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Prime intersection graph of ideals of a ring

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Abstract. Let R be a ring. The prime intersection graph of ideals of R, denoted by $G_P(R)$, is the graph whose vertex set is the collection of all non-trivial (left) ideals of R with two distinct vertices I and J are adjacent if and only if $I \cap J \neq 0$ and either one of I or J is a prime ideal of R. We discuss connectedness in $G_P(R)$. The concepts of bipartition, planarity and colorability are interpreted. Finally, we introduce the idea of traversability in $G_P(\mathbb{Z}_n)$. The core part of this paper is observed in the ring $\mathbb{Z}_n$.

Keywords. Prime intersection graph; ring; prime ideal; connected graph.

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1. Introduction

In 2008, Chakrabarty *et al.* [8] introduced the motivating insight of graphical aspect of algebraic structures, namely intersection graphs of ideals of rings. In that intersection graph, the vertex set is the collection of non-trivial ideals of a ring and any two vertices are adjacent if their intersection is non-zero. They observed almost all fundamental concepts of the intersection graphs of ideals of rings. The central part of their interpretation depended upon the ring $\mathbb{Z}_n$. After that introduction, Akbari *et al.* [1] studied some more results of intersection graph of ideals of rings. They interestingly noticed characteristics between the graph-theoretic properties of this graph and some algebraic properties of rings. In [2], Akbari *et al.* extended the concept of intersection graphs of ideals of rings into intersection graphs of submodules of modules. Akbari *et al.* [3] also discussed on the complement of the intersection graph of submodules of a module. Rajkhowa and Saikia [18] generalized the intersection graph introduced by Chakrabarty *et al.* [8] and investigated the features of the center in that graph. Some more discussions of intersection graphs can be found in [4,9,14,19].

In this paper, we introduce the prime intersection graph of ideals of rings. Let R be a ring. Then the prime intersection graph of R, denoted by $G_P(R)$, is an undirected graph with vertex set as the collection of non-trivial (left) ideals of R and any two vertices I, J are adjacent if and only if $I \cap J \neq 0$ and one of I and J is a prime ideal of R. The vital part of this discussion is based on the ring $\mathbb{Z}_n$ and continues for generalized interpreta-

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tion. We first study the aspects of connectedness in the prime intersection graph $G_P(\mathbb{Z}_n)$ and then proceed for the same for an arbitrary ring. We also find the diameter, completeness character for the prime intersection graph $G_P(\mathbb{Z}_n)$. The finding for completeness character moves to the rings which are not integral domains with the property that every prime ideal is maximal. In the section 3, we analyze the idea of bipartition, planarity and colorability. The girth is also obtained for prime intersection graphs of some rings. In section 4, we discuss what is the degree of a vertex in the prime intersection graph $G_P(\mathbb{Z}_n)$. Results of Eulerian and Hamiltonian graphs for $G_P(\mathbb{Z}_n)$ are given in this section 5.

Unless otherwise specified here, R is any ring and $G_P(R)$ is the corresponding prime intersection graph of R.

Any undefined terminology can be obtained in [5–7,10–13,15–17].

2. Connectedness in $G_P(R)$

In this section, we discuss the characteristics of connectedness of prime intersection graph of ideals of R. We basically emphasize the role of prime intersection graph of $\mathbb{Z}_n$.

The prime intersection graph $G_P(R)$ is not defined if R is a field, as a field contains exactly two trivial ideals. Henceforth, R is not a field. We observe that the prime intersection graph $G_P(\mathbb{Z}_n)$ is disconnected, in fact totally disconnected, if we draw the graphs for $n = 6, 10, 14, 15, 21, \ldots$. This urges us to give the following theorem.

Theorem 2.1. *$G_P(\mathbb{Z}_n)$ is disconnected if and only if $n = pq$, where p and q are distinct primes.*

Proof. One part follows exactly the same way as in Theorem 2.1 of [8]. For the converse part, assume that $G_P(\mathbb{Z}_n)$ is disconnected. Let $n = p_1 p_2 \cdots p_i$, where all p_j's are prime but may not be all distinct, for $j = 1, 2, \ldots, i$, $i > 1$. Suppose that $i \geq 3$ and $I = (p_1)$ and J be any vertex of $G_P(\mathbb{Z}_n)$ which is distinct from I. Then we get $J = (m)$, $m|n$, $m < n$. If $p_1|m$, then I and J are adjacent (I adj J), as I is a prime ideal of $\mathbb{Z}_n$. Again if p_1 does not divide m, then m has at least one prime factor, say q, distinct from p_1 and so $(p_1, q) = 1$. Let $K = (q)$. Since K is a prime ideal, it is easy to check that I adj K (since $p_1 q \in I \cap K$ and $p_1 q < n$) and K adj J (since $q|m$). Thus $I - K - J$ is a path from I to J. Therefore, if we consider two vertices, say J_1 and J_2, different from I, then from the above discussion, we can see that both J_1 and J_2 are connected to I. Hence $G_P(\mathbb{Z}_n)$ is connected. Thus for $i \geq 3$, $G_P(\mathbb{Z}_n)$ is connected. So the only option left here is $i = 2$ and $p_1 \neq p_2$. □

We now proceed to generalize the above theorem for an arbitrary ring. Before this, we see the following examples.

Example 2.2. Consider the ring $R = \{0, a, b, c\}$ with addition and multiplication which are defined as follows:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

and

.	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	a	b	c
c	0	a	b	c

Here we have three left ideals as $\{0, a\}, \{0, b\}, \{0, c\}$. Clearly, $G_P(R)$ is a disconnected graph.

Example 2.3. Consider the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$ and $R = \mathbb{Z}_2 \times \mathbb{Z}_3$. The proper ideals of R are $I = \{(0, 0), (0, 1), (0, 2)\}$ and $J = \{(0, 0), (1, 0)\}$. Both I and J are minimal and so $G_P(R)$ is a disconnected graph.

Observe that it is true for every product of two fields.

Before going to the following discussion, it is important to mention that in case of an arbitrary ring we agree that every maximal ideal is left as well as right.

Theorem 2.4. *$G_P(R)$ is a disconnected graph for a ring R if and only if R contains at least two minimal left ideals and every nontrivial left ideal of R is minimal* (*as well as maximal*).

Proof. The proof can be obtained in the same fashion as given in the proof of Theorem 2.4 by Chakrabarty *et al.* in [8]. □

COROLLARY 2.5

If $G_P(R)$ is a disconnected graph for a ring R, then it contains no edges, i.e., a null graph.

COROLLARY 2.6

If $G_P(R)$ is a disconnected graph for a ring R, then every pair of maximal ideals have non-trivial intersection.

Next we notice the following theorem whose proof is omitted since it follows the same way as in Theorem 2.7 of [8].

Theorem 2.7. *$G_P(R)$ is a disconnected graph for a commutative ring R if and only if R is a direct product of two simple commutative rings, i.e., $R = R_1 \times R_2$, where each R_i $(i = 1, 2)$ is either a field or a null ring with prime number of elements.*

COROLLARY 2.8

$G_P(R)$ is a disconnected graph for a commutative ring R with unity if and only if R is a direct product of two fields.

Now we first classify the completeness of $\mathbb{Z}_n$ and then the same for commutative ring with unity.

Theorem 2.9. *$G_P(\mathbb{Z}_n)$ is complete if and only if $n = p^2$ or p^3.*

Proof. Let $n = p^2$ or p^3.

Case I. If $n = p^2$, then $G_P(\mathbb{Z}_n)$ has only one non-trivial ideal which is (p), thus a complete graph.
Case II. If $n = p^3$, then the non-trivial ideals of $G_P(\mathbb{Z}_n)$ are (p) and (p^2). Observe that $(p) \cap (p^2) = (p^2) \neq 0$. Also, (p) is prime and so (p) adj (p^2). Hence $G_P(\mathbb{Z}_n)$ is complete.

Conversely, assume that $G_P(\mathbb{Z}_n)$ is complete. Consider $n = p_1^{m_1} p_2^{m_2} \cdots p_i^{m_i}$, where p_j's are all distinct primes, $m_j \in \mathbb{N}$ and $i \geq 2$. Then we see that $(p_1^{m_1}) \cap (p_2^{m_2} \cdots p_i^{m_i}) = 0$. This implies that $G_P(\mathbb{Z}_n)$ can not be complete. Thus $i = 1$ and so n is of the form p^m, i.e. $n = p^m$, $m > 1$. Therefore the non-trivial ideals of $\mathbb{Z}_{p^m}$ are $(p), (p^2), \ldots, (p^{m-1})$. Thus (p^l) nadj (p^k) for $l, k \in \{2, 3, \ldots, m-1\}$. If $m > 4$, then since (p^2) nadj (p^3), we get a contradiction, as $G_P(\mathbb{Z}_{p^m})$ is complete. Hence $m \leq 3$, i.e. $n = p^2$ or p^3. Notice that m can not be p. This completes the proof. □

Remark 2.10. Now we study the completeness for commutative principal ideal ring which is not an integral domain. We consider this particular case as we stick to the ring for which every prime ideal is maximal. Thus for the following theorem, we take into account that in the commutative ring R with unity which is not an integral domain, every prime ideal is maximal.

Theorem 2.11. *$G_P(R)$ is complete if and only if $|V(G_P(R))| \leq 2$ and R has exactly one minimal ideal.*

Proof. Let $|V(G_P(R))| \leq 2$ and R has exactly one minimal ideal. Then it is easy to verify that $G_P(R)$ is a complete graph.

Next, suppose that $G_P(R)$ is a complete graph. If possible, assume that $|V(G_P(R))| \geq 3$ or R has at least two minimal ideals, say M_1 and M_2. Then both M_1 and M_2 are not maximal, otherwise we get a disconnected graph by Theorem 2.4. Let M_1 be not maximal. Thus we have a maximal ideal M_3 such that $M_1 \subseteq M_3$. But then M_2 must be a prime ideal, as $G_P(R)$ is a complete graph. This is not possible as then M_2 is maximal, a contradiction. This contradiction concludes that $|V(G_P(R))| \leq 2$ and R has exactly one minimal ideal. Hence the theorem. □

In the integral domain $\mathbb{Z}$, we see that $G_P(R)$ is a connected graph. This inspires us to interpret the connectedness in an integral domain which is not a field. The immediate succeeding theorem is the consequence of the same.

Theorem 2.12. *If R is an integral domain which is not a field, then $G_P(R)$ is connected.*

Proof. Let R be an integral domain which is not a field. Then $V(G_P(R)) \neq \Phi$. Let I and J be two vertices of $G_P(R)$. Suppose that $a \in I$ and $b \in J$ with $a, b \neq 0$. As R is an integral, we obtain $ab \neq 0$. This implies that $ab \in I \cap J$, i.e. $I \cap J \neq 0$. If one of I or J is prime then I adj J. If it is not so, then I and J are not maximal ideals of R. Thus we have at least one maximal ideal, say M, which contains at least one of I and J. Suppose that $I \subseteq M$. Also every maximal ideal is prime. Let c be a non-zero element in M. Thus $bc(\neq 0) \in J \cap M$. Hence we get $I - M - J$ is a path from I to J. This concludes that $G_P(R)$ is a connected graph. □

But the converse of the above theorem is not true. As an example, $G_P(\mathbb{Z}_8)$ is a connected graph, though $\mathbb{Z}_8$ is not an integral domain.

Next we find the diameter of $G_P(\mathbb{Z}_n)$ whenever $G_P(\mathbb{Z}_n)$ is a connected graph.

Theorem 2.13. *If $G_P(\mathbb{Z}_n)$ is a connected graph*, $\mathrm{diam}(G_P(\mathbb{Z}_n)) \leq 2$.

Proof. Since $G_P(\mathbb{Z}_n)$ is a connected graph, so $n = pq$ is not possible, where p and q are distinct primes. If $n = p^2$, then $\mathrm{diam}(G_P(\mathbb{Z}_n)) = 0$. Again, if $n = p^3$ then $G_P(\mathbb{Z}_n)$ is a complete graph. Thus $\mathrm{diam}(G_P(\mathbb{Z}_n)) = 1$. If $n = p^2q$, then we have the vertices as $(p), (p^2), (q), (pq)$. It is easy to see that $(p^2) - (p) - (pq)$ and $(p^2) - (p) - (q)$ are the longest induced paths of length 2, and so $\mathrm{diam}(G_P(\mathbb{Z}_n)) = 2$. Next we consider $n = p_1^{m_1} p_2^{m_2} \cdots p_i^{m_i}$, $i \geq 2$ with $m_1, m_2 > 1$. Then (p_j) adj $(p_1^{l_1} p_2^{l_2} \cdots p_j^{l_j} p_{j+1}^{l_{j+1}} \cdots p_i^{l_i})$, where $1 \leq j \leq i, 1 < l_j \leq m_i$. Thus we obtain the largest path as $(p_j) - (p_1^{l_1} p_2^{l_2} \cdots p_j^{l_j} p_{j+1}^{l_{j+1}} \cdots p_k^{l_k} p_{k+1}^{l_{k+1}} \cdots p_i^{l_i}) - (p_k)$, $j \neq k$, $1 \leq j \leq i, 1 < l_j \leq m_i$, $1 \leq k \leq i, 1 < l_k \leq m_i$. Therefore $\mathrm{diam}(G_P(\mathbb{Z}_n)) = 2$. Hence the theorem. □

Now we establish that the diameter of the prime intersection graph of a commutative ring with unity is at most 2.

Theorem 2.14. *Let R be a commutative ring with unity. If $G_P(R)$ is a connected graph, then* $\mathrm{diam}(G_P(R)) \leq 2$.

Proof. Let I and J be two non-trivial distinct ideals of R. If $I \cap J \neq 0$ and either I or J is a prime ideal of R, then $d(I, J) = 1$. Again assume that $I \cap J \neq 0$ and neither I nor J is a prime ideal of R. In that case, if I and J are comparable, then both I and J will be contained in a maximal ideal M, say. This gives that $d(I, J) = 2$, as $I - M - J$ is the shortest path between the vertices I and J. Again if I and J are not comparable, then $I + J$ is contained in a maximal ideal M. This asserts that $d(I, J) = 2$. Now assume that $I \cap J = 0$ and $I + J \neq R$, then $I - M - J$ is a path of length 2, where M is the maximal ideal which contains $I + J$. This gives $\mathrm{diam}(G_P(R)) \leq 2$. Next, let $I \cap J = 0$ and $I + J = R$. Since $G_P(R)$ is connected, it provides that either one of I or J is not a maximal ideal of R. Let I be not a maximal ideal of R. Thus there is a maximal ideal K with $I \subsetneqq K \subsetneqq R$. We claim that J adj K. As K is prime, we have to show only that $J \cap K \neq 0$. If possible, consider $J \cap K = 0$. As $R = I + J$, so for $k \in K$, we obtain $k = i + j$, where $i \in I$ and $j \in J$. Thus $k - i = j \in J \cap K = 0$ and hence $k = i$. Thus $K \subseteq I$, a contradiction. This contradiction concludes that $J \cap K \neq 0$ and so J adj K. Therefore, $I - K - J$ is a path of length 2. Hence $d(I, J) \leq 2$ and thus $\text{diam}(G_P(R)) \leq 2$. This completes the proof. □

3. Bipartition, planarity and colorability

In this section, we characterize the bipartite and planar graphs and colorability in $G_P(\mathbb{Z}_n)$.

Theorem 3.1. *$G_P(\mathbb{Z}_n)$ is a star graph if and only if $n = p^m$, $m > 1$.*

Proof. Let $G_P(\mathbb{Z}_n)$ be a star graph. Then the vertex set of $G_P(\mathbb{Z}_n)$ can be partitioned into two subsets, say V_1 and V_2, with one of V_1 and V_2 containing exactly one vertex. Let $|V_1| = 1$. Notice that every vertex of V_2 is adjacent to the single vertex of V_1 and no two vertices of V_2 are adjacent, as $G_P(\mathbb{Z}_n)$ is a complete bipartite graph. We note that the vertex $v \in V_1$, say, is a prime ideal of $\mathbb{Z}_n$, i.e., $v = (p)$, p being a prime number. Also a vertex of V_2 is not generated by a prime number. If so, then we may have at least two vertices in V_2 which are adjacent. This also asserts that no vertex of V_2 is generated by multiple of a prime, say q, distinct from p. If so, we can have a prime ideal (q) in V_2, and then either power p or q is at least 2, otherwise, we get a totally disconnected graph. So, if power of p is at least two, then $(p^2) \in V_2$ and $(p) - (q) - (pq) - (p)$ is a triangle, a contradiction. Similarly, if power of q is at least 2, then $(q^2) \in V_2$ and $(p) - (q) - (pq) - (p)$ is a triangle, again a contradiction. Thus every vertex of V_2 is a multiple of p. As every vertex represents an ideal, whose number of elements is a divisor of n, hence $n = p^m$, $m > 1$.

Next, let $n = p^m$, $m > 1$. Then the vertices of $G_P(\mathbb{Z}_n)$ are $(p), (p^2), (p^3), \ldots, (p^{m-1})$. It is easy to check that (p) adj (p^i), $i = 1, 2, \ldots, m-1$ and (p^j) nadj (p^k), $j, k \in \{2, 3, \ldots, m-1\}$. Thus $G_P(\mathbb{Z}_n)$ is a star graph. Hence $G_P(\mathbb{Z}_n)$ is a star graph if and only if $n = p^m$, $m > 1$. □

For the following theorem, we consider the ring as in Theorem 2.11.

Theorem 3.2. *$G_P(R)$ is a star graph if and only if the ideals of R form a chain.*

Proof. Let $G_P(R)$ be a star graph $K_{1,n}$, say. Then we have exactly one prime ideal v which belongs to that vertex set V_1 of $K_{1,n}$ containing exactly one vertex. This asserts that there is a unique maximal ideal v in R, as every prime ideal is maximal. Observe that no two vertices of the other vertex set, say V_2 of $K_{1,n}$ are adjacent. Let $v_1, v_2, \ldots, v_n$ be the vertices in V_2. Suppose that $v_1, v_2, \ldots, v_n$ do not form a chain. Then we have two non-comparable vertices, say v_i and v_j in V_2, where $i, j \in \{1, 2, \ldots, n\}$. Let $v_i = (x_i)$ and $v_j = (x_j)$. Then $x_i \notin v_j, x_j \notin v_i$. It is clear that neither x_i nor x_j is a prime element in R. If so, then one of v_i or v_j is maximal, a contradiction. Therefore x_i and x_j are reducible. Thus $x_i = p_1 p_2 \cdots p_k$ $x_j = p_1 p_2 \cdots p_l$. Then in these representations of x_i and x_j we have at least two distinct primes, otherwise $v_i = (p^{m_1})$ and $v_j = (p^{m_2})$ for some m_1, m_2, where p is the unique prime element of R. This will imply that v_i and v_j are comparable, a contradiction. Thus we have at least two distinct primes, say p and q. Then (p) and (q) are two distinct maximal ideals in R, again a contradiction. This contradiction concludes that the ideals of R form a chain. The converse part is easily obtainable.

Now our next attempt is to find out the girth of the prime intersection graph $G_P(\mathbb{Z}_n)$ and then continue the same for a commutative ring with unity.

Theorem 3.3. *If $G_P(\mathbb{Z}_n)$ contains a cycle, then* $\mathrm{girth}(G_P(\mathbb{Z}_n)) = 3$.

Proof. Let $n = p_1^{m_1} p_2^{m_2} \cdots p_i^{m_i}$. Observe that if in the factorization of n there is exactly one prime, then precisely the power of that prime must not be equal to 1. In this case the prime intersection graph $G_P(\mathbb{Z}_n)$ is a star graph. If there are exactly two distinct primes with both the powers of the distinct primes, say p_1 and p_2, are equal to 1, then we obtain a 2-vertex prime intersection graph $G_P(\mathbb{Z}_n)$ which is totally disconnected. Thus we have two distinct primes in which both the powers are not equal to 1 in the factorization of n. Let $m_1 \neq 1$, then we have at least 4 vertices in $G_P(\mathbb{Z}_n)$ specifically (p_1), (p_2), $(p_1 p_2)$ and (p_1^2). It is easy to notice that $(p_1) - (p_2) - (p_1 p_2) - (p_1)$ is a cycle of length 3. Hence girth of $G_P(\mathbb{Z}_n)$ is 3. □

Now we will find the girth of prime intersection graph of a commutative ring with unity. For the following theorem, we again consider the ring as in Theorem 2.11.

Theorem 3.4. *If $G_P(R)$ contains a cycle for a ring R, then* $\mathrm{girth}(G_P(R) = 3$.

Proof. Let $G_P(R)$ contain a cycle for a ring R. Let $C : I_1 - I_2 - I_3 \cdots I_n - I_1$ be a cycle in $G_P(R)$. On the contrary, suppose that $\mathrm{girth}(G_P(R) \geq 4$. We will consider three cases: (i) I_1 and I_2 are prime ideals of R, or (ii) I_1 is a prime ideal and I_2 is not a prime ideal, but I_3 is a prime ideal of R, or (iii) I_1 is not a prime ideal, I_2 is a prime ideal, I_3 is not a prime ideal, but I_4 is a prime ideal of R. If (i) holds, then immediately we obtain a triangle $I_1 - I_2 - I_1 \cap I_2 - I_1$, a contradiction. Next, if (ii) holds, then observe that $I_1 \cap I_2 \neq 0$ and thus we get the triangle as $I_1 - I_2 - I_3 - I_1$, a contradiction. Then if (iii) holds, then again we can show that $I_2 - I_3 - I_4 - I_2$ is a triangle, a contradiction. These contradictions imply that $\mathrm{girth}(G_P(R) = 3$. □

Theorem 3.5. *$G_P(\mathbb{Z}_n)$ is a bipartite graph if and only if $n = p^m, m > 1$ or pq.*

Proof. Let $G_P(\mathbb{Z}_n)$ be a bipartite graph. If possible assume that $n \neq p^m$, $m > 1$ and $n \neq pq$. Then in the factorization of n there are at least two distinct primes, say p, q, in which both the powers of p, q are not equal to 1 or all primes are distinct. For the first case, let power of p be at least 2. Then we obtain the triangle as $(p) - (q) - (pq) - (p)$ in $G_P(\mathbb{Z}_n)$. But, as $G_P(\mathbb{Z}_n)$ is bipartite, we reach a contradiction. For the second case also we can reach a contradiction easily. This contradiction asserts that either $n = p^m, m > 1$ or pq.

Now, let $n = p^m, m > 1$ or pq. If $n = p^m, m > 1$, then by the previous theorem $G_P(\mathbb{Z}_n)$ is a star graph. Also, if $n = pq$, then $G_P(\mathbb{Z}_n)$ is a 2-vertex totally disconnected graph, and thus a bipartite graph. Hence the theorem. □

Remark 3.6. For the ring in Theorem 2.11, we have that $G_P(R)$ is a bipartite graph if and only if the ideals of R form a chain or $G_P(R)$ is disconnected.

Theorem 3.7. *$G_P(\mathbb{Z}_n)$ is planar if and only if $n = p_1^{m_1}$ or $p_1^{m_1}p_2^{m_2}$ or $p_1p_2p_3$.*

Proof. Let $G_P(\mathbb{Z}_n)$ be a planar graph. If possible assume that n is not of the given forms. So let $n = p_1^{m_1}p_2^{m_2}\cdots p_i^{m_i}$, where $i \geq 3$, $m_j \geq 2$ for at least one $j = 1, 2, 3$; or $i \geq 4$. Then we see $K_{3,3}$ as a subgraph in $G_P(\mathbb{Z}_n)$ with vertex sets $V_1 = \{(p_1), (p_2), (p_3)\}$ and $V_2 = \{(p_1p_2), (p_2p_3), (p_1p_3)\}$. Observe that since $m_j \geq 2$, for at least one $j = 1, 2, 3$, every vertex of V_1 is adjacent to every vertex of V_2 and conversely. Thus we obtain $K_{3,3}$ as a subgraph in $G_P(\mathbb{Z}_n)$ if $n = p_1^{m_1}p_2^{m_2}\cdots p_i^{m_i}$, where $i \geq 3$, $m_j \geq 2$ for at least one $j = 1, 2, 3$. This is a contradiction to the fact that $G_P(\mathbb{Z}_n)$ is planar. This contradiction implies that $n = p_1^{m_1}$ or $p_1^{m_1}p_2^{m_2}$ or $p_1p_2p_3$.

Next, let $n = p_1^{m_1}$ or $p_1^{m_1}p_2^{m_2}$ or $p_1p_2p_3$. (i) When $n = p_1^{m_1}$, we have the connected graph with one vertex and no edges whenever $m_1 = 1$, otherwise a star graph, and hence a planar graph. (ii) When $n = p_1^{m_1}p_2^{m_2}$, then it is seen that $\deg((p_j))$ is at most $(m_1 + 1)(m_2+1)-2$, for $j = 1, 2$ and all other vertices are of degree at most 2. Thus we can not have a subgraph $K_{3,3}$ or K_5 in $G_P(\mathbb{Z}_n)$. Observe that there can not exist a subdivision of $K_{3,3}$ or K_5. Therefore, $G_P(\mathbb{Z}_n)$ is a planar graph. (iii) When $n = p_1p_2p_3$, then $\deg((p_j))$ is 5 for $j = 1, 2, 3$ and all other vertices, i.e., (p_1p_2), (p_1p_3) and (p_2p_3) are of degree 2. Thus in this case also we do not have $K_{3,3}$ or K_5 as a subgraph of $G_P(\mathbb{Z}_n)$. Again, observe that there can not exist a subdivision of $K_{3,3}$ or K_5 and so $G_P(\mathbb{Z}_n)$ is a planar graph.

Hence $G_P(\mathbb{Z}_n)$ is planar if and only if $n = p_1^{m_1}$ or $p_1^{m_1}p_2^{m_2}$ or $p_1p_2p_3$. □

Remark 3.8. $G_P(\mathbb{Z}_n)$ is non-planar if and only if $n = p_1^{m_1}p_2^{m_2}\cdots p_i^{m_i}$, where $i \geq 3$, $m_j \geq 2$ for at least one $j = 1, 2, 3, \ldots, n$.

Remark 3.9. Now our aim is to first discuss some characteristics of coloring of $G_P(\mathbb{Z}_n)$. It is noticed that $G_P(\mathbb{Z}_n)$ is 1-chromatic if and only if $n = p^2$ or pq. In general, for a ring R, $G_P(R)$ is 1-chromatic if and only if R contains exactly one non-trivial ideal or R contains at least two minimal left ideals and every nontrivial left ideal of R is minimal (as well as maximal). Again we see that $G_P(\mathbb{Z}_n)$ is 2-colorable if and only if $n = p^m$, $m > 2$ or pq. Also for the ring of Theorem 2.11, we check that $G_P(R)$ is 2-colorable if and only if ideals of R form a chain or R contains exactly one non-trivial ideal or R contains at least two minimal left ideals and every nontrivial left ideal of R is minimal (as well as maximal). Next, we establish the following two interesting results in $G_P(\mathbb{Z}_n)$.

Theorem 3.10. *$\chi(G_P(\mathbb{Z}_n)) = 3$ if and only if (i) $n = p_1^{m_1}p_2^{m_2}$, where both m_1 and m_2 are not equal to 1, or (ii) $n = p_1p_2p_3$.*

Proof. Let $\chi(G_P(\mathbb{Z}_n)) = 3$. Suppose both the conditions (i) and (ii) are not satisfied. Let $n = p_1^{m_1}p_2^{m_2}\cdots p_i^{m_i}$, $m_j \geq 1$, $j = 1, 2, \ldots, i$. It is necessary to mention here that $i \neq 1$, as in the case $i = 1$, we have $\chi(G_P(\mathbb{Z}_n)) = 1$ if $m_1 = 2$ and $\chi(G_P(\mathbb{Z}_n)) = 2$ if $m_1 > 2$. Also, since (i) is not satisfied, so $i \neq 2$. Note that if $i = 2$, then since (i) is not satisfied, we must have $n = p_1p_2$. In this case, $\chi(G_P(\mathbb{Z}_n)) = 1$. Thus $i \geq 3$ with at least one m_1, m_2, m_3 not equal to 1. Assume that $m_1 > 1$. Then it is easy to observe that $G_P(\mathbb{Z}_n)$ contains a subgraph K_4. For example, we can get a K_4 with vertices (p_1), (p_2), (p_3), (p_1p_2). In the remaining cases also, we can find out K_4 as a subgraph in $G_P(\mathbb{Z}_n)$. Hence we obtain a contradiction as $\chi(G_P(\mathbb{Z}_n)) = 3$. This implies that either (i) or (ii) is satisfied.

Next, let (i) or (ii) be satisfied. If (i) is satisfied, say $m_1 \neq 1$, then we get the vertices of $G_P(\mathbb{Z}_n)$ as $(p_1), (p_2^2), \ldots, (p_1^{m-1}), (p_2), (p_1p_2), (p_1^2p_2), \ldots, (p_1^{m-1}p_2)$. As (p_1) adj (p_1^l), $l = 2, 3, \ldots, m-1$; (p_1) adj $(p_1^s p_2)$, $s = 1, 2, \ldots, m-1$; (p_1^t) adj (p_2), $t = 1, 2, \ldots, m-1$ and $(p_1^s p_2)$ adj (p_2), $s = 1, 2, \ldots, m-1$. Then we get K_3 as a subgraph as $(p_1)-(p_2)-(p_1^s p_2)-(p_1)$, $s = 1, 2, \ldots, m-1$. Notice that there can not exist subgraph K_n, $n \geq 4$, as in the factorization of n, we have only two distinct primes. Therefore, in this case, $\chi(G_P(\mathbb{Z}_n)) = 3$. If (ii) is satisfied, i.e., when $n = p_1p_2p_3$, then we have the vertices of $G_P(\mathbb{Z}_n)$ as $(p_1), (p_2), (p_3), (p_1p_2), (p_1p_3), (p_2p_3)$. It is easy to verify that $\chi(G_P(\mathbb{Z}_n)) = 3$. Hence the theorem. □

Theorem 3.11. *Let $i > 3$. Then $\chi(G_P(\mathbb{Z}_n)) = i + 1$ if and only if $n = p_1^{m_1} p_2^{m_2} \cdots p_i^{m_i}$.*

Proof. Let $n = p_1^{m_1} p_2^{m_2} \cdots p_i^{m_i}$. Then we notice that the vertices $(p_1), (p_2), \ldots, (p_i)$, (p_1p_2) (or (p_1p_3) or (p_2p_3)) form the complete graph K_{i+1} as $i > 3$. It is necessary to state that we can not obtain a subgraph K_{i+j}, $j > 1$ as we have only i prime ideals in $\mathbb{Z}_n$, i.e., every vertex which is not prime is adjacent to at most i prime ideals. Hence $\chi(G_P(\mathbb{Z}_n)) = i + 1$.

Next, let $\chi(G_P(\mathbb{Z}_n)) = i + 1$. It asserts that there are at least i prime ideals in $\mathbb{Z}_n$. If we have only $i-1$ prime ideals, then $\chi(G_P(\mathbb{Z}_n)) = i$. We claim that there are exactly i prime ideals. If not, then we have $i+1$ prime ideals (other cases are not possible) and we get $i \leq 3$ as $\chi(G_P(\mathbb{Z}_n)) = i+1$. Hence we must have i prime ideals in $\mathbb{Z}_n$. Let $(p_1), (p_2), \ldots, (p_i)$ be prime ideals in $\mathbb{Z}_n$. Each of these primes $p_1, p_2, \ldots, p_i$ is a divisor of n. Also the ideal generated by each power (finite) of p_j, $j = 1, 2, \ldots$ is adjacent to exactly i prime ideals $(p_1), (p_2), \ldots, (p_i)$ in $G_P(\mathbb{Z}_n)$. So in this case also we obtain a complete subgraph of size $i + 1$ in $G_P(\mathbb{Z}_n)$. Since prime power p_j^m, $j = 1, 2, \ldots, n$, m(finite) $= 1, 2, \ldots$ is a divisor of n. Hence $n = p_1^{m_1} p_2^{m_2} \cdots p_i^{m_i}$, $m_j \in \{1, 2, \ldots\}$, $j = 1, 2, \ldots, i$. Hence the result. □

4. Traversability in $G_P(\mathbb{Z}_n)$

First, we highlight how to count the degree of a vertex in $G_P(\mathbb{Z}_n)$. In traversability part, this insight is implemented. Let $n = p_1^{m_1} p_2^{m_2} \cdots p_i^{m_i}$, where p_i's are distinct and $m_1, m_2, \ldots, m_i \in \mathbb{N}$. Then using number theoretic concept, we have

$$T = |V(G_P(\mathbb{Z}_n))| = (m_1 + 1)(m_2 + 1) \cdots (m_i + 1) - 2.$$

Case I. Suppose $m_j \neq 1$ for all $j = 1, 2, \ldots, i$. Then (p_j) adj $(p_1^{l_1} p_2^{l_2} \cdots p_j^{l_j} p_{j+1}^{l_{j+1}} \cdots p_i^{l_i})$, $j = 1, 2, \ldots, i$, $l_j \leq m_j$. Thus in this case $\deg((p_j)) = T - 1$ for $j = 1, 2, \ldots, i$ and $\deg(v) = i$, if v is not a prime ideal of $\mathbb{Z}_n$. Here we have i vertices each of which has degree $T - 1$ and $T - i$ vertices each of which has degree i. Thus the total degree of all T vertices is

$$2e = i(T - 1) + (T - i)i,$$

where e is the number of edges in $G_P(\mathbb{Z}_n)$. Thus $e = \frac{1}{2}(2Ti - i - i^2)$.

Example 4.1. Let us interpret Case I with an example. Let $n = p_1^3 p_2^7 p_3^4 p_4^5$. Then the number of vertices T in $G_P(\mathbb{Z}_n)$ is $(3 + 1)(7 + 1)(4 + 1)(5 + 1) - 2 = 838$. Here $i = 4$ and so there are $i = 4$ vertices each of degree $T - 1 = 837$, $T - i = 834$ vertices each of degree 4. Thus the number of edges e in $G_P(\mathbb{Z}_n)$ is $\frac{1}{2}(2Ti - i - i^2) = 3342$.

Case II. Suppose there are l primes whose powers are equal to 1 for $l = 1, 2, \ldots, i$. Then we have additionally (p_j) nadj $(p_1^{m_1} p_2^{m_2} \cdots p_{j-1}^{m_{j-1}} p_{j+1}^{m_{j+1}} \cdots p_i^{m_i})$ for $j = 1, 2, \ldots, l$ as

$$(p_j) \cap (p_1^{m_1} p_2^{m_2} \cdots p_{j-1}^{m_{j-1}} p_{j+1}^{m_{j+1}} \cdots p_i^{m_i}) = 0.$$

In this case,

$$\deg(p_j) = \begin{cases} T-2, & \text{for } j = 1, 2, \ldots, l, \\ T-1, & \text{for } j = l+1, l+1, \ldots, i. \end{cases}$$

Also, $\deg((p_1^{m_1} p_2^{m_2} \cdots p_{j-1}^{m_{j-1}} p_{j+1}^{m_{j+1}} \cdots p_i^{m_i})) = i - 1$ for each $j = 1, 2, \ldots, l$ and $\deg(v) = i$ if $v (\neq (p_1^{m_1} p_2^{m_2} \cdots p_{j-1}^{m_{j-1}} p_{j+1}^{m_{j+1}} \cdots p_i^{m_i}))$ is not a prime ideal of $\mathbb{Z}_n$. Thus $\deg((p_j)) = T - 2$ for $j = 1, 2, \ldots, l$. Thus there are l vertices each of which has degree $T - 2$ and $i - l$ vertices each of which has degree $T - 1$. If v is not a prime ideal of $\mathbb{Z}_n$, then

$$\deg(v) = \begin{cases} i-1, & \text{if } v = (p_1^{m_1} p_2^{m_2} \cdots p_{j-1}^{m_{j-1}} p_{j+1}^{m_{j+1}} \cdots p_i^{m_i}), \quad j = 1, 2, \ldots, l; \\ i, & \text{otherwise.} \end{cases}$$

Hence we have l vertices each of which has degree $i - 1$, and $T - i - l$ vertices each of which has degree i. Thus total degree of T vertices is

$$2e = (T-2)l + (T-1)(i-l) + l(i-1) + i(T-i-l),$$

where e is the number of edges in $G_P(\mathbb{Z}_n)$. Thus

$$e = \frac{1}{2}((T-2)l + (T-1)(i-l) + l(i-1) + i(T-i-l)).$$

If $l = 0$, then we see that $e = \frac{1}{2}(2Ti - i - i^2)$, which is Case I.

Example 4.2. Now we interpret Case II with an example. Let $n = p_1 p_2^7 p_3 p_4^5$. Then the number of vertices T in $G_P(\mathbb{Z}_n)$ is $(1+1)(7+1)(1+1)(5+1) - 2 = 190$. Here $i = 4$, $m_1 = 1, m_2 = 7, m_3 = 1, m_4 = 5, l = 2$. So there are $l = 2$ vertices each of which has degree $T - 2 = 188$, $i - l = 2$ vertices each of which has degree $T - 1 = 189$, 2 vertices each of which has degree $i - 1 = 3$ and $T - i - l = 184$ vertices each of which has degree 4. Thus the number of edges e in $G_P(\mathbb{Z}_n)$ is

$$\begin{aligned} &\frac{1}{2}((T-2)l + (T-1)(i-l) + l(i-1) + i(T-i-l)) \\ &\quad = \frac{1}{2}(188 \times 2 + 189 \times 2 + 2 \times 3 + 4 \times 184) = 748. \end{aligned}$$

Finally, we produce the following interesting results of traversability.

Theorem 4.3. *$G_P(\mathbb{Z}_n)$ is Eulerian if and only if $n = p_1^{m_1} p_2^{m_2} \ldots p_i^{m_i}$, where i is even and m_j is even for each $j = 1, 2, \ldots, i$ or $n = p_1 p_2 p_3$.*

Proof. Let $G_P(\mathbb{Z}_n)$ be an Eulerian graph. Let $n = p_1^{m_1} p_2^{m_2} \cdots p_i^{m_i}$, $i \geq 1$, $m_j \in \mathbb{N}$, $j = 1, 2, \ldots, i$. Then there are T vertices in $G_P(\mathbb{Z}_n)$, where $T = (m_1 + 1)(m_2 + 1) \cdots (m_i + 1) - 2$. Since $G_P(\mathbb{Z}_n)$ is Eulerian, each of these T vertices is of even degree. Here we have i prime ideal in $\mathbb{Z}_n$. Now any vertices which are not prime is adjacent to only prime vertices. Thus each vertex is adjacent to at most i prime vertices. For an Eulerian graph, each vertex is of even degree, so either i is even or odd. If i is odd, then it is just a routine work to notice that $n = p_1 p_2 p_3$. When i is even, it is necessary to mention that each $m_j > 1$, $j = 1, 2, \ldots, i$. If not, when $m_k = 1$ for some $k \in \{1, 2, \ldots, i\}$, the vertex $p_1^{m_1} p_2^{m_2} \cdots p_{k-1}^{m_{k-1}} p_{k+1}^{m_{k+1}} \cdots p_i^{m_i}$ is not adjacent to $p_k^{m_k}$, i.e., p_k (as $m_k = 1$). So we obtain an odd degree vertex $(p_1^{m_1} p_2^{m_2} \cdots p_{k-1}^{m_{k-1}} p_{k+1}^{m_{k+1}} \cdots p_i^{m_i})$ in $G_P(\mathbb{Z}_n)$, a contradiction. This contradiction concludes that no m_j, $j = 1, 2, \ldots, i$ is 1. This asserts that

$$\begin{aligned} \deg((p_j)) &= T - 1, \quad \forall j = 1, 2, \ldots, i \\ &= (m_1 + 1)(m_2 + 1) \cdots (m_i + 1) - 3. \end{aligned}$$

Since $G_P(\mathbb{Z}_n)$ is Eulerian, therefore,

$$\begin{aligned} &\deg((p_j)) = \text{even number} \quad \forall j = 1, 2, \ldots, i \\ &\Rightarrow (m_1 + 1)(m_2 + 1) \cdots (m_i + 1) - 3 = \text{even number} \\ &\Rightarrow (m_j + 1) \text{ is odd}, \quad \forall j = 1, 2, \ldots, i \\ &\Rightarrow m_j \text{ is even}, \quad \forall j = 1, 2, \ldots, i. \end{aligned}$$

Hence $n = p_1^{m_1} p_2^{m_2} \cdots p_i^{m_i}$, where i is even and m_j is even for each $j = 1, 2, \ldots, i$. Next, let $n = p_1^{m_1} p_2^{m_2} \cdots p_i^{m_i}$, where i is even and m_j is even for each $j = 1, 2, \ldots, i$. Here there are even number of prime ideals in $\mathbb{Z}_n$. Every vertex of $G_P(\mathbb{Z}_n)$ which is not a prime ideal is of degree at most i. As m_j is even, for each $j = 1, 2, \ldots, i$, $\deg(v)$ is i, where v is not a prime ideal of $\mathbb{Z}_n$. Also, each m_j is even implies that $j = 1, 2, \ldots, i$, the total number of vertices $T = (m_1 + 1)(m_2 + 1) \cdots (m_i + 1) - 2$ is odd, as $m_j + 1$ is odd, for each $j = 1, 2, \ldots, i$. Since each prime is of degree $T - 1$, $\deg(P)$ is even, where P is a prime ideal in $\mathbb{Z}_n$, as T is odd. Thus each vertex of $G_P(\mathbb{Z}_n)$ is of even degree. Hence $G_P(\mathbb{Z}_n)$ is an Eulerian graph. □

Theorem 4.4. *$G_P(\mathbb{Z}_n)$ is not Hamiltonian.*

Proof. The proof starts with the conclusion to check the result if the number of vertices T is greater than 3. It is obvious that case $T = 3$ does not arise. If $T < 3$, then $G_P(\mathbb{Z}_n)$ does not contain a circuit. As $T > 3$, so $n \neq p^2, pq$. Also, if $T = 4$, then $n = p^2 q$ and we see that $G_P(\mathbb{Z}_n)$ is not Hamiltonian. Again, if $T = 6$, then $n = pqr$ and so $G_P(\mathbb{Z}_n)$ is not Hamiltonian. Thus we assume that $T > 6$ and for this case consider $n = p_1^{m_1} p_2^{m_2} \cdots p_i^{m_i}$. If possible, suppose that $G_P(\mathbb{Z}_n)$ is Hamiltonian. Then we have a Hamiltonian circuit in $G_P(\mathbb{Z}_n)$. For a Hamiltonian cycle we require vertices corresponding to prime ideals to be arranged alternatively between vertices corresponding to non-prime ideals as no two non-prime ideals are adjacent. Now the total number vertices $T = (m_1 + 1)(m_2 + 1) \cdots (m_i + 1) - 2 \geq 2^i - 2$. Since there are i prime ideals, the above required situation never arises when $2^i - 2 > 2i$ and that holds for all $i \geq 4$. Now since we are considering the case that $T > 6$, we have $T > 2i$ for $i \leq 3$. Hence the theorem. □

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