) = 2$, $(-\alpha_1 + \alpha_2 + \alpha_3, \alpha_4) = 1$, $(-\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_4) = 2$ and $(-\alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_4) = 2$, by using SES and Lemma 2.3(2), we have

$$H^0(s_4s_2s_3s_4s_1s_2, \alpha_2) = C_{-(\alpha_2+2\alpha_3)} \oplus C_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+\alpha_3+2\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_3+\alpha_4)}.$$ (5.7.1)

Since $C_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$ is the indecomposable $\tilde{R}_{\alpha_3}$-module, by using Lemma 2.4(1), we have $C_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} = V \otimes C_{-\alpha_3}$ (where $V$ is the standard two-dimensional irreducible $\tilde{L}_{\alpha_3}$-module) and $\langle -\alpha_2 + 2\alpha_3, \alpha_3 \rangle = -2$, by using SES and Lemma 2.3(3), we have

$$H^1(s_3, H^0(s_4s_2s_3s_4s_1s_2, \alpha_2)) = C_{-(\alpha_2+\alpha_3)}.$$ 

By using SES and Corollary 5.6(1), we have

$$H^0(s_3, H^1(s_4s_2s_3s_4s_1s_2, \alpha_2)) = 0.$$

Thus we have

$$H^1(s_3s_4s_2s_3s_4s_1s_2, \alpha_2) = C_{-(\alpha_2+\alpha_3)}.$$

**Proof of (2).** Recall from (5.1.9) that

$$H^0(s_2s_3s_4w_1, \alpha_2) = C_{-(\alpha_2+2\alpha_3+2\alpha_4)} \oplus C_{-(\alpha_2+2\alpha_3+3\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+3\alpha_3+3\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_3+2\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_3+\alpha_4)}.$$ 

Since $C_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}$ is the standard two-dimensional irreducible $\tilde{L}_{\alpha_4}$-module, $\langle -\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_4 \rangle = 0$, $\langle -\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_4 \rangle = 0$, $\langle -\alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_4 \rangle = -2$, $\langle -\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_4 \rangle = -2$, $\langle -\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_4 \rangle = -2$, by using Lemma 2.3, we have

$$H^0(s_4s_2s_3s_4w_1, \alpha_2) = C_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} \oplus C_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}.$$ (5.7.2)

By Corollary 5.6(2), we have

$$H^1(s_4s_2s_3s_4w_1, \alpha_2) = C_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus C_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus C_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus C_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$
Since \( \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} \) is an indecomposable \( \tilde{B}_{\alpha_3} \)-module, by Lemma 2.4(1), we have

\[
\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} = V \otimes \mathbb{C}_{-\omega_3}
\]

where \( V \) is the standard two-dimensional irreducible \( \tilde{L}_{\alpha_3} \)-module.

Further, since \( \langle -(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4), \alpha_3 \rangle = 0, \langle -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), \alpha_3 \rangle = -1 \), by using Lemma 2.3, we have

\[
H^1(s_3, H^0(s_4s_2s_3s_4w_1, \alpha_2)) = 0.
\]

Since \( \mathbb{C}_{-(\alpha_1+\alpha_2+3\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \) is the standard two-dimensional irreducible \( \tilde{L}_{\alpha_3} \)-module, \( \langle -(\alpha_1+\alpha_2+\alpha_3), \alpha_3 \rangle = 0, \langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_3 \rangle = 1, \langle -(\alpha_2+2\alpha_3+\alpha_4), \alpha_3 \rangle = -1 \), by using Lemma 2.3(2), we have

\[
H^0(s_3, H^1(s_4s_2s_3s_4w_1, \alpha_2)) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+3\alpha_4)}
\]
\[
\oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}
\]
\[
\oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.
\]

Thus we have

\[
H^1(s_3s_4s_2s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+3\alpha_4)}
\]
\[
\oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}
\]
\[
\oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.
\]

Proof of (3). Recall from (5.1.14) that

\[
H^0(s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)} = \mathbb{C}_{-\omega_1}.
\]

Since \( \alpha_4 \) is orthogonal to \( \omega_1 \), by using Lemma 2.3(2), we have

\[
H^0(s_4s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)} = \mathbb{C}_{-\omega_1}.
\]

Since \( \alpha_3 \) is orthogonal to \( \omega_1 \), by using Lemma 2.3(2), we have

\[
H^1(s_3, H^0(s_4s_2s_3s_4w_2, \alpha_2)) = 0.
\]

On the other hand, by Corollary 5.6(3), we have

\[
H^1(s_4s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.
\]

Since \( \langle -\omega_4, \alpha_3 \rangle = 0 \) and \( \langle -\omega_4+\alpha_4, \alpha_3 \rangle = -1 \), by using Lemma 2.3, we have

\[
H^0(s_3, H^1(s_4s_2s_3s_4w_2, \alpha_2)) = \mathbb{C}_{-\omega_4}.
\]

Thus we have

\[
H^1(s_3s_4s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-\omega_4}.
\]
Proof of (4). By Lemma 4.1, we have $H^0(w_r, \alpha_2) = 0$ for $r = 3, 4$. Therefore, we have $H^0(s_4s_2s_3s_4w_r, \alpha_2) = 0$ for $r = 3, 4$. Hence we have $H^1(s_3, H^0(s_4s_2s_3s_4w_r, \alpha_2)) = 0$ for $r = 3, 4$. On the other hand, by Corollary 5.6(4), we have $H^0(s_3, H^1(s_4s_2s_3s_4w_r, \alpha_2)) = 0$ for $r = 3, 4$. Thus by using SES, we have $H^1(s_3s_4s_2s_3s_4w_r, \alpha_2) = 0$ for $r = 3, 4$. □

Lemma 5.8.

(1) $H^1(s_4s_3s_4s_2, \alpha_2) = 0$.

(2) $H^1(s_4s_3s_4s_2s_3s_4s_1s_2, \alpha_2) = C_-(-\alpha_2+\alpha_3) \oplus C_-(-\alpha_2+\alpha_3+\alpha_4) \oplus C_-(-\alpha_2+2\alpha_3+\alpha_4)$.

(3) $H^1(s_4s_3s_4s_2s_3s_4s_1s_2, \alpha_2) = C_-(-\alpha_1+\alpha_2+\alpha_3) \oplus C_-(-\alpha_1+\alpha_2+\alpha_3+\alpha_4) \oplus C_-(-\alpha_1+\alpha_2+2\alpha_3+\alpha_4) \oplus C_-(-\alpha_1+2\alpha_2+2\alpha_3+\alpha_4) \oplus C_-\alpha_4 \oplus C_-\alpha_4$.

(4) $H^1(s_4s_3s_4s_2s_3s_4w_r, \alpha_2) = 0$ for $r = 2, 3$.

Proof.

Proof of (1). By using SES, it is easy to see that

$$H^0(s_3s_4s_2, \alpha_2) = Ch(\alpha_2) \oplus C_-\alpha_2 \oplus C_-(-\alpha_2+\alpha_3)$$

and

$$H^1(s_3s_4s_2, \alpha_2) = C_-\alpha_2+\alpha_3.$$ 

Since $H^0(s_3s_4s_2, \alpha_2)_\mu \neq 0$ implies $\mu, \alpha_4 \geq 0$, by using Lemma 2.3(2), we have

$$H^1(s_4, H^0(s_3s_4s_2, \alpha_2)) = 0.$$ 

Since $(-\alpha_2+\alpha_3, \alpha_4) = -1$, by using Lemma 2.3(4), we have $H^0(s_4, H^1(s_3s_4s_2, \alpha_2)) = 0$. Therefore, by using SES, we have $H^1(s_4s_3s_4s_2, \alpha_2) = 0$.

Proof of (2). By the Corollary 5.7(1), we have

$$H^1(s_3s_4s_2s_3s_4s_1s_2, \alpha_2) = C_-(-\alpha_2+\alpha_3).$$

Since $(-\alpha_2+\alpha_3, \alpha_4) = 1$, by using Lemma 2.3(2), we have

$$H^0(s_4, H^1(s_3s_4s_2s_3s_4s_1s_2, \alpha_2)) = C_-(-\alpha_2+\alpha_3) \oplus C_-(-\alpha_2+\alpha_3+\alpha_4).$$

Recall from (5.7.1) that

$$H^0(s_4s_2s_3s_4s_1s_2, \alpha_2) = C_-(-\alpha_2+2\alpha_3) \oplus C_-(-\alpha_2+2\alpha_3+\alpha_4) \oplus C_-(-\alpha_2+2\alpha_3+2\alpha_4) \oplus C_-(-\alpha_1+\alpha_2+\alpha_3+\alpha_4) \oplus C_-(-\alpha_1+\alpha_2+2\alpha_3+\alpha_4) \oplus C_-(-\alpha_1+\alpha_2+2\alpha_3+2\alpha_4) \oplus C_-\alpha_4 \oplus C_-\alpha_4.$$
Therefore, by using SES, we have

\[ \langle -\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \alpha_3 \rangle = 0, \langle -\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \alpha_3 \rangle = 1, \langle -\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \alpha_3 \rangle = 0, \langle -\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \alpha_3 \rangle = 1, \langle -\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \alpha_3 \rangle = 1, \langle -\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \alpha_3 \rangle = 0, \langle -\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \alpha_3 \rangle = 0, \langle -\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \alpha_3 \rangle = 0. \]

On the other hand, from (5.7.2), we have

\[ \hat{B}_{\alpha_3} \text{-indecomposable summands} \ V \text{ of } H^0(s_3 s_4 s_2 s_3 s_4 s_1 s_2, \alpha_2) \text{ for which } H^1(s_4, V) \neq 0 \text{ is } C_{-(\alpha_2 + 2\alpha_3 + 2\alpha_4)}. \text{ Thus by using SES and Lemma } 2.3(3), \text{ we have} \]

\[ H^1(s_4, H^0(s_3 s_4 s_2 s_3 s_4 s_1 s_2, \alpha_2)) = C_{-(\alpha_2 + 2\alpha_3 + \alpha_4)}. \]

Therefore, by using SES, we have

\[ H^1(s_4 s_3 s_4 s_2 s_3 s_4 s_1 s_2, \alpha_2) = C_{-(\alpha_2 + \alpha_3)} \oplus C_{-(\alpha_2 + \alpha_3 + \alpha_4)} \oplus C_{-(\alpha_2 + 2\alpha_3 + \alpha_4)} \]

**Proof of (3).** Recall that from Corollary 5.7(2), we have

\[ H^1(s_3 s_4 s_2 s_3 s_4 w_1, \alpha_2) = C_{-(\alpha_1 + \alpha_2 + \alpha_3)} \oplus C_{-(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)}. \]

Since \( C_{-(\alpha_1 + \alpha_2 + \alpha_3)} \oplus C_{-(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)} \) is the standard two-dimensional irreducible \( \hat{L}_{\alpha_3} \)-module, \( \langle -\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_4 \rangle = 0 \), \( \langle -\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \alpha_4 \rangle = 0 \) and \( \langle -\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \alpha_4 \rangle = 1 \), by using SES and Lemma 2.3, we have

\[ H^0(s_4, H^1(s_3 s_4 s_2 s_3 s_4 w_1, \alpha_2)) = C_{-(\alpha_1 + \alpha_2 + \alpha_3)} \oplus C_{-(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)}. \]

On the other hand, from (5.7.2), we have

\[ H^0(s_4 s_2 s_3 s_4 w_1, \alpha_2) = C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)}. \]

Since \( C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)} = V \otimes C_{-\alpha_3} \), where \( V \) is the standard two-dimensional irreducible \( \hat{L}_{\alpha_3} \)-module, \( \langle -\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_3 \rangle = 0 \) and \( \langle -\alpha_1 + 2\alpha_2 + 4\alpha_3 + \alpha_4 . \]
2\alpha_2 + 3\alpha_3 + \alpha_4, \alpha_3) = -1, \text{ by using SES and Lemma 2.3, we have}

\[ H^0(s_3 s_4 s_2 s_3 s_4 w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}. \]

Since \((-\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4), \alpha_4) = 0, \text{ by using SES and Lemma 2.3, we have}

\[ H^1(s_4, H^0(s_3 s_4 s_2 s_3 s_4 w_1, \alpha_2)) = 0. \]

Therefore by SES, we have

\[ H^1(s_4 s_3 s_4 s_2 s_3 s_4 w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}. \]

**Proof of (4).** For \( r = 2 \), we recall that from (5.1.14) that \( H^0(s_2 s_3 s_4 w_2, \alpha_2) = \mathbb{C}_{-\omega_1}. \) Since \( \alpha_4, \alpha_3 \) are orthogonal to \( \omega_1 \), by using SES, we have \( H^0(s_3 s_3 s_4 s_4 w_2, \alpha_2) = \mathbb{C}_{-\omega_1}. \) Further, using the orthogonality of \( \alpha_4 \) and \( \omega_1 \), we have \( H^1(s_4, H^0(s_3 s_3 s_4 s_4 w_2, \alpha_2)) = 0. \) On the other hand, by Corollary 5.7(3), we have \( H^0(s_4, H^1(s_3 s_4 s_4 s_3 s_4 w_2, \alpha_2)) = 0. \) Thus we have \( H^1(s_4 s_3 s_4 s_2 s_3 s_4 w_2, \alpha_2) = 0. \) For \( r = 3, \) by Lemma 4.1, we have \( H^0(w_3, \alpha_2) = 0. \) Therefore, \( H^0(s_3 s_3 s_4 s_2 s_3 s_4 w_3, \alpha_2) = 0. \) Hence we have \( H^1(s_4, H^0(s_3 s_4 s_2 s_3 s_4 w_3, \alpha_2)) = 0. \) On the other hand, by Corollary 5.7(4), we have \( H^0(s_4, H^1(s_3 s_4 s_2 s_3 s_4 w_3, \alpha_2)) = 0. \) Thus by using SES, we have \( H^1(s_4 s_3 s_4 s_2 s_3 s_4 w_r, \alpha_2) = 0. \)

We denote \( v_r = [1, 4]^r \) for \( 1 \leq r \leq 6 \) and \( \tau_r = [1, 4]^r \) for \( 1 \leq r \leq 5. \)

**Lemma 5.9.** We have

1. \( H^i(\tau_r, \alpha_1) = 0 \) for all \( i \geq 0, 1 \leq r \leq 5. \)
2. \( H^i(v_r, \alpha_4) = 0 \) for all \( i \geq 0, 2 \leq r \leq 6. \)

**Proof.**

**Proof of (1).** By [15, Corollary 6.4, p. 780], we have

\[ H^i(\tau_r, \alpha_1) = 0 \] for all \( i \geq 2, r \geq 1. \)

Note that \( H^i(s_1 s_2 s_3 s_4 s_1, \alpha_1) = H^i(s_1 s_2 s_1, \alpha_1) = H^i(s_2 s_1 s_2, \alpha_1) = 0 \) for \( i = 0, 1 \) (see Lemma 2.3(4)). Now by using SES repeatedly, we have the required result.

**Proof of (2).** By [15, Corollary 6.4, p. 780], we have

\[ H^i(v_r, \alpha_4) = 0 \] for all \( i \geq 2, r \geq 1. \)

We note that

\[ H^i(s_4 s_1 s_2 s_3 s_4, \alpha_4) = H^i(s_1 s_2 s_4 s_3 s_4, \alpha_4) = H^i(s_1 s_2 s_3 s_4 s_3, \alpha_4) = 0 \]

for \( i = 0, 1 \) (see Lemma 2.3(4)).

Since \( 2 \leq r \leq 6, \) we have \( v_r = u s_4 s_1 s_2 s_3 s_4 \) for some \( u \in W \) such that \( l(v_r) = l(u) + 5. \) Thus by using SES repeatedly, we have the required result. \( \square \)
COROLLARY 5.10

We have the following:

(1) $H^i(s_4 \tau_r, \alpha_1) = 0$ for $i \geq 0$, $1 \leq r \leq 5$.

$H^i(s_4 v_r, \alpha_4) = 0$ for $i \geq 0$, $1 \leq r \leq 5$.

(2) $H^i(s_3 s_4 \tau_r, \alpha_1) = 0$ for $i \geq 0$, $1 \leq r \leq 5$.

$H^i(s_3 s_4 v_r, \alpha_4) = 0$ for $i \geq 0$, $1 \leq r \leq 5$.

(3) $H^i(s_2 s_3 s_4 \tau_r, \alpha_1) = 0$ for $i \geq 0$, $1 \leq r \leq 5$.

$H^i(s_2 s_3 s_4 v_r, \alpha_4) = 0$ for $i \geq 0$, $1 \leq r \leq 5$.

(4) $H^i(s_4 s_3 s_4 \tau_r, \alpha_1) = 0$ for $i \geq 0$, $1 \leq r \leq 4$.

$H^i(s_4 s_3 s_4 v_r, \alpha_4) = 0$ for $i \geq 0$, $1 \leq r \leq 4$.

(5) $H^i(s_4 s_2 s_3 s_4 \tau_r, \alpha_1) = 0$ for $i \geq 0$, $1 \leq r \leq 3$.

$H^i(s_4 s_2 s_3 s_4 v_r, \alpha_4) = 0$ for $i \geq 0$, $1 \leq r \leq 3$.

Proof.

Proof of (1). By using SES and Lemma 5.9(1), we have $H^i(s_4 \tau_r, \alpha_1) = 0$ for all $1 \leq r \leq 5$, $i \geq 0$.

By (5.9.1), we have $H^i(s_4 v_1, \alpha_4) = 0$ for $i \geq 0$. On the other hand, by using SES and Lemma 5.9(2), we have $H^i(s_4 v_r, \alpha_4) = 0$ for $i \geq 0$, $2 \leq r \leq 5$. Thus, by combining, we have $H^i(s_4 v_r, \alpha_4) = 0$ for $i \geq 0$, $1 \leq r \leq 5$.

Proofs of (2), (3), (4), (5), and (6). Follow by using SES and (1).

6. Surjectivity of some maps

Let $w \in W$ and let $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced expression for $w$ and let $\underline{i} = (i_1, i_2, \ldots, i_r)$. Let $\tau = s_{i_1} s_{i_2} \cdots s_{i_{r-1}}$ and $\underline{i} = (i_1, i_2, \ldots, i_{r-1})$.

Recall the following long exact sequence of $B$-modules from [13] (see [13, Proposition 3.1, p. 673]):

$$
0 \to H^0(w, \alpha_i_r) \to H^0(Z(w, \underline{i}), T(w, \underline{i})) \to H^0(Z(\tau, \underline{i}_r), T(\tau, \underline{i}_r)) \to \\
H^1(w, \alpha_i_r) \to H^1(Z(w, \underline{i}), T(w, \underline{i})) \to H^1(Z(\tau, \underline{i}_r), T(\tau, \underline{i}_r)) \to \\
T(\tau, \underline{i}_r) \to H^2(w, \alpha_i_r) \to \\
H^2(Z(w, \underline{i}), T(w, \underline{i})) \to H^2(Z(\tau, \underline{i}_r), T(\tau, \underline{i}_r)) \to H^3(w, \alpha_i_r) \to \cdots.
$$

By [15, Corollary 6.4, p. 780], we have $H^j(w, \alpha_i_r) = 0$ for every $j \geq 2$. Thus we have the following exact sequence of $B$-modules:

$$
0 \to H^0(w, \alpha_i_r) \to H^0(Z(w, \underline{i}), T(w, \underline{i})) \to H^0(Z(\tau, \underline{i}_r), T(\tau, \underline{i}_r)) \to \\
H^1(w, \alpha_i_r) \to H^1(Z(w, \underline{i}), T(w, \underline{i})) \to H^1(Z(\tau, \underline{i}_r), T(\tau, \underline{i}_r)) \to 0.
$$

From now onwards, we call this exact sequence by LES.

Let $w_0 = s_{j_1} s_{j_2} \cdots s_{j_N}$ be a reduced expression of $w_0$. Let $w = s_{j_1} s_{j_2} \cdots s_{j_r}$, $\underline{i} = (j_1, j_2, \ldots, j_r)$, and $\underline{j} = (j_1, j_2, \ldots, j_N)$. 

Lemma 6.1. The natural homomorphism 
\[ f : H^0(Z(w_0, j), T_{(w_0, j)}) \rightarrow H^0(Z(w, i), T_{(w, i)}) \]
of $B$-modules is injective if and only if $w^{-1}(\alpha_0) < 0$.

Proof. Suppose $w^{-1}(\alpha_0) < 0$. By [13, Lemma 6.2, p. 667], we have $H^0(Z(w, i), T_{(w, i)}) - \alpha_0 = f$ is injective. Then by [13, Theorem 7.1], $H^0(Z(w_0, i), T_{(w_0, i)})$ is a parabolic subalgebra of $g$ and hence there is a unique $B$-stable line in $H^0(Z(w_0, i), T_{(w_0, i)})$, namely $g - \alpha_0$. Therefore, we conclude that the natural homomorphism 
\[ H^0(Z(w_0, i), T_{(w_0, i)}) \rightarrow H^0(Z(w, i), T_{(w, i)}) \]
is injective.

Conversely, suppose the natural homomorphism 
\[ H^0(Z(w_0, i), T_{(w_0, i)}) \rightarrow H^0(Z(w, i), T_{(w, i)}) \]
is injective. Then by [13, Lemma 6.2, p. 667], we have $w^{-1}(\alpha_0) < 0$. \qed

Lemma 6.2. The natural homomorphism 
\[ f : H^1(Z(w_0, j), T_{(w_0, j)}) \rightarrow H^1(Z(w, i), T_{(w, i)}) \]
of $B$-modules is surjective.

Proof. See [14, Lemma 7.1, p. 459]. For $1 \leq r \leq 5$, let $\tau_r$ be the reduced expression of $\tau_r = [1, 4]'s_1, i_r = (j_r, 2)$ be the reduced expression of $w_r = [1, 4]'s_1s_2$ and $j_r = (i_r, 3)$ be the reduced expression of $w_r s_3 = [1, 4]'s_1s_2s_3$. \qed

Lemma 6.3.

(1) We have $\dim H^0(Z(\tau_4, j_4), T_{(\tau_4, j_4)}) - \omega_4 = 2$. Further, the natural map 
\[ H^0(Z(\tau_4, j_4), T_{(\tau_4, j_4)}) \rightarrow H^1(w_4, \alpha_2) \]
is surjective.

(2) We have $\dim H^0(Z(\tau_3, j_3), T_{(\tau_3, j_3)}) - \omega_4 + \alpha_4 = 2$. Further, the natural map 
\[ H^0(Z(\tau_3, j_3), T_{(\tau_3, j_3)}) \rightarrow H^1(w_3, \alpha_2) \]
is surjective.

Proof.

Proof of (1). Since $w_4^{-1}(\alpha_0) < 0$, by Lemma 6.1, we conclude that the natural homomorphism 
\[ H^0(Z(w_0, i), T_{(w_0, i)}) \rightarrow H^0(Z(w_4, i_4), T_{(w_4, i_4)}) \]
is injective.

Since $\alpha_3$ is a short simple root, by [15, Corollary 5.6, p. 778] we have $H^1(w_r s_3, \alpha_3) = 0$ for $r = 4, 5$. On the other hand, by Lemma 5.1, we have $H^1(w_5, \alpha_2) = 0$ and by Lemma 5.9, $H^1(w_r, \alpha_4) = 0$ and $H^1(\tau_r, \alpha_1) = 0$ for $r = 4, 5$. 

Proof of (2).
Thus from above observations and using LES the natural map
\[ H^0(Z(w_0, i), T_{(w_0, i)}) \rightarrow H^0(Z(w_4, i^4), T_{(w_4, i^4)}) \]  \hspace{1cm} (6.3.1)
is surjective, hence an isomorphism.

By [13, Theorem 7.1], \( H^0(Z(w_0, i), T_{(w_0, i)}) \) is a parabolic subalgebra of \( g \). Hence for any \( \mu \in X(T) \setminus \{0\} \), we have
\[
\dim H^0(Z(w_0, i), T_{(w_0, i)}) \mu \leq 1.
\]

By using LES repeatedly and using Lemma 5.9, we have
\[ H^0(Z(\tau_4, j_4), T_{(\tau_4, j_4)}) = H^0(Z(w_3s_3, l_3), T_{(l_3, j_3)}). \] \hspace{1cm} (6.3.2)

By using LES and [15, Corollary 5.6, p. 778] we have an exact sequence
\[
0 \rightarrow H^0(w_3s_3, \alpha_3) \rightarrow H^0(Z(w_3s_3, l_3), T_{(w_3s_3, l_3)}) \rightarrow H^0(Z(w_3, i_3), T_{(w_3, i_3)}) \rightarrow 0
\] \hspace{1cm} (6.3.3)
of \( B \)-modules.

Since \( w_3^{-1}(\alpha_0) < 0 \), by using Lemma 6.1, we conclude that the natural homomorphism
\[ H^0(Z(w_0, i), T_{(w_0, i)}) \rightarrow H^0(Z(w_3, i_3), T_{(w_3, i_3)}) \] \hspace{1cm} (6.3.4)
is injective.

Thus by (6.3.4), we have \( \dim H^0(Z(w_3, i_3), T_{(w_3, i_3)}) - \omega_4 \geq 1. \) Hence by Lemma 4.2(2) and (6.3.3), we have
\[
\dim H^0(Z(w_3s_3, l_3), T_{(w_3s_3, l_3)}) - \omega_4 \geq 2. \] \hspace{1cm} (6.3.5)

By (6.3.1), we have \( \dim H^0(Z(w_4, i^4), T_{(w_4, i^4)}) - \omega_4 \leq 1. \) Therefore, by using LES, we see that \( \dim H^0(Z(\tau_4, j_4), T_{(\tau_4, j_4)}) - \omega_4 \leq 2. \)

Thus by (6.3.2) and (6.3.5), we have \( \dim H^0(\tau_4, j_4) - \omega_4 = 2. \) Therefore, by LES, the natural map \( H^0(\tau_4, j_4) - \omega_4 \rightarrow H^1(w_4, \alpha_2 - \omega_4) \) is surjective. Hence by Lemma 5.1(3), the natural map \( H^0(\tau_4, j_4) \rightarrow H^1(w_4, \alpha_2) \) is surjective.

**Proof of (2).** By using LES repeatedly and using Lemma 5.9, we have
\[ H^0(Z(\tau_3, j_3), T_{(\tau_3, j_3)}) = H^0(Z(w_2s_3, l_2), T_{(w_2s_3, l_2)}). \] \hspace{1cm} (6.3.6)

By using LES and [15, Corollary 5.6, p. 778], we have an exact sequence
\[
0 \rightarrow H^0(w_2s_3, \alpha_3) \rightarrow H^0(Z(w_2s_3, l_2), T_{(w_2s_3, l_2)}) \rightarrow H^0(Z(w_2, i_2), T_{(w_2, i_2)}) \rightarrow 0
\] \hspace{1cm} (6.3.7)
of \( B \)-modules.
Since \( w_2^{-1}(\alpha_0) < 0 \), by using Lemma 6.1, we conclude that the natural homomorphism

\[
H^0(Z(w_0, i), T_{(w_0, i)}) \to H^0(Z(w_2, i_2), T_{(w_2, i_2)})
\]

is injective. Thus by (6.3.8), we have \( \dim H^0(Z(w_2, i_2), T_{(w_2, i_2)})_{-\alpha_4 + \alpha_4} \geq 1 \). Hence by Lemma 4.2(1) and (6.3.7), we have

\[
\dim H^0(Z(w_2, i_2), T_{(w_2, i_2)})_{-\alpha_4 + \alpha_4} \geq 2.
\]

By Lemma 5.1, we have \( H^1(w_4, \alpha_2)_{-\alpha_4 + \alpha_4} = 0 \). Since \( \alpha_3 \) is a short simple root, by [15, Corollary 5.6, p. 778], we have \( H^1(w_3, \alpha_3) = 0 \). On the other hand, by Lemma 5.9, we have \( H^1(w_4, \alpha_4) = 0 \) and \( H^1(\tau_4, \alpha_1) = 0 \). Thus by using LES and from the above discussion we have the natural map

\[
H^0(Z(w_4, i_4), T_{(w_4, i_4)})_{-\alpha_4 + \alpha_4} \to H^0(Z(w_3, i_3), T_{(w_3, i_3)})_{-\alpha_4 + \alpha_4}
\]

is surjective. Thus by using (6.3.1) and the above surjectivity we have \( \dim H^0(Z(w_3, i_3), T_{(w_3, i_3)})_{-\alpha_4 + \alpha_4} \leq 1 \). Therefore, by using LES, we see that \( \dim H^0(Z(\tau_3, j_3), T_{(\tau_3, j_3)})_{-\alpha_4 + \alpha_4} \leq 2 \). Thus by (6.3.6) and (6.3.9), we have \( H^0(Z(\tau_3, j_3), T_{(\tau_3, j_3)})_{-\alpha_4 + \alpha_4} = 2 \). Therefore, by LES, the natural map \( H^0(Z(\tau_3, j_3), T_{(\tau_3, j_3)})_{-\alpha_4 + \alpha_4} \to H^1(w_3, \alpha_2)_{-\alpha_4 + \alpha_4} \) is surjective. Hence by Lemma 5.1(2), the natural map \( H^0(Z(\tau_3, j_3), T_{(\tau_3, j_3)}) \to H^1(w_3, \alpha_2) \) is surjective.

\[\square\]

Lemma 6.4.

(1) Let \( \mu = -\alpha_4, -\omega_4 + \alpha_4 \). Then we have \( \dim H^0(Z(s_4\tau_3, (4, j_3)), T_{(s_4\tau_3, (4, j_3))})_\mu = 2 \). Further, the natural map \( H^0(Z(s_4\tau_3, (4, j_3)), T_{(s_4\tau_3, (4, j_3))}) \to H^1(s_4\tau, \alpha_2) \) is surjective.

(2) Let \( \mu = -(\alpha_2 + 2\alpha_3 + \alpha_4), -(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4) \). Then we have \( \dim H^0(Z(s_4\tau_2, (4, j_3)), T_{(s_4\tau_2, (4, j_3))})_\mu = 2 \). Further, the natural map \( H^0(Z(s_4\tau_2, (4, j_3)), T_{(s_4\tau_2, (4, j_3))}) \to H^1(s_4\tau_2, \alpha_2) \) is surjective.

\[\text{Proof.}\] Since \( (s_4w_3)^{-1}(\alpha_0) < 0 \), by Lemma 6.1, we conclude that the natural homomorphism

\[
H^0(Z(w_0, (4, l_5)), T_{(w_0, (4, l_5))}) \to H^0(Z(s_4w_3, (4, i_3)), T_{(s_4w_3, (4, i_3))})
\]

is injective. Since \( \alpha_3 \) is a short simple root, by [15, Corollary 5.6, p. 778] we have \( H^1(s_4w_r, \omega_3) = 0 \) for \( r = 3, 4, 5 \). On the other hand, by Corollary 5.2, we have \( H^1(s_4w_r, \alpha_2) = 0 \) for \( r = 4, 5 \), and by Corollary 5.10(1), we have \( H^1(s_4w_r, \alpha_4) = 0 \) and \( H^1(s_4\tau_r, \alpha_1) = 0 \) for \( r = 4, 5 \). Thus from the above observations and using LES, the natural map

\[
H^0(Z(w_0, (4, l_5)), T_{(w_0, (4, l_5))}) \to H^0(Z(s_4w_3, (4, i_3)), T_{(s_4w_3, (4, i_3))})
\]
is surjective, hence an isomorphism.

**Proof of (1).** By using LES repeatedly and using Corollary 5.10(1), we have

\[ H^0(Z(s_4 \tau_3, (4, j_3)), T(s_4 \tau_3, (4, j_3))) = H^0(Z(s_4 w_2 s_3, (4, l_2)), T(s_4 w_2 s_3, (4, l_2))). \]  

(6.4.2)

By using LES and [15, Corollary 5.6, p. 778], we have an exact sequence

\[ 0 \to H^0(s_4 w_2 s_3, \alpha_3) \to H^0(Z(s_4 w_2 s_3, (4, l_2)), T(s_4 w_2 s_3, (4, l_2))) \]

\[ \to H^0(Z(s_4 w_2, (4, i_2)), T(s_4 w_2, (4, i_2))) \to 0 \]  

(6.4.3)

of \(B\)-modules. On the other hand, since \((s_4 w_2)^{-1}(\alpha_0) < 0\), by using Lemma 6.1, we conclude that the natural homomorphism

\[ H^0(Z(w_0, (4, i_2)), T(w_0, (4, i_2))) \to H^0(Z(s_4 w_2, (4, i_2)), T(s_4 w_2, (4, i_2))) \]  

(6.4.4)

is injective.

Let \(\mu = -\omega_4, -\omega_4 + \alpha_4\). Thus by (6.4.4), we have \(\dim H^0(Z(s_4 w_2, (4, i_2)), T(s_4 w_2, (4, i_2)))_\mu \geq 1\). Hence by (6.4.2) and by Corollary 4.3(2), we have

\[ \dim H^0(Z(s_4 w_2 s_3, (4, l_2)), T(s_4 w_2 s_3, (4, l_2)))_\mu \geq 2. \]  

(6.4.5)

By (6.4.1), \(\dim H^0(Z(s_4 w_3, (4, i_3)), T(s_4 w_3, (4, i_3)))_\mu \leq 1\).

By using LES, we have \(\dim H^0(Z(s_4 \tau_3, (4, j_3)), T(s_4 \tau_3, (4, j_3)))_\mu \leq 2\) Thus by (6.4.2) and (6.4.5), we have \(\dim H^0(Z(s_4 \tau_3, (4, j_3)), T(s_4 \tau_3, (4, j_3)))_\mu = 2\). Therefore, by LES, the natural map \(H^0(Z(s_4 \tau_3, (4, j_3)), T(s_4 \tau_3, (4, j_3)))_\mu \to H^1(s_4 w_3, \alpha_2)_\mu\) is surjective. Hence by Corollary 5.2(4), the natural map \(H^0(Z(s_4 \tau_3, (4, j_3)), T(s_4 \tau_3, (4, j_3))) \to H^1(s_4 w_3, \alpha_2)\) is surjective.

**Proof of (2).** By using LES repeatedly and using Corollary 5.10(1), we have

\[ H^0(Z(s_4 \tau_2, (4, j_2)), T(s_4 \tau_2, (4, j_2))) = H^0(Z(s_4 w_1 s_3, (4, l_1)), T(s_4 w_1 s_3, (4, l_1))). \]  

(6.4.6)

By using LES and [15, Corollary 5.6, p. 778], we have an exact sequence

\[ 0 \to H^0(s_4 w_1 s_3, \alpha_3) \to H^0(Z(s_4 w_1 s_3, (4, l_1)), T(s_4 w_1 s_3, (4, l_1))) \]

\[ \to H^0(Z(s_4 w_1, (4, i_1)), T(s_4 w_1, (4, i_1))) \to 0 \]  

(6.4.7)

of \(B\)-modules.

Let \(\mu = -2\alpha_3 + \alpha_4, -(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)\). Since \(H^1(s_4 w_2, \alpha_2)_\mu \neq 0\), by Corollary 5.2 and (5.1.5), the same weight appears in \(H^0(s_4 w_1, \alpha_2)\), i.e. \(H^0(s_4 w_1, \alpha_2)_\mu \neq 0\). This implies \(H^0(Z(s_4 w_1, (4, i_1)), T(s_4 w_1, (4, i_1)))_\mu \neq 0\).

Thus by (6.4.7) and Corollary 4.3(1), we have

\[ \dim H^0(Z(s_4 w_1 s_3, (4, l_1)), T(s_4 w_1 s_3, (4, l_1)))_\mu \geq 2. \]  

(6.4.8)
Since $H^1(s_4w_2, \alpha_2)_\mu \neq 0$, by Corollary 5.2, we have $H^1(s_4w_3, \alpha_2)_\mu = 0$. Since $\alpha_3$ is a short simple root, by [15, Corollary 5.6, p. 778], we have $H^1(s_4w_2s_3, \alpha_3) = 0$. On the other hand, by using Corollary 5.10(1), we have $H^1(s_4w_3, \alpha_4) = 0$ and $H^1(s_4w_3, \alpha_1) = 0$. Thus by using LES and from the above discussion, we have the natural map

$$H^0(Z(s_4w_3, (4, \underline{j}_3)), T_{(s_4w_3, (4, j_3))})_\mu \longrightarrow H^0(Z(s_4w_2, (4, i_2)), T_{(s_4w_2, (4, i_2))})_\mu$$

is surjective.

By (6.4.1) and the above surjectivity, we have $\dim H^0(Z(s_4w_2, (4, i_2)), T_{(s_4w_2, (4, i_2))})_\mu \leq 1$.

By using LES, we see that $\dim H^0(Z(s_4w_2, (4, j_2)), T_{(s_4w_2, (4, j_2))})_\mu \leq 2$. Thus by (6.4.6) and (6.4.8), we have $\dim H^0(Z(s_4w_2, (4, j_2)), T_{(s_4w_2, (4, j_2))})_\mu = 2$. Therefore, by LES, the natural map $H^0(Z(s_4w_2, (4, j_2)), T_{(s_4w_2, (4, j_2))})_\mu \longrightarrow H^1(s_4w_2, \alpha_2)_\mu$ is surjective. Hence by Corollary 5.2(3), the natural map $H^0(Z(s_4w_2, (4, j_2)), T_{(s_4w_2, (4, j_2))}) \longrightarrow H^1(s_4w_2, \alpha_2)$ is surjective.

Lemma 6.5.

(1) We have $\dim H^0(Z(s_3s_4w_3, (3, 4, \underline{3})), T_{(s_3s_4w_3, (3, 4, j_3))})_{-\alpha_4} = 2$. Further, the natural map $H^0(Z(s_3s_4w_3, (3, 4, j_3)), T_{(s_3s_4w_3, (3, 4, j_3))}) \longrightarrow H^1(s_3s_4w_3, \alpha_2)_\mu$ is surjective.

(2) Let $\mu = -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)$, $-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)$. Then we have $\dim H^0(Z(s_3s_4w_2, (3, 4, \underline{3})), T_{(s_3s_4w_2, (3, 4, j_3))})_\mu = 2$. Further, the natural map $H^0(Z(s_3s_4w_2, (3, 4, j_3)), T_{(s_3s_4w_2, (3, 4, j_3))}) \longrightarrow H^1(s_3s_4w_2, \alpha_2)_\mu$ is surjective.

(3) Let $\mu = -(\alpha_2 + \alpha_3), -(\alpha_1 + \alpha_2 + \alpha_3)$. Then we have $\dim H^0(Z(s_3s_4w_1, (3, 4, \underline{1})), T_{(s_3s_4w_1, (3, 4, j_1))})_\mu = 2$. Further, the natural map $H^0(Z(s_3s_4w_1, (3, 4, j_1)), T_{(s_3s_4w_1, (3, 4, j_1))}) \longrightarrow H^1(s_3s_4w_1, \alpha_2)_\mu$ is surjective.

Proof. The proofs of Lemma 6.5(1), Lemma 6.5(2) and Lemma 6.5(3) are similar to that of Lemma 6.4 by using [15, Corollary 5.6, p. 778], Corollary 5.3 and Corollary 5.10(2) appropriately.

Lemma 6.6.

(1) We have $\dim H^0(Z(s_2s_3s_4w_3, (2, 3, 4, \underline{3})), T_{(s_2s_3s_4w_3, (2, 3, 4, j_3))})_{-\alpha_4} = 2$. Further, the natural map $H^0(Z(s_2s_3s_4w_3, (2, 3, 4, j_3)), T_{(s_2s_3s_4w_3, (2, 3, 4, j_3))}) \longrightarrow H^1(s_2s_3s_4w_3, \alpha_2)_\mu$ is surjective.

(2) We have $\dim H^0(Z(s_2s_3s_4w_2, (2, 3, 4, \underline{3})), T_{(s_2s_3s_4w_2, (2, 3, 4, j_3))})_{-\alpha_4+\alpha_3} = 2$. Further, the natural map $H^0(Z(s_2s_3s_4w_2, (2, 3, 4, j_3)), T_{(s_2s_3s_4w_2, (2, 3, 4, j_3))}) \longrightarrow H^1(s_2s_3s_4w_2, \alpha_2)_\mu$ is surjective.

(3) We have $\dim H^0(Z(s_2s_3s_4w_1, (2, 3, 4, \underline{1})), T_{(s_2s_3s_4w_1, (2, 3, 4, j_1))})_{-(\alpha_1+\alpha_2+\alpha_3)} = 2$. Further, the natural map $H^0(Z(s_2s_3s_4w_1, (2, 3, 4, j_1)), T_{(s_2s_3s_4w_1, (2, 3, 4, j_1))}) \longrightarrow H^1(s_2s_3s_4w_1, \alpha_2)_\mu$ is surjective.

Proof. The proofs of Lemma 6.6(1), Lemma 6.6(2) and Lemma 6.6(3) are similar to that of Lemma 6.4 by using [15, Corollary 5.6, p. 778], Corollary 5.4 and Corollary 5.10(3) appropriately.
Lemma 6.7.

(1) Let \( \mu = - (\alpha_1 + 2 \alpha_2 + 2 \alpha_3 + \alpha_4), -\omega_4 + \alpha_4, -\omega_4. \) Then we have \( \dim H^0(Z(s_4 s_3 s_4 \tau_2, (4, 3, 4, j_2)), T_{(s_4 s_3 s_4 \tau_2, (4, 3, 4, j_2))} \mu = 2. \) Further, the natural map \( H^0(Z(s_4 s_3 s_4 \tau_2, (4, 3, 4, j_2)), T_{(s_4 s_3 s_4 \tau_2, (4, 3, 4, j_2))}) \rightarrow H^1(s_4 s_3 s_4 w_2, \alpha_2) \) is surjective.

(2) Let \( \mu = -(\alpha_2 + \alpha_3), -\omega_2 + \alpha_3, -\omega_2 + \alpha_3). \) Then we have \( \dim H^0(Z(s_4 s_3 s_4 \tau_1, (4, 3, 4, j_1)), T_{(s_4 s_3 s_4 \tau_1, (4, 3, 4, j_1))} \mu = 2. \) Further, the natural map \( H^0(Z(s_4 s_3 s_4 \tau_1, (4, 3, 4, j_1)), T_{(s_4 s_3 s_4 \tau_1, (4, 3, 4, j_1))}) \rightarrow H^1(s_4 s_3 s_4 w_1, \alpha_2) \) is surjective.

Proof. The proofs of Lemma 6.7(1) and Lemma 6.7(2) are similar to that of Lemma 6.4 by using [15, Corollary 5.6, p. 778], Corollary 5.5 and Corollary 5.10(4) appropriately.

Lemma 6.8.

(1) Let \( \mu = -\omega_4 + \alpha_4, -\omega_4. \) Then we have \( \dim H^0(Z(s_4 s_2 s_3 s_4 \tau_2, (4, 2, 3, 4, j_2)), T_{(s_4 s_2 s_3 s_4 \tau_2, (4, 2, 3, 4, j_2))} \mu = 2. \) Further, the natural map \( H^0(Z(s_4 s_2 s_3 s_4 \tau_2, (4, 2, 3, 4, j_2)), T_{(s_4 s_2 s_3 s_4 \tau_2, (4, 2, 3, 4, j_2))}) \rightarrow H^1(s_4 s_2 s_3 s_4 w_2, \alpha_2) \) is surjective.

(2) Let \( \mu = -(\alpha_1 + 2 \alpha_2 + 2 \alpha_3 + \alpha_4), -(\alpha_1 + 2 \alpha_2 + 3 \alpha_3 + \alpha_4), -(\alpha_1 + 2 \alpha_2 + 3 \alpha_3 + \alpha_4), -(\alpha_1 + 2 \alpha_2 + 3 \alpha_3 + \alpha_4). \) Then we have \( \dim H^0(Z(s_4 s_2 s_3 s_4 \tau_1, (4, 3, 4, j_2)), T_{(s_4 s_2 s_3 s_4 \tau_1, (4, 3, 4, j_1))} \mu = 2. \) Further, the natural map \( H^0(Z(s_4 s_2 s_3 s_4 \tau_1, (4, 2, 3, 4, j_1)), T_{(s_4 s_2 s_3 s_4 \tau_1, (4, 2, 3, 4, j_1))}) \rightarrow H^1(s_4 s_2 s_3 s_4 w_1, \alpha_2) \) is surjective.

Proof. Proofs of Lemma 6.8(1) and Lemma 6.8(2) are similar to that of Lemma 6.4 by using [15, Corollary 5.6, p. 778], Corollary 5.6 and Corollary 5.10(5) appropriately.

Lemma 6.9. Let \( j'_1 = (4, 3, 4, 2, 3, 4, j_1) \) and \( j'_2 = (4, 3, 4, 2, 3, 4, 1). \)

(1) Let \( \Lambda = \{ - (\alpha_1 + 2 \alpha_2 + 2 \alpha_3 + \alpha_4), -(\alpha_1 + 2 \alpha_2 + 3 \alpha_3 + \alpha_4), -(\alpha_1 + 2 \alpha_2 + 3 \alpha_3 + 2 \alpha_4). \} \) Then we have \( \dim H^0(Z(s_4 s_3 s_4 s_2 s_3 s_4 \tau_1, j'_1), T_{(s_4 s_3 s_4 s_2 s_3 s_4 \tau_1, j'_1)} \mu = 2 \) for all \( \mu \in \Lambda. \) Further, the natural map \( H^0(Z(s_4 s_3 s_4 s_2 s_3 s_4 \tau_1, j'_1), T_{(s_4 s_3 s_4 s_2 s_3 s_4 \tau_1, j'_1)} \) \( \rightarrow H^1(s_4 s_3 s_4 s_2 s_3 s_4 w_1, \alpha_2) \) is surjective.

(2) Let \( \Pi = \{ -(\alpha_2 + \alpha_3), -(\alpha_2 + 2 \alpha_3 + \alpha_4), -(\alpha_2 + 3 \alpha_3 + \alpha_4). \} \) Then we have \( \dim H^0(Z(s_4 s_3 s_4 s_2 s_3 s_4 \tau_1, j'_1), T_{(s_4 s_3 s_4 s_2 s_3 s_4 \tau_1, j'_1)} \mu = 2 \) for all \( \mu \in \Pi. \) Further, the natural map \( H^0(Z(s_4 s_3 s_4 s_2 s_3 s_4 \tau_1, j'_1), T_{(s_4 s_3 s_4 s_2 s_3 s_4 \tau_1, j'_1)} \) \( \rightarrow H^1(s_4 s_3 s_4 s_2 s_3 s_4 s_1, \alpha_2) \) is surjective.

Proof. Let \( u_1 = s_4 s_3 s_4 s_2 s_3 s_4 \tau_1, \) and \( u = s_4 s_3 s_4 s_2 s_3 s_4 w_1. \) Note that \( u_0 = s_4 s_3 s_4 s_2 s_3 s_4 w_1. \) Let \( i'_1 \) be the reduced expression of \( u_0. \) By Lemma 4.1(2) and Corollary 5.2(2), we have \( H^i(s_4 s_3 s_4 s_2 s_3 s_4 \tau_1, \omega_2) = 0 \) for \( i \geq 0. \) Since \( s_4 \) commutes with \( s_1, s_2, \) we have \( H^i(s_4 s_3 s_4 s_2 s_3 s_4 \tau_2, \omega_2) = H^i(s_4 s_3 s_4 s_2 s_3 s_4 \tau_2, \omega_2) = 0. \) Thus we have \( H^i(s_4 s_3 s_4 s_2 s_3 s_4 s_1, \alpha_1) = 0. \) Therefore by using SES, we have \( H^i(s_4 s_3 s_4 s_2 s_3 s_4 w_1, \alpha_2) = 0. \) Since \( s_3 \) commutes with \( s_1, \) we have \( H^i(s_4 s_3 s_4 s_2 s_3 s_4 w_1, \alpha_1) = H^i(s_4 s_3 s_4 s_2 s_3 s_4 s_1, \alpha_1) \)
for $i \geq 0$, $H^i(s_4s_3s_4s_2s_3s_4w_3s_1, \alpha_1) = H^i(s_4s_3s_4s_2s_3s_4[1, 4]^3s_2s_1s_2, \alpha_1) = 0$ for $i \geq 0$ (see Lemma 2.3(4)). Thus we have $H^i(s_4s_3s_4s_2s_3s_4w_3s_1, \alpha_1) = 0$ for $i \geq 0$. By Lemma 5.8(4), we have $H^1(s_4s_3s_4s_2s_3s_4w_r, \alpha_2) = 0$ for $r = 2, 3$. Since $\alpha_3$ is a short simple root, by [15, Corollary 5.6, p. 778], we have $H^1(s_4s_3s_4s_2s_3s_4w_r, \alpha_3) = 0$ for $r = 1, 2, 3$. On the other hand, by using Corollary 5.10(6), we have $H^1(s_4s_3s_4s_2s_3s_4v_r, \alpha_4) = 0$ and $H^1(s_4s_3s_4s_2s_3s_4\tau_r, \alpha_1) = 0$ for $r = 2, 3$.

Thus by using LES and the above discussion, we have the natural map

$$H^0(Z(w_0, i'), T_{(w_0, i')}) \to H^0(Z(s_4s_3s_4s_2s_3s_4w_1, (j_1, 2)), T_{(s_4s_3s_4s_2s_3s_4w_1, (j_1, 2))})$$

(6.9.1)

which is surjective.

**Proof of (1).** By using LES repeatedly and Corollary 5.10(6), we have

$$H^0(Z(u_1, j'_1), T_{(u_1, j'_1)}) = H^0(Z(u_2s_3), (j'_2, 3)), T_{(u_2s_3, (j'_2, 3))}).$$

(6.9.2)

By using LES and [15, Corollary 5.6, p. 778], we have an exact sequence

$$0 \to H^0(u_2s_3, \alpha_3) \to H^0(Z(u_2s_3), T_{(u_2s_3, (j'_2, 3))}) \to H^0(Z(u_2), T_{(u_2, (j'_2, 3))}) \to 0$$

(6.9.3)

of $B$-modules.

Let $\Lambda_1 = \{-(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)\}$. Let $\Lambda_2 = \{-(\alpha_1 + \alpha_2 + 3\alpha_3), -(\alpha_1 + \alpha_2 + 3\alpha_3 + \alpha_4), -(\alpha_1 + \alpha_2 + 3\alpha_3 + 2\alpha_4)\}$. By (5.8.1), we have

$$H^0(s_3s_4s_2s_3s_4s_1s_2, \alpha_2) = C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)} \oplus C_{-(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)}.$$

Thus by using SES, we see that $H^0(u_2s_2, \alpha_2) \neq 0$ for all $\mu \in \Lambda_1$. By using LES and Lemma 5.8(2), we have an exact sequence

$$0 \to H^0(u_2s_2, \alpha_2) \to H^0(Z(u_2s_2, (j'_2, 2)), T_{(u_2s_2, (j'_2, 2))}) \mu \to H^0(Z(u, j'), T_{(u, (j'))}) \mu \to 0$$

for all $\mu \in \Lambda$.

Note that $H^0(u, \alpha_1) = H^0(s_4s_3s_2s_3s_1, \alpha_1)$. Now it is easy to see that $H^0(s_4s_3s_2s_3s_1, \alpha_1) \neq 0$ for $\mu \in \Lambda_2$. Therefore, we have $H^0(Z(u, j'), T_{(u, j')}) \mu \neq 0$ for all $\mu \in \Lambda_2$. Thus combining the above discussion, we have

$$H^0(Z(u_2s_2, (j'_2, 2)), T_{(u_2s_2, (j'_2, 2))}) \mu \neq 0$$

(6.9.4)
for all $\mu \in \Lambda$. Therefore, by using (6.9.3), (6.9.4) and Corollary 4.9(2), we have

$$\dim H^0(Z(us_2s_3, (j', 2, 3)), T_{(us_2s_3, (j', 2, 3))}) \geq 2$$

(6.9.5)

for all $\mu \in \Lambda$. By (6.9.1), we have $H^0(Z(s_4s_3s_4s_2s_3s_4w_1, (j_1', 2)), T_{(s_4s_3s_4s_2s_3s_4w_1, (j_1', 2))}) \leq 1$ for all $\mu \in \Lambda$.

Therefore by using LES and Lemma 5.8(3), we have $H^0(Z(u_1, j_1'), T_{(u_1, j_1')}) \mu \leq 2$ for all $\mu \in \Lambda$.

Thus by (6.9.2) and (6.9.5), we have $H^0(Z(u_1, j_1'), T_{(u_1, j_1')}) \mu = 2$ for all $\mu \in \Lambda$.

By using LES, we have $H^0(Z(u_1, j_1'), T_{(u_1, j_1')}) \mu \rightarrow H^1(s_4s_3s_4s_2s_3s_4w_1, \alpha_2) \mu$ is surjective for all $\mu \in \Lambda$. Hence by Lemma 5.8(3), the natural map $H^0(Z(u_1, j_1'), T_{(u_1, j_1')}) \rightarrow H^1(s_4s_3s_4s_2s_3s_4w_1, \alpha_2)$ is surjective.

**Proof of (2).** It is easy to see that $H^1(u, \alpha_1) = H^1(s_4s_3s_2s_1, \alpha_1) = 0$ and $H^0(u, \alpha_1) = H^0(s_4s_3s_2s_1, \alpha_1) \mu = 0$ for all $\mu \in \Pi$.

Further, we have $H^i(s_4s_3s_4s_2s_3s_4, \alpha_4) = H^i(s_4s_3s_2s_3s_4, \alpha_3) = 0$ for all $i \geq 0$ (see Lemma 2.3(4)).

From the above discussions and using LES repeatedly, we have

$$H^0(Z(u, j'), T_{(u, j')}) \mu = H^0(Z(s_4s_3s_4s_2s_3, (4, 3, 4, 2, 3)), T_{(s_4s_3s_4s_2s_3, (4, 3, 4, 2, 3))}) \mu$$

(6.9.6)

for all $\mu \in \Pi$.

By using LES and [15, Corollary 5.6, p. 778], we have an exact sequence

$$0 \rightarrow H^0(s_4s_3s_4s_2s_3, \alpha_3) \rightarrow H^0(Z(s_4s_3s_4s_2s_3), T_{(s_4s_3s_4s_2s_3, (4, 3, 4, 2, 3))})$$

$$\rightarrow H^0(Z(s_4s_3s_4s_2, T_{(s_4s_3s_4s_2, (4, 3, 4, 2))}) \rightarrow 0.$$  

(6.9.7)

It is easy to see that $H^0(s_4s_3s_4s_2, \alpha_2) \mu \neq 0$ for all $\mu \in \Pi$. Therefore, we have $H^0(Z(s_4s_3s_4s_2), T_{(s_4s_3s_4s_2, (4, 3, 4, 2))}) \mu \neq 0$ for all $\mu \in \Pi$. Thus from (6.9.7) and Corollary 4.9(1), we have

$$\dim H^0(Z(s_4s_3s_4s_2s_3), T_{(s_4s_3s_4s_2s_3, (4, 3, 4, 2, 3))}) \mu \geq 2$$

for all $\mu \in \Pi$.

Since $\alpha_3$ is a short simple root, by [15, Corollary 5.6, p. 778], we have $H^1(us_2s_3, \alpha_3) = 0$.

By using Corollary 5.10(6), we have

$H^1(s_4s_3s_4s_2s_3s_4t_1, \alpha_1) = 0$ and $H^1(s_4s_3s_4s_2s_3s_4w_1, \alpha_4) = 0$.

By Lemma 5.8, we have $H^1(s_4s_3s_4s_2s_3s_4w_1, \alpha_2) \mu = 0$ for all $\mu \in \Pi$. Thus combining the above discussion, we have the natural map

$$H^0(Z(s_4s_3s_4s_2s_3s_4w_1, (j_1', 2)), T_{(s_4s_3s_4s_2s_3s_4w_1, (j_1', 2))}) \mu \rightarrow H^0(Z(us_2, (j', 2)), T_{(us_2, (j', 2))}) \mu,$$

is surjective for all $\mu \in \Pi$. 


Now, using (6.9.1) and the above surjectivity, we have $H^0(Z(us_2, (j', 2)), T_{(us_2,(j',2))})_\mu \leq 1$ for all $\mu \in \Pi$. Further, by Lemma 5.8(2), $H^1(us_2, \alpha_2)_\mu = 1$ for all $\mu \in \Pi$.

Therefore, by using LES,

$$0 \rightarrow H^0(us_2, \alpha_2) \rightarrow H^0(Z(us_2, (j', 2)), T_{(us_2,(j',2))}) \rightarrow H^1(us_2, \alpha_2) \rightarrow H^1(Z(us_2, (j', 2)), T_{(us_2,(j',2))}) \rightarrow 0,$$

we have $H^0(Z(u, j'), T_{(u,j')})_\mu \leq 2$ for all $\mu \in \Pi$.

Therefore by (6.9.6) and (6.9.8), we have $\dim H^0(Z(s_4 s_3 s_4 s_2 s_3), T_{(s_4 s_3 s_4 s_2 s_3, (4,3,4,2,3))})_\mu = 2$ for all $\mu \in \Pi$.

Therefore, $H^0(Z(u, j'), T_{(u,j')})_\mu \rightarrow H^1(us_2, \alpha_2)_\mu$ is surjective for all $\mu \in \Pi$.

Hence by Lemma 5.8(2), the natural map $H^0(Z(u, j'), T_{(u,j')}) \rightarrow H^1(us_2, \alpha_2)$ is surjective.

\section{Main theorem}

In this section, we prove the main theorem. Let $c$ be a Coxeter element of $W$. Then there exists a decreasing sequence $4 \geq a_1 > a_2 > \cdots > a_k = 1$ of positive integers such that $c = [a_1, 4][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$, where for $i \leq j$ denotes $[i, j] = s_i s_{i+1} \cdots s_j$.

\begin{theorem}
$H^j(Z(w_0, i), T_{(w_0, i)}) = 0$ for all $j \geq 1$ if and only if $a_1 \neq 3$ or $a_2 \neq 2$.
\end{theorem}

\begin{proof}
From [13, Proposition 3.1, p. 673], we have $H^j(Z(w_0, i), T_{(w_0, i)}) = 0$ for all $j \geq 2$. It is enough to prove the following: $H^1(Z(w_0, i), T_{(w_0, i)}) = 0$ if and only if $c$ is of the form $[a_1, 4][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$ with $a_1 \neq 3$ or $a_2 \neq 2$.

\textit{Proof of $(\Rightarrow)$:} If $a_1 = 3$ and $a_2 = 2$, then $c = s_3 s_4 s_2 s_1$. Let $u = s_3 s_4 s_2$. Then $c = us_1$. Let $j = (3, 4, 2)$ be the sequence corresponding to $u$. Then using LES, we have

$$0 \rightarrow H^0(u, \alpha_2) \rightarrow H^0(Z(u, j), T_{(u,j)}) \rightarrow H^0(Z(s_3 s_4, (3, 4)), T_{(s_3 s_4, (3,4))})$$

$$\rightarrow H^1(u, \alpha_2) \rightarrow H^1(Z(u, j), T_{(u,j)}) \rightarrow H^1(Z(s_3 s_4, (3, 4)), T_{(s_3 s_4, (3,4))}) \rightarrow 0.$$

We see that $H^1(u, \alpha_2) = \mathbb{C}_{a_2 + a_3}, H^0(s_3, \alpha_3)_{a_2 + a_3} = 0$ and $H^0(s_3 s_4, \alpha_4)_{a_2 + a_3} = 0$.

Therefore by LES, we have $H^0(Z(s_3 s_4, (3, 4)), T_{(s_3 s_4, (3,4))})_{a_2 + a_3} = 0$. Hence $f$ is a non zero homomorphism. Hence $H^1(Z(u, j), T_{(u,j)}) \neq 0$. By Lemma 6.2, the natural homomorphism

$$H^1(Z(w_0, i), T_{(w_0, i)}) \rightarrow H^1(Z(u, j), T_{(u,j)})$$

is surjective.
Hence we have $H^1(Z(w_0, i), T_{(w_0, i)}) \neq 0$.

Proof of ($\implies$): Assume that $\alpha_1 \neq 3$ or $\alpha_2 \neq 2$. We prove the result by studying case by case. Note that by using Lemma 2.3(4), we have $H^1(w_0, \alpha_i) = 0$ for $i = 1, 2, 3, 4$. In each of the following cases we use these appropriately.

Case 1: $c = s_1s_2s_3s_4$. Then in this case we have $w_0 = v_6 = [1, 4]^6$. By using LES and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(w_0, i), T_{(w_0, i)}) = H^1(Z(w_5, i_5), T_{(w_5, i_5)}).$$

By using LES, Lemma 5.1, Lemma 5.9 and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(w_5, i_5), T_{(w_5, i_5)}) = H^1(Z(w_4, i_4), T_{(w_4, i_4)}).$$

By using LES and Lemma 6.3(1), we have

$$H^1(Z(w_4, i_4), T_{(w_4, i_4)}) = H^1(Z(\tau_4, j_4), T_{(\tau_4, j_4)}).$$

By using LES, Lemma 5.9 and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(\tau_4, j_4), T_{(\tau_4, j_4)}) = H^1(Z(w_3, i_3), T_{(w_3, i_3)}).$$

By using LES and Lemma 6.3(2), we have

$$H^1(Z(w_3, i_3), T_{(w_3, i_3)}) = H^1(Z(\tau_3, j_3), T_{(\tau_3, j_3)}).$$

By using LES, Lemma 5.9 and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(\tau_3, j_3), T_{(\tau_3, j_3)}) = H^1(Z(w_2, i_2), T_{(w_2, i_2)}).$$

By using LES, Lemma 5.1, Lemma 5.9 and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(w_2, i_2), T_{(w_2, i_2)}) = H^1(Z(w_1, i_1), T_{(w_1, i_1)}).$$

By using LES, Lemma 5.1, Lemma 5.9 and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(w_1, i_1), T_{(w_1, i_1)}) = H^1(Z(s_1s_2, (1, 2)), T_{(s_1s_2, (1, 2))}).$$

We see that $H^1(s_1, \alpha_1) = 0$, $H^1(s_1s_2, \alpha_2) = 0$. Thus by using LES, we have $H^1(Z(s_1s_2, (1, 2)), T_{(s_1s_2, (1, 2))}) = 0$. Thus by combining all, we have $H^1(Z(w_0, i), T_{(w_0, i)}) = 0$.

Case 2: $c = s_4s_1s_2s_3$. Then in this case we have $w_0 = s_4w_5s_3$. By using LES and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(w_0, (4, i_5)), T_{(w_0, (4, i_5))}) = H^1(Z(s_4w_5, (4, i_5)), T_{(s_4w_5, (4, i_5))}).$$

By using LES, Corollary 5.2, Corollary 5.10(1) and [15, Corollary 5.6, p. 778], we have

$$H^1(Z(s_4w_5, (4, i_5)), T_{(s_4w_5, (4, i_5))}) = H^1(Z(s_4w_4, (4, i_4)), T_{(s_4w_4, (4, i_4))}).$$
By using LES, Corollary 5.2, Corollary 5.10(1) and [15, Corollary 5.6, p. 778], we have
\[ H^1(Z(s_4w_4, (4, i_4)), T_{(s_4w_4, (4, i_4)}) = H^1(Z(s_4w_3, (4, i_3)), T_{(s_4w_3, (4, i_3)}) \].

By using LES and Lemma 6.4(1), we have
\[ H^1(Z(s_4w_3, (4, i_3)), T_{(s_4w_3, (4, i_3)}) = H^1(Z(s_4\tau_3, (4, j_3)), T_{(s_4\tau_3, (4, j_3)}) \].

By using LES, Corollary 5.10(1) and [15, Corollary 5.6, p. 778], we have
\[ H^1(Z(s_4\tau_3, (4, j_3)), T_{(s_4\tau_3, (4, j_3)}) = H^1(Z(s_4w_2, (4, i_2)), T_{(s_4w_2, (4, i_2)}) \].

By using LES and Lemma 6.4(2), we have
\[ H^1(Z(s_4w_2, (4, i_2)), T_{(s_4w_2, (4, i_2)}) = H^1(Z(s_4\tau_2, (4, j_2)), T_{(s_4\tau_2, (4, j_2)}) \].

By using LES, Corollary 5.10(1) and [15, Corollary 5.6, p. 778], we have
\[ H^1(Z(s_4\tau_2, (4, j_2)), T_{(s_4\tau_2, (4, j_2)}) = H^1(Z(s_4w_1, (4, i_1)), T_{(s_4w_1, (4, i_1)}) \].

By using LES, Corollary 5.2, Corollary 5.10(1) and [15, Corollary 5.6, p. 778], we have
\[ H^1(Z(s_4w_1, (4, i_1)), T_{(s_4w_1, (4, i_1)}) = H^1(Z(s_4s_1s_2, (4, 1, 2)), T_{(s_4s_1s_2, (4, 1, 2)}) \].

We see that \( H^1(s_4s_1, \alpha_1) = 0 \), \( H^1(s_4s_1s_2, \alpha_2) = 0 \). Thus by using LES, we have
\[ H^1(Z(s_4s_1s_2, (4, 1, 2)), T_{(s_4s_1s_2, (4, 1, 2)}) = 0. \]

Thus combining all we have \( H^1(Z(w_0, (4, l_3)), T_{(w_0, (4, l_3)}) = 0. \)

Case 3: \( c = s_3s_4s_1s_2 \). Then we have \( w_0 = s_3s_4w_5 \). By using LES, Corollary 5.3, Corollary 5.10(2) and [15, Corollary 5.6, p. 778], we have
\[ H^1(Z(s_3s_4w_5, (3, 4, i_5)), T_{(s_3s_4w_5, (3, 4, i_5)}) \]
\[ = H^1(Z(s_3s_4w_4, (3, 4, i_5)), T_{(s_3s_4w_4, (3, 4, i_5)}) \].

By using LES, Corollary 5.3, Corollary 5.10(2) and [15, Corollary 5.6, p. 778], we have
\[ H^1(Z(s_3s_4w_4, (3, 4, i_4)), T_{(s_3s_4w_4, (3, 4, i_4)}) \]
\[ = H^1(Z(s_3s_4w_3, (3, 4, i_3)), T_{(s_3s_4w_3, (3, 4, i_3)}) \].

By using LES and Lemma 6.5(1), we have
\[ H^1(Z(s_3s_4w_3, (3, 4, i_3)), T_{(s_3s_4w_3, (3, 4, i_3)}) \]
\[ = H^1(Z(s_3s_4\tau_3, (3, 4, j_3)), T_{(s_3s_4\tau_3, (3, 4, j_3)}) \].

By using LES, Corollary 5.10(2) and [15, Corollary 5.6, p. 778] we have
\[ H^1(Z(s_3s_4\tau_3, (3, 4, j_3)), T_{(s_3s_4\tau_3, (3, 4, j_3)}) \]
\[ = H^1(Z(s_3s_4w_2, (3, 4, i_2)), T_{(s_3s_4w_2, (3, 4, i_2)}) \].
By using LES and Lemma 6.5(2), we have

\[ H^1(Z(s_3 s_4 w_2, (3, 4, i_2)), T_{(s_3 s_4 w_2, (3, 4, i_2))}) = H^1(Z(s_3 s_4 \tau_2, (3, 4, j_2)), T_{(s_3 s_4 \tau_2, (3, 4, j_2))}). \]

By using LES, Corollary 5.10(2) and [15, Corollary 5.6, p. 778], we have

\[ H^1(Z(s_3 s_4 \tau_2, (3, 4, j_2)), T_{(s_3 s_4 \tau_2, (3, 4, j_2))}) = H^1(Z(s_3 s_4 w_1, (3, 4, i_1)), T_{(s_3 s_4 w_1, (3, 4, i_1))}). \]

By using LES and Lemma 6.5(3), we have

\[ H^1(Z(s_3 s_4 w_1, (3, 4, i_1)), T_{(s_3 s_4 w_1, (3, 4, i_1))}) = H^1(Z(s_3 s_4 \tau_1, (3, 4, j_1)), T_{(s_3 s_4 \tau_1, (3, 4, j_1))}). \]

By using LES, Corollary 5.10(2) and [15, Corollary 5.6, p. 778], we have

\[ H^1(Z(s_3 s_4 \tau_1, (3, 4, j_1)), T_{(s_3 s_4 \tau_1, (3, 4, j_1))}) = H^1(Z(s_3 s_4 s_1 s_2, (3, 4, 1, 2)), T_{(s_3 s_4 s_1 s_2, (3, 4, 1, 2))}). \]

We see that \( H^1(s_3 s_4, \alpha_4) = 0 \) (see [15, Corollary 5.6, p. 778]), \( H^1(s_3 s_4 s_1, \alpha_1) = 0 \), \( H^1(s_3 s_4 s_1 s_2, \alpha_2) = 0 \). Thus by using LES, we have

\[ H^1(Z(s_3 s_4 s_1 s_2, (3, 4, 1, 2)), T_{(s_3 s_4 s_1 s_2, (3, 4, 1, 2))}) = 0. \]

Thus combining all we have \( H^1(Z(w_0, (3, 4, i_5)), T_{(w_0, (3, 4, i_5))}) = 0. \)

Case 4: \( c = s_2 s_3 s_4 s_1 \). Then \( w_0 = s_2 s_3 s_4 \tau_5 \). Let \( t_1 = s_2 s_3 s_4 \). By using LES, Corollary 5.10(3) and [15, Corollary 5.6, p. 778], we have

\[ H^1(Z(w_0, (2, 3, 4, j_5)), T_{(w_0, (2, 3, 4, j_5))}) = H^1(Z(t_1 w_4, (2, 3, 4, i_4)), T_{(t_1 w_4, (2, 3, 4, i_4))}). \]

By using LES, Corollary 5.4, Corollary 5.10(3) and [15, Corollary 5.6, p. 778], we have

\[ H^1(Z(t_1 w_4, (2, 3, 4, i_4)), T_{(t_1 w_4, (2, 3, 4, i_4))}) = H^1(Z(t_1 w_3, (2, 3, 4, i_3)), T_{(t_1 w_3, (2, 3, 4, i_3))}). \]

By using LES and Lemma 6.6(1), we have

\[ H^1(Z(t_1 w_3, (2, 3, 4, i_3)), T_{(t_1 w_3, (2, 3, 4, i_3))}) = H^1(Z(t_1 \tau_3, (2, 3, 4, j_3)), T_{(t_1 \tau_3, (2, 3, 4, j_3))}). \]

By using LES, Corollary 5.10(3) and [15, Corollary 5.6, p. 778], we have

\[ H^1(Z(t_1 \tau_3, (2, 3, 4, j_3)), T_{(t_1 \tau_3, (2, 3, 4, j_3))}) = H^1(Z(t_1 w_2, (2, 3, 4, i_2)), T_{(t_1 w_2, (2, 3, 4, i_2))}). \]
By using LES and Lemma 6.6(2), we have
\[ H^1(Z(t_1w_2, (2, 3, 4, i_2)), T(t_1w_2, (2, 3, 4, i_2))) = H^1(Z(t_1\tau_2, (2, 3, 4, j_2)), T(t_1\tau_2, (2, 3, 4, j_2))). \]

By using LES, Corollary 5.10(3) and [15, Corollary 5.6, p. 778], we have
\[ H^1(Z(t_1\tau_2, (2, 3, 4, j_2)), T(t_1\tau_2, (2, 3, 4, j_2))) = H^1(Z(t_1w_1, (2, 3, 4, i_1)), T(t_1w_1, (2, 3, 4, i_1))). \]

By using LES and Lemma 6.6(3), we have
\[ H^1(Z(t_1w_1, (2, 3, 4, i_1)), T(t_1w_1, (2, 3, 4, i_1))) = H^1(Z(t_1\tau_1, (2, 3, 4, j_1)), T(t_1\tau_1, (2, 3, 4, j_1))). \]

By using LES, Corollary 5.10(3) and [15, Corollary 5.6, p. 778], we have
\[ H^1(Z(t_1\tau_1, (2, 3, 4, j_1)), T(t_1\tau_1, (2, 3, 4, j_1))) = H^1(Z(t_1s_1s_2, (2, 3, 4, 1, 2)), T(t_1s_1s_2, (2, 3, 4, 1, 2))). \]

It is easy to see that \( H^1(t_1s_1, \alpha_1) = H^1(s_2s_1, \alpha_1) = 0 \). We see that \( H^1(s_2s_3, \alpha_3) = 0, H^1(t_1, \alpha_4) = 0 \) by [15, Corollary 5.6, p. 778]. \( H^1(t_1s_1s_2, \alpha_2) = 0 \) by Corollary 5.4.
Thus by using LES, we have
\[ H^1(Z(t_1s_1s_2, (2, 3, 4, 1, 2)), T(t_1s_1s_2, (2, 3, 4, 1, 2))) = 0. \]

Thus combining all we have \( H^1(Z(w_0, (2, 3, 4, j_5)), T(w_0, (2, 3, 4, j_5))) = 0. \)

Case 5: \( c = s_4s_3s_1s_2 \). In this case we have \( w_0 = s_4s_3s_4w_4s_3s_1s_2 \). Let \( t_2 = s_4s_3s_4 \). Since \( s_3 \) commutes with \( s_1 \), we have \( H^1(t_2w_4s_3s_3s_2, \alpha_1) = H^1(t_2w_4s_3s_3s_2s_1s_2, \alpha_1) = 0 \) for \( i \geq 0 \) (see Lemma 2.3(4)).
Thus by using LES, Corollary 5.10(4) and [15, Corollary 5.6, p. 778], we have
\[ H^1(Z(w_0, (4, 3, 4, i_4), 3, 1, 2)), T(w_0, (4, 3, 4, i_4), 3, 1, 2)) = H^1(Z(t_2w_4, (4, 3, 4, i_4)), T(t_2w_4, (4, 3, 4, i_4))). \]

By using LES, Corollary 5.5, Corollary 5.10(4) and [15, Corollary 5.6, p. 778], we have
\[ H^1(Z(t_2w_4, (4, 3, 4, i_4)), T(t_2w_4, (4, 3, 4, i_4))) = H^1(Z(t_2w_3, (4, 3, 4, i_3)), T(t_2w_3, (4, 3, 4, i_3))). \]

By using LES, Corollary 5.5, Corollary 5.10(4) and [15, Corollary 5.6, p. 778], we have
\[ H^1(Z(t_2w_3, (4, 3, 4, i_3)), T(t_2w_3, (4, 3, 4, i_3))) = H^1(Z(t_2w_2, (4, 3, 4, i_2)), T(t_2w_2, (4, 3, 4, i_2))). \]
By using LES and Lemma 6.7(1), we have

\[ H^1(Z(t_2w_2, (4, 3, 4, i_2)), T_{(t_2w_2,(4,3,4,i_2))}) \]
\[ = H^1(Z(t_2\tau_2, (4, 3, 4, j_2)), T_{(t_2\tau_2,(4,3,4,j_2))}). \]

By using LES, Corollary 5.10(4) and [15, Corollary 5.6, p. 778], we have

\[ H^1(Z(t_2\tau_2, (4, 3, 4, j_2)), T_{(t_1\tau_2,(4,3,4,j_2))}) \]
\[ = H^1(Z(t_2w_1, (4, 3, 4, i_1)), T_{(t_2w_1,(4,3,4,i_1))}). \]

By using LES and Lemma 6.7(2), we have

\[ H^1(Z(t_2w_1, (4, 3, 4, i_1)), T_{(t_2w_1,(4,3,4,i_1))}) \]
\[ = H^1(Z(t_2\tau_1, (4, 3, 4, j_1)), T_{(t_2\tau_1,(4,3,4,j_1))}). \]

By using LES, Corollary 5.10(4) and [15, Corollary 5.6, p. 778], we have

\[ H^1(Z(t_2\tau_1, (4, 3, 4, j_1)), T_{(t_2\tau_1,(4,3,4,j_1))}) \]
\[ = H^1(Z(t_2s_1s_2, (4, 3, 4, 1, 2)), T_{(t_2s_1s_2,(4,3,4,1,2))}). \]

We see that \( H^1(s_4s_3, \alpha_3) = 0, H^1(t_2, \alpha_4) = 0 \) by [15, Corollary 5.6, p. 778]. Since \( s_3, s_4 \) commutes with \( s_1 \) we have \( H^1(t_2s_1, \alpha_1) = H^1(s_1, \alpha_1) = 0 \). By Corollary 5.5, we have \( H^1(t_2s_1s_2, \alpha_2) = 0 \).

Thus combining all we have \( H^1(Z(w_0, (4, 3, 4, 1, 2)), T_{(w_0,(4,3,4,1,2))}) = 0 \). Thus combining all we have \( H^1(Z(w_0, (4, 3, 4, i_4, 3, 1)), T_{(w_0,(4,3,4,i_4,3,1))}) = 0 \).

Case 6: \( c = s_4s_2s_3s_1 \). In this case we have \( w_0 = s_4s_2s_3s_4w_4s_3s_1 \). Let \( t_3 = s_4s_2s_3s_4 \). By using LES and [15, Corollary 5.6, p. 778], we have

\[ H^1(Z(t_3w_0, (4, 2, 3, 4, i_4, 3, 1)), T_{(t_3w_0,(4,2,3,4,i_4,3,1))}) \]
\[ = H^1(Z(t_3w_0, (4, 2, 3, 4, i_4)), T_{(t_3w_0,(4,2,3,4,i_4))}). \]

By using LES, Corollary 5.6, Corollary 5.10(5) and [15, Corollary 5.6, p. 778], we have

\[ H^1(Z(t_3w_0, (4, 2, 3, 4, i_4)), T_{(t_3w_0,(4,2,3,4,i_4))}) \]
\[ = H^1(Z(t_3w_3, (4, 2, 3, 4, i_3)), T_{(t_3w_3,(4,2,3,4,i_3))}). \]

By using LES, Corollary 5.6, Corollary 5.10(5) and [15, Corollary 5.6, p. 778], we have

\[ H^1(Z(t_3w_3, (4, 2, 3, 4, i_3)), T_{(t_3w_3,(4,2,3,4,i_3))}) \]
\[ = H^1(Z(t_3w_2, (4, 2, 3, 4, i_2)), T_{(t_3w_2,(4,2,3,4,i_2))}). \]

By using LES and Lemma 6.8(1), we have

\[ H^1(Z(t_3w_2, (4, 2, 3, 4, i_2)), T_{(t_3w_2,(4,2,3,4,i_2))}) \]
\[ = H^1(Z(t_3\tau_2, (4, 2, 3, 4, j_2)), T_{(t_3\tau_2,(4,2,3,4,j_2))}). \]
By using Corollary 5.10(5) and [15, Corollary 5.6, p. 778], we have
\[ H^1(Z(t_3 \tau_2, (4, 2, 3, 4, j^i_3)), T_{(t_3 \tau_2, (4, 2, 3, 4, j^i_3))}) = H^1(Z(t_3 w_1, (4, 2, 3, 4, i^i_1)), T_{(t_3 w_1, (4, 2, 3, 4, i^i_1))}). \]

By using LES and Lemma 6.8(2), we have
\[ H^1(Z(t_3 w_1, (4, 2, 3, 4, i^i_1)), T_{(t_3 w_1, (4, 2, 3, 4, i^i_1))}) = H^1(Z(t_3 \tau_1, (4, 2, 3, 4, j^i_1)), T_{(t_3 \tau_1, (4, 2, 3, 4, j^i_1))}). \]

By using LES, Corollary 5.10(5) and [15, Corollary 5.6, p. 778], we have
\[ H^1(Z(t_3 \tau_1, (4, 2, 3, 4, j^i_1)), T_{(t_3 \tau_1, (4, 2, 3, 4, j^i_1))}) = H^1(Z(t_3 s_1 s_2, (4, 2, 3, 4, 1, 2)), T_{(t_3 s_1 s_2, (4, 2, 3, 4, 1, 2))}). \]

We see that \( H^1(s_4 s_2, \alpha_2) = 0 \), \( H^1(t_3 s_1, \alpha_1) = 0 \). Further, by using [15, Corollary 5.6, p. 778], we have \( H^1(s_4 s_2 s_3, \alpha_3) = 0 \), \( H^1(t_3, \alpha_4) = 0 \). By Corollary 5.6, we have \( H^1(t_3 s_1 s_2, \alpha_2) = 0 \).

Therefore by using LES we have \( H^1(Z(t_3 s_1 s_2, (4, 2, 3, 4, 1, 2)), T_{(t_3 s_1 s_2, (4, 2, 3, 4, 1, 2))}) = 0 \). Thus combining all we have \( H^1(Z(w_0, (4, 2, 3, 4, t^i_4, 3, 1)), T_{(w_0, (4, 2, 3, 4, t^i_4, 3, 1))}) = 0 \).

**Case 7:** \( c = s_4 s_3 s_2 s_1 \). In this case we have \( w_0 = s_4 s_3 s_2 s_3 s_4 w_3 s_3 s_1 s_2 s_1 \). Let \( t_4 = s_4 s_3 s_2 s_3 s_4 \). Let \( i^i_r = (4, 3, 4, 2, 3, 4, 1) \). Recall that \( l_r = (i_r, 3) \). Let \( i^i_r = (4, 3, 4, 2, 3, 4, i^i_r) \) be the reduced expressions of \( t_4 w_r \) for \( r = 1, 2, 3 \). Let \( j^i_r = (4, 3, 4, 2, 3, 4, j^i_r) \) be the reduced expression of \( t_4 \tau_r \) for \( r = 1, 2, 3 \). Let \( i^i_1 = (4, 3, 4, 2, 3, 4, 1) \) be the reduced expression of \( t_4 s_1 \).

By Lemma 4.1(2) and Corollary 5.2(2), we have \( H^i(s_4 w_4, \alpha_2) = 0 \) for \( i \geq 0 \). Since \( s_4 \) commutes with \( s_1, s_2 \), we have \( H^i(s_4 w_4, \alpha_4) = H^i(s_4 s_3 s_2 s_1, \alpha_2) = H^i(s_4 w_3 s_3 s_2, \alpha_2) \) for \( i \geq 0 \). Thus we have \( H^i(s_4 s_3 s_2 s_1, \alpha_2) = 0 \) for \( i \geq 0 \). Therefore, by using LES, we have \( H^i(t_4 w_3 s_3 s_2, \alpha_2) = 0 \) for \( i \geq 0 \). Since \( s_3 \) commutes with \( s_1 \) we have \( H^i(t_4 w_3 s_3 s_1, \alpha_1) = H^i(t_4 w_3 s_1, \alpha_1) \) for \( i \geq 0 \). Thus we have \( H^i(t_4 w_3 s_3 s_1, \alpha_1) = 0 \) for \( i \geq 0 \) (see Lemma 2.3(4)). Thus we have \( H^i(t_4 w_3 s_3 s_1, \alpha_1) = 0 \) for \( i \geq 0 \). Thus by using LES, above discussion, and [15, Corollary 5.6, p.778] we have
\[ H^1(Z(w_0, i^i_r), T_{(w_0, i^i_r)}) = H^1(Z(t_4 w_3, i^i_r), T_{(t_4 w_3, i^i_r)}). \]

By using LES, Lemma 5.8(4), Corollary 5.10(6) and [15, Corollary 5.6, p. 778], we have
\[ H^1(Z(t_4 w_3, i^i_r), T_{(t_4 w_3, i^i_r)}) = H^1(Z(t_4 w_2, i^i_r), T_{(t_4 w_2, i^i_r)}). \]

By using LES, Lemma 5.8(4), Corollary 5.10(6) and [15, Corollary 5.6, p. 778], we have
\[ H^1(Z(t_4 w_2, i^i_r), T_{(t_4 w_2, i^i_r)}) = H^1(Z(t_4 w_1, i^i_r), T_{(t_4 w_1, i^i_r)}). \]
By using LES and Lemma 6.9(1), we have
\[ H^1(Z(t_4 w_1, i'_1), T_{(t_1 w_1, i'_2)}) = H^1(Z(t_4 t_1, j'_1), T_{(t_4 t_1, j'_2)}). \]

By using LES, Corollary 5.10(6) and [15, Corollary 5.6, p. 778], we have
\[ H^1(Z(t_4 t_1, j'_1), T_{(t_1 t_1, j'_2)}) = H^1(Z(t_4 s_1 s_2, (j'_2, 2)), T_{(t_4 s_1 s_2, (j'_2, 2))}). \]

By using LES and Lemma 6.9(2), we have
\[ H^1(Z(t_4 s_1 s_2, (j'_2, 2)), T_{(t_4 s_1 s_2, (j'_2, 2))}) = H^1(Z(t_4 s_1, j'_2), T_{(t_4 s_1, j'_2)}). \]

By [15, Corollary 5.6, p. 778], we see that \( H^1(s_4 s_3, \alpha_3) = 0, H^1(s_4 s_3 s_4, \alpha_4) = 0, H^1(s_4 s_3 s_4 s_3, \alpha_3) = 0 \) and \( H^1(t_4, \alpha_4) = 0 \). By Lemma 5.8(1), we have \( H^1(s_4 s_3 s_4 s_2, \alpha_2) = 0 \).

Since \( s_3, s_4 \) commute with \( s_1 \), we have \( H^1(t_4 s_1, \alpha_1) = H^1(s_4 s_3 s_2 s_1, \alpha_1) \). It is easy to see by using SES that \( H^1(s_4 s_3 s_2 s_1, \alpha_1) = 0 \). Thus we have \( H^1(t_4 s_1, \alpha_1) = 0 \). Therefore, by using LES, we have \( H^1(Z(t_4 s_1, j'_2), T_{(t_4 s_1, j'_2)}) = 0 \). Thus combining all we have \( H^1(Z(w_0, i'_2), T_{(w_0, i'_2)}) = 0 \).

**COROLLARY 7.2**

Let \( c \) be a Coxeter element such that \( c \) is of the form \([a_1, 4][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]\) with \( a_1 \neq 3 \) or \( a_2 \neq 2 \) and \( a_k = 1 \). Let \((w_0, i)\) be a reduced expression of \( w_0 \) in terms of \( c \) as in Theorem 7.1. Then, \( Z(w_0, i) \) has no deformations.

**Proof.** By Theorem 7.1 and [13, Proposition 3.1, p. 673], we have \( H^i(Z(w_0, i), T_{(w_0, i)}) = 0 \) for all \( i > 0 \). Hence, by [11, Proposition 6.2.10, p. 272], we see that \( Z(w_0, i) \) has no deformations.

**8. Non rigidity for \( G_2 \)**

Now onwards, we will assume that \( G \) is of type \( G_2 \). Note that the longest element \( w_0 \) of the Weyl group \( W \) of \( G \) is equal to \(-id\). We recall the following proposition from [17, Proposition 1.3, p. 858]. We use Proposition 3.1 and the notation as in [17] to deduce the following.

**Lemma 8.1.** Let \( c \in W \) be a Coxeter element. Then we have

1. \( w_0 = c^3 \).
2. For any sequence \( i = (i^1, i^2, i^3) \) of reduced expressions of \( c \); the sequence \( i = (i^1, i^2, i^3) \) is a reduced expression of \( w_0 \).

**Proof.**

**Proof of (1).** Let \( \eta : S \rightarrow S \) be the involution of \( S \) defined by \( i \rightarrow i^* \), where \( i^* \) is given by \( \omega_{i^*} = -w_0(\omega_i) \). Since \( G \) is of type \( G_2 \), \( w_0 = -id \), therefore, we have \( i = i^* \) for every
Let $h$ be the Coxeter number. By [17, Proposition 1.7], we have $h(i, c) + h(i^*, c) = h$. Since $h = 2|R^+|/2$ (see [9, Proposition 3.18]) and $i = i^*$, we have $h(i, c) = h/2 = 3$, as $|R^+| = 6$. By Proposition 3.1, we have $c^6(\omega_i) = -\omega_i$ for all $i = 1, 2$. Since $\{\omega_i : i = 1, 2\}$ forms an $\mathbb{R}$-basis of $X(T) \otimes \mathbb{R}$, it follows that $c^3 = -id$. Hence, we have $w_0 = c^3$. The assertion (2) follows from the fact that $l(c) = 2$ and $l(w_0) = |R^+| = 6$ (see [7, p. 66, Table 1]).

Let $c$ be a Coxeter element of $W$. Then $c = s_1s_2$ or $c = s_2s_1$. Then from Lemma 8.1, we have $w_0 = s_1s_2s_1s_2s_1s_2$ or $w_0 = s_2s_1s_2s_1s_2s_1$ accordingly as $c = s_1s_2$ or $c = s_2s_1$.

Let $i_1$ (respectively, $i_2$) be the the reduced expression of $w_0 = s_1s_2s_1s_2s_1s_2$ (respectively, $w_0 = s_2s_1s_2s_1s_2s_1$). Then we have as follows.

**Theorem 8.2.** $H^1(Z(w_0, i_r), T_{(w_0,i_r)}) \neq 0$ for $r = 1, 2$.

**Proof.** Let $c = s_1s_2$. Let $i = (1, 2)$ be the sequence corresponding to $c$. Then using LES, we have

$$0 \rightarrow H^0(c, \alpha_2) \rightarrow H^0(Z(c, i), T_{(c,i)}) \rightarrow H^0(s_1, \alpha_1) \rightarrow H^1(c, \alpha_2) \rightarrow H^1(Z(c, i), T_{(c,i)}) \rightarrow 0.$$  

By using SES, we see that $H^1(s_1s_2, \alpha_2) = \mathbb{C}_{\alpha_2+\alpha_1} \oplus \mathbb{C}_{\alpha_2+2\alpha_1}$. Now $H^0(s_1, \alpha_1)_{\alpha_2+\alpha_1} = 0$. Hence $g$ is a non zero homomorphism. Hence $H^1(Z(c, i), T_{(c,i)}) \neq 0$. By Lemma 6.2, the natural homomorphism

$$H^1(Z(w_0, i_1), T_{(w_0,i_1)}) \rightarrow H^1(Z(c, i), T_{(c,i)})$$

is surjective.

Hence we have $H^1(Z(w_0, i_1), T_{(w_0,i_1)}) \neq 0$.

Let $c = s_2s_1$. $u = s_2s_1s_2$. Let $i = (2, 1, 2)$ be the sequence corresponding to $u$. Then using LES, we have

$$0 \rightarrow H^0(u, \alpha_2) \rightarrow H^0(Z(u, j), T_{(u,j)}) \rightarrow H^0(s_2s_1, (2, 1)) \rightarrow H^1(u, \alpha_2) \rightarrow H^1(Z(u, j), T_{(u,j)}) \rightarrow H^1(Z(s_2s_1, (2, 1)), T_{(s_2s_1,(2,1))}) \rightarrow 0.$$  

We see that $H^1(u, \alpha_2) = \mathbb{C}_{\alpha_1} \oplus \mathbb{C}_{\alpha_2+\alpha_1} \oplus \mathbb{C}_{\alpha_2+2\alpha_1}$, $H^0(s_1, \alpha_1)_{\alpha_2+\alpha_1} = 0$ and $H^0(s_2s_1, \alpha_1)_{\alpha_2+\alpha_1} = 0$.

Therefore, by LES, we have $H^0(Z(s_2s_1, (2, 1)), T_{(s_2s_1,(2,1))})_{\alpha_2+\alpha_1} = 0$. Hence $h$ is a non zero homomorphism. Hence $H^1(Z(u, j), T_{(u,j)}) \neq 0$. By Lemma 6.2, the natural homomorphism

$$H^1(Z(w_0, i_2), T_{(w_0,i_2)}) \rightarrow H^1(Z(u, j), T_{(u,j)})$$

is surjective.

Hence we have $H^1(Z(w_0, i_2), T_{(w_0,i_2)}) \neq 0$.  

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References

[16] Senthamarai Kannan S and Saha Pinakinath, Rigidity of Bott–Samelson–Demazure–Hansen variety for $PSO(2n + 1, \mathbb{C})$, preprint

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