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Dimension formula for the space of relative symmetric polynomials of D_n with respect to any irreducible representation

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Abstract. For positive integers d and n, the vector space $H_d(x_1, x_2, \ldots, x_n)$ of homogeneous polynomials of degree d is a representation of the symmetric group S_n acting by permutation of variables. Regarding this as a representation for the dihedral subgroup D_n, we prove a formula for the dimension of all the isotypical subrepresentations. Our formula is simpler than the existing one found by Zamani and Babaei (*Bull. Iranian Math. Soc.* **40(4)** (2014) 863–874). By varying the degrees d we compute the generating functions for these dimensions. Further, our formula leads us naturally to a specific supercharacter theory of D_n. It turns out to be a $*$-product of a specific supercharacter theory studied in depth by Fowler *et al.* (The Ramanujan Journal (2014)), with the unique supercharacter theory of a group of order 2.

Keywords. Relative symmetric polynomials; dihedral groups; invariants; supercharacters.

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1. Introduction

Every complex representation of a finite group has a canonical decomposition into the direct sum of isotypical components. Serre's textbook [7, page 21] gives the formula for the projection map to all these components. We recall the formula here.

Given the isotypical decomposition $V = \bigoplus_{\chi \in \mathrm{Irr}(G)} V_\chi$, the projection to the component V_χ is given by

$$p = \frac{\deg \chi}{|G|} \sum_{g \in G} \bar{\chi}(g)\rho(g). \tag{1.1}$$

When χ is the trivial 1-dimensional representation, this projection is the Reynold's operator.

In this paper, we focus on the natural action of S_n (and its subgroups) on the complex polynomial algebra of n variables, by permuting the variables.

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We denote by $H_d(x_1, x_2, \ldots, x_n)$ the complex vector space of all homogeneous polynomials of degree d in the n variables, $x_1, x_2, \ldots, x_n$, sometimes denoted simply by H_d.

The image of the Reynold's operator will be the space of all symmetric polynomials of degree d.

Shahryari [8] introduced the notion of relative symmetric polynomials for any subgroup $G \subset S_n$ with respect to any irreducible character χ of G.

The vector space of relative symmetric polynomials of G relative to χ, denoted by $H_d(G, \chi)$, is defined as the image of the projection operator defined in equation (1.1) as

$$T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g) g. \tag{1.2}$$

Finding the dimension of this vector space of relative symmetric polynomials for various subgroups of the symmetric group of S_n is a fundamental question.

Later, Babaei *et al.* [1] found the dimension of the space of relative invariants for S_n and its subgroup A_n [8] and for Young subgroups [9].

In a series of papers, Babaei and Zamani have given corresponding formula for the cyclic group in [11], for the dicyclic group in [12] and for the dihedral group D_n in [13].

Here we relook at the formula for dihedral groups. Babaei–Zamani formula for characters of degree 2 is a summation involving cosine values which are in general irrational numbers. So they may be inconvenient to calculate dimensions which are nonnegative integers. By using appropriate theorems from elementary number theory along with combinatorial counting arguments, we have provided an alternative dimensional formula as a summation of integer terms. Another advantage of our formula is that it allows us to write down the generating function (see Theorems 5.1 and 5.2 for precise statements).

A notable feature is that this formula gives this dimension as a *summation over the divisors of n* involving the Möbius and Euler ϕ-functions, even though this problem is not apparently connected with number theory.

A careful inspection of these dimensions when they coincide for different isotypical components led us to construct what is now called as supercharacter theory. This specific supercharacter theory that is an offshoot of our result is described as a $*$-product of a D_n-invariant supercharacter theory of the cyclic normal subgroup C_n of D_n with the unique supercharacter theory of D_n/C_n. It is really a pleasant surprise for us to learn that a specific supercharacter theory of cyclic groups found by Fowler *et al.* [3] to have many applications in discovering number-theoretic identities, which appears in our context too.

Our paper is organized as follows: in section 2, a few number-theoretic preliminaries are assembled, in section 3 the character table of the dihedral group is given following Serre [7]. Section 4 is a recollection of definitions from supercharacter theory of finite groups [4]. In section 5, we state and prove our main result. In section 6, we provide a specific supercharacter theory arising out of our formula. A congruence for some binomial coefficients deduced from our formula is also provided in this section. Finally in section 7, an example is shown illustrating how easy it is to compute with our formulæ.

2. Preliminaries

First we set up the notations:

- We use the standard notations $\phi(n)$ and $\mu(n)$ respectively for the Euler's totient function and Möbius function.
- For any n positive integer and r a divisor of n, we denote by $S_r(n)$ the set of integers between 1 and n having r as their gcd.

$$S_r(n) := \{k : 1 \leq k \leq n, \gcd(k, n) = r\}.$$

We state below, without proofs, some well-known facts from elementary number theory as lemmas. These results were known to Ramanujan [6] and von Sterneck [10].

Lemma 2.1. *With the notation as above, we have* $S_r(n) = \{rk : k \in S_1(n/r)\}$, *i.e.*, $S_r(n) = rS_1(n/r)$. *In particular*, $|S_r(n)| = |S_1(n/r)|$.

Lemma 2.2. *The sum of all the primitive n-th roots of unity is* $\mu(n)$, *i.e.*,

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} \exp(\tfrac{2\pi i k}{n}) = \mu(n).$$

In fact, we need the following variation of Lemma 2.

Lemma 2.3.

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} \cos(2\pi k/n) = \mu(n).$$

Lemma 2.4. *For any two positive integers n and m*,

$$\sum_{k=1}^{n} \exp\left(\tfrac{2\pi i m k}{n}\right) = \mu\left(\frac{n}{\gcd(m,n)}\right)\frac{\phi(n)}{\phi(n/\gcd(m,n))}$$

3. Characters of the dihedral group D_n

We write the elements of D_n as $D_n = \{1, \sigma, \sigma^2, \ldots, \sigma^{n-1}, \tau, \tau\sigma, \tau\sigma^2, \ldots, \tau\sigma^{n-1}\}$. The dihedral group D_n has only degree 1 and degree 2 irreducible representations.

3.1 *One-dimensional representations*

- When n is odd, there are two irreducible representations of degree 1 namely, χ_1 and χ_2 and the character table for those representations is as follows:

Character	σ^k	$\tau\sigma^k$
χ_1	1	1
χ_2	1	-1

- When n is even, there are four irreducible representations of degree 1 namely, χ_1, χ_2, χ_3 and χ_4 and the character table for those representations is as follows:

Character	σ^k	$\tau\sigma^k$
χ_1	1	1
χ_2	1	-1
χ_3	$(-1)^k$	$(-1)^k$
χ_4	$(-1)^k$	$(-1)^{k+1}$

3.2 *Two-dimensional representations*

Let h be a positive integer with $h < n/2$. A representation ρ_h of D_n has the character given by $\psi_h(\sigma^k) = 2\cos\frac{2\pi hk}{n}$ and $\psi_h(\tau\sigma^k) = 0$.

4. Supercharacter theory for finite groups

In this section, we recall definitions of supercharacter theory and their $*$-products. For full details, the reader is referred to the works of Diaconis and Isaacs [2] or Hendrickson [4] and Lamar [5].

In rough terms, a supercharacter theory for a group G is a setup consisting of (i) a partition of G where each part, called a superclass, is a union of conjugacy classes and (ii) a set of characters of G called supercharacters satisfying the main requirement that each supercharacter as a function is constant on each superclass. The precise complete definition is given below.

DEFINITION 4.1

A supercharacter theory of a finite group G is an ordered pair $\mathcal{S} = (\mathcal{K}, \mathcal{X})$, where $\mathcal{K}$ is a partition of G into unions of conjugacy classes, $\mathcal{X}$ is a partition of $\mathrm{Irr}(G)$, such that the following conditions are met:

- $|\mathcal{K}| = |\mathcal{X}|$.
- If e denotes the identity of G, then $\{e\} \in \mathcal{K}$.
- For each $X \in \mathcal{X}$, the character $\sigma_X = \sum_{\chi\in X} \chi(e)\chi$ is constant on the parts of $\mathcal{K}$.

If $\mathcal{S} = (\mathcal{K}, \mathcal{X})$ is a supercharacter theory, then we call the parts of $\mathcal{K}$ the superclasses of $\mathcal{S}$, or $\mathcal{S}$-superclasses, and we call the functions σ_X the supercharacters of $\mathcal{S}$, or $\mathcal{S}$-supercharacters.

DEFINITION 4.2

Let N be a normal subgroup of G. Let $\mathcal{S} = (\mathcal{K}, \mathcal{X})$ be a G-invariant supercharacter theory of N and $\mathcal{T} = (\mathcal{L}, \mathcal{Y})$ be a supercharacter theory of G/N. Then the supercharacter theory of the group G is given by the $*$-product of $\mathcal{S}$ and $\mathcal{T}$ is given by $\mathcal{S} *_N \mathcal{T} = (\mathcal{M}, \mathcal{Z})$, whose superclass partition is

$$\mathcal{M} = \mathcal{K} \cup \left\{ \bigcup_{gN \in L} gN : L \in \mathcal{L} \setminus \{\{eN\}\} \right\}$$

and whose supercharacter partition is

$$\mathcal{Z} = \left\{ X^G : X \in \mathcal{X} \setminus \{1_N\} \right\} \cup \mathcal{Y},$$

where $X^G = \{\mathrm{Irr}(G|\chi) : \chi \in X\}$ and the elements of $\mathrm{Irr}(G/N)$are identified as elements of $\mathrm{Irr}(G)$ through inflation.

5. Dimension formulæ and generating functions

We state the main result of our paper providing formulæ for the dimensions for various isotypical components. There is one formula for isotypical components corresponding to degree 2 irreducible representations of D_n. In the case of one-dimensional irreducible representations the formulæ are seperate for the two or four irreducible characters according as n is odd or even.

Theorem 5.1. *Let ψ_h be the irreducible character of degree 2 of the dihedral group D_n as above. Then the dimension of $H_d(D_n, \psi_h)$, the vector space of relative symmetric polynomials is described as in the below two cases.*

Case (*i*). *h is coprime to n*:

$$\dim H_d(D_n, \psi_h) = \frac{2}{n} \sum_{r|n} \binom{r + \frac{d}{n/r} - 1}{r-1} \mu\left(\frac{n}{r}\right). \tag{5.1}$$

The generating function in this case is given by

$$\sum_{d=0}^{\infty} \dim H_d(D_n, \psi_h) t^d = \frac{2}{n} \sum_{r|n} \mu\left(\frac{n}{r}\right) (1 - t^{\frac{n}{r}})^{-r}.$$

(*We follow the convention that $\binom{m}{k}$ is zero if m or k is not an integer.*)

Case (*ii*). *h is not coprime to n*:

$$\dim H_d(D_n, \psi_h) = \frac{2}{n} \sum_{r|n} \binom{r + \frac{d}{n/r} - 1}{r-1} \mu\left(\frac{n/r}{g_r}\right) \frac{\phi\,(n/r)}{\phi\left(\frac{n/r}{g_r}\right)}, \tag{5.2}$$

where $\mu(n)$ is the Möbius function and $g_r = \gcd(h, \frac{n}{r})$.

The generating function in this case is given by

$$\sum_{d=0}^{\infty} \dim H_d(D_n, \psi_h) t^d = \frac{2}{n} \sum_{r|n} \mu\left(\frac{n/r}{g_r}\right) \frac{\phi\,(n/r)}{\phi\left(\frac{n/r}{g_r}\right)} (1 - t^{\frac{n}{r}})^{-r}.$$

Before moving to the proof, we would like to record the following corollary.

COROLLARY 5.1.1

If $\gcd(d, n) = 1$, *then* $\dim H_d(D_n, \psi_h) = \frac{2}{n}\binom{d+n-1}{n-1}$, *a positive integer. In particular, for* $d = 1$, *there always exist linear homogeneous polynomials that are relative invariants with respect to any given* 2*-dimensional representation.*

Proof of Theorem 5.1. It suffices to prove the formula for the dimension of $H_d(D_n, \psi)$ for a general d. The formula for the generating function is a straight forward consequence.

Case (*i*). For definiteness, we fix the embedding of D_n in S_n with the generators of D_n as below: $D_n = \langle \sigma, \tau \rangle$, where σ is the n-cycle given by $(1, 2, 3, \ldots, n)$ and $\tau(j) = n + 1 - j$ is the reversal permutation. In fact, D_n can be embedded in S_n uniquely up to conjugacy. Now in the case of a 2-dimensional irreducible character ψ of D_n, $\psi(\tau\sigma^k) = 0$ for all k. So the dimension formula reduces to the summation over the cyclic subgroup of all rotations in D_n.

$$\dim H_d(D_n, \psi) = \frac{\psi(1)}{|D_n|} \sum_{k=1}^{n} \psi(\sigma^k)\, \mathrm{Tr}\,(\sigma^k) = \frac{2}{2n} \sum_{k=1}^{n} 2\cos\frac{2\pi k}{n}\, \mathrm{Tr}\,(\sigma^k). \tag{5.3}$$

Note that $\mathrm{Tr}\,(\sigma^k)$ is the trace of the k-th power of the n-cycle σ in the vector space of homogeneous polynomials in n variables, with σ permuting the variables cyclically. As the set of all monomials of degree d in n variables is a basis of this vector space, its dimension is $\binom{n+d-1}{n-1}$. Being a permutation action $\mathrm{Tr}\,(\sigma^k)$ is the number of monomials of degree d in n variables fixed by σ^k. So the calculation boils down to finding the number of invariant monomials of degree d. To calculate $\mathrm{Tr}\,(\sigma^k)$, let $r = \gcd(n, k)$. Then σ^k decomposes into a product of r number of disjoint cycles of length $\frac{n}{r}$. For a monomial to be invariant under σ^k, degree of all the variables within an $\frac{n}{r}$-cycle should be a constant. Call these degrees $d_1, d_2, \ldots, d_r$, i.e.,

$$d = \frac{n}{r}d_1 + \frac{n}{r}d_2 + \cdots + \frac{n}{r}d_r.$$

Therefore,

$$d_1 + d_2 + \cdots + d_r = \frac{d}{n/r}.$$

A necessary condition d must be a multiple of $\frac{n}{r}$. Let us assume this holds. So $\mathrm{Tr}\,(\sigma^k)$ is the number of ordered partitions of $\frac{d}{n/r}$ into r parts. This is known to be $\binom{r+\frac{d}{n/r}-1}{r-1}$. Substituting this value of trace into (5.3), we get

$$\dim H_d(D_n,\psi)=\frac{1}{n}\sum_{k=1}^{n}2\binom{r+\frac{d}{n/r}-1}{r-1}\cos\frac{2\pi k}{n}. \tag{5.4}$$

Note that for two terms of the summation on the right-hand side, if $1\le k_1,k_2\le n$ are such that $\gcd(k_1,n)=\gcd(k_2,n)$, then the coefficient of $\cos\frac{2\pi k_1}{n}$ equals that of $\cos\frac{2\pi k_2}{n}$. As the gcd of any number with n is a divisor of n, the dimension formula can be rewritten as a summation over the divisors of n. We treat the above summation as a sum of binomial coefficients $\binom{r+\frac{d}{n/r}-1}{r-1}$, one for each divisor r of n with some weights. These weights are sums of cosine values. So the dimensional formula takes the form

$$\dim H_d(D_n,\psi)=\frac{1}{n}\sum_{r|n}\Big[\sum_{k\in S_r(n)}2\cos\big(2\pi k/n\big)\Big]\binom{r+\frac{d}{n/r}-1}{r-1}. \tag{5.5}$$

That is,

$$\dim H_d(D_n,\psi)=\frac{1}{n}\sum_{r|n}\Big[\sum_{a\in S_1(n/r)}2\cos\big(2\pi a/\tfrac{n}{r}\big)\Big]\binom{r+\frac{d}{n/r}-1}{r-1}\quad\text{(by Lemma 2.1).} \tag{5.6}$$

Using Lemma 2.3, the above equation reduces to

$$\dim H_d(D_n,\psi)=\frac{2}{n}\sum_{r|n}\mu\left(\frac{n}{r}\right)\binom{r+\frac{d}{n/r}-1}{r-1}$$

Case (*ii*). Proceeding as in Case (i), we have

$$\dim H_d(D_n,\psi)=\frac{2}{n}\sum_{r|n}\left[\sum_{a\in S_1(n/r)}\cos\frac{2\pi ah}{n/r}\right]\binom{r+\frac{d}{n/r}-1}{r-1}. \tag{5.7}$$

The inner summation inside the square brackets in the above equation is actually the sum of the real parts of h-th powers of all primitive $\left(\frac{n}{r}\right)$-th roots of unity. Defining $g_r=\gcd(h,n/r)$, we see that the above is the same as the sum of all the real parts of g-th powers of all primitive $\left(\frac{n}{r}\right)$-th roots of unity. Now we can apply Lemma 2.4 which makes Equation (5.7) as

$$\dim H_d(D_n,\psi)=\frac{2}{n}\sum_{r|n}\left[\sum_{a\in S_1(n/g_r r)}\cos\frac{2\pi a}{n/g_r r}\right]\frac{\phi(n/r)}{\phi(n/g_r r)}\binom{r+\frac{d}{n/r}-1}{r-1}.$$

Again using Lemma 2.3,

$$\dim H_d(D_n,\psi)=\frac{2}{n}\sum_{r|n}\mu\left(\frac{n/r}{g_r}\right)\frac{\phi(n/r)}{\phi(n/g_r r)}\binom{r+\frac{d}{n/r}-1}{r-1}.$$

□

Theorem 5.2.

Case (i). *When n is odd*: *Let* χ_1 *and* χ_2 *be two irreducible characters of degree* 1 *with* χ_1 *being the trivial character and* ψ_2 *taking* $+1$ *on rotations and* -1 *on reflections. The dimensions of* $H_d(D_n, \chi_1)$ *and* $H_d(D_n, \chi_2)$ *are given by*

$$\dim H_d(D_n, \chi_1) = \frac{1}{2n}\left[\sum_{r|n}\binom{r+\frac{d}{n/r}-1}{r-1}\phi\left(\frac{n}{r}\right) + n\sum_{l=0}^{\lfloor d/2\rfloor}\binom{\frac{n-1}{2}+l-1}{l}\right]$$

and

$$\dim H_d(D_n, \chi_2) = \frac{1}{2n}\left[\sum_{r|n}\binom{r+\frac{d}{n|r}-1}{r-1}\phi\left(\frac{n}{r}\right) - n\sum_{l=0}^{\lfloor d/2\rfloor}\binom{\frac{n-1}{2}+l-1}{l}\right].$$

The generating functions for the above two cases are given by

$$\sum_{d=0}^{\infty}\dim H_d(D_n, \chi_1)t^d = \frac{1}{2n}\left[\sum_{r|n}\phi\left(\frac{n}{r}\right)(1-t^{\frac{n}{r}})^{-r} + \frac{n(1-t^2)^{-(n-1)/2}}{1-t}\right],$$

$$\sum_{d=0}^{\infty}\dim H_d(D_n, \chi_2)t^d = \frac{1}{2n}\left[\sum_{r|n}\phi\left(\frac{n}{r}\right)(1-t^{\frac{n}{r}})^{-r} - \frac{n(1-t^2)^{-(n-1)/2}}{1-t}\right].$$

Case (ii). *When n is even*: *Let* χ_1, χ_2, χ_3 *and* χ_4 *be the four irreducible characters of degree* 1. *Let* $H_d(D_n, \chi_1)$, $H_d(D_n, \chi_2)$, $H_d(D_n, \chi_3)$ *and* $H_d(D_n, \chi_4)$ *be the space of relative symmetric polynomials with respect to* χ_1, χ_2, χ_3 *and* χ_4. *Then the dimensions* $H_d(D_n, \chi_1)$, $H_d(D_n, \chi_2)$, $H_d(D_n, \chi_3)$ *and* $H_d(D_n, \chi_4)$ *are given by*

$$\dim H_d(D_n, \chi_1) = \frac{1}{2n}\left\{\sum_{r|n}\binom{r+\frac{d}{n/r}-1}{r-1}\phi\left(\frac{n}{r}\right) + \frac{n}{2}\left[\binom{\frac{n}{2}+\frac{d}{2}-1}{\frac{d}{2}} + \sum_{l=0}^{d/2}\binom{\frac{n-2}{2}+l-1}{l}(d-2l+1)\right]\right\},$$

$$\dim H_d(D_n, \chi_2) = \frac{1}{2n}\left\{\sum_{r|n}\binom{r+\frac{d}{n/r}-1}{r-1}\phi\left(\frac{n}{r}\right) - \frac{n}{2}\left[\binom{\frac{n}{2}+\frac{d}{2}-1}{\frac{d}{2}} + \sum_{l=0}^{d/2}\binom{\frac{n-2}{2}+l-1}{l}(d-2l+1)\right]\right\},$$

$$\dim H_d(D_n, \chi_3) = \frac{1}{2n}\left\{\sum_{r|n}(-1)^r\phi\left(\frac{n}{r}\right)\binom{r+\frac{d}{n/r}-1}{r-1} + \frac{n}{2}\left[\binom{\frac{n}{2}+\frac{d}{2}-1}{\frac{d}{2}} - \sum_{l=0}^{\lfloor d/2\rfloor}\binom{\frac{n-2}{2}+l-1}{l}(d-2l+1)\right]\right\},$$

$$\dim H_d(D_n,\chi_4)=\frac{1}{2n}\left\{\sum_{r|n}(-1)^r\phi\left(\frac{n}{r}\right)\binom{r+\frac{d}{n/r}-1}{r-1}\right.$$
$$\left.-\frac{n}{2}\left[\binom{\frac{n}{2}+\frac{d}{2}-1}{\frac{d}{2}}-\sum_{l=0}^{\lfloor d/2\rfloor}\binom{\frac{n-2}{2}+l-1}{l}(d-2l+1)\right]\right\}.$$

The generating functions for the above four cases are given by

$$\sum_{d=0}^{\infty}\dim H_d(D_n,\chi_1)t^d$$
$$=\frac{1}{2n}\left[\sum_{r|n}\phi\left(\frac{n}{r}\right)(1-t^{\frac{n}{r}})^{-r}+\frac{n}{2}(1-t^2)^{-(n+2)/2}(2+t^2)(1+t)\right],$$
$$\sum_{d=0}^{\infty}\dim H_d(D_n,\chi_2)t^d$$
$$=\frac{1}{2n}\left[\sum_{r|n}\phi\left(\frac{n}{r}\right)(1-t^{\frac{n}{r}})^{-r}-\frac{n}{2}(1-t^2)^{-(n+2)/2}(2+t^2)(1+t)\right],$$
$$\sum_{d=0}^{\infty}\dim H_d(D_n,\chi_3)t^d$$
$$=\frac{1}{2n}\left[\sum_{r|n}\phi\left(\frac{n}{r}\right)(-1)^r(1-t^{\frac{n}{r}})^{-r}-\frac{n}{2}(1-t^2)^{-(n+2)/2}(1+t+t^2)\right],$$
$$\sum_{d=0}^{\infty}\dim H_d(D_n,\chi_4)t^d$$
$$=\frac{1}{2n}\left[\sum_{r|n}\phi\left(\frac{n}{r}\right)(-1)^r(1-t^{\frac{n}{r}})^{-r}+\frac{n}{2}(1-t^2)^{-(n+2)/2}(1+t+t^2)\right].$$

Proof.

Case (i). *When n is odd*: By definition

$$\dim H_d(D_n,\chi_1)=\frac{1}{2n}\left[\sum_{k=1}^{n}\mathrm{Tr}\,(\sigma^k)\psi_1(\sigma^k)+\sum_{k=1}^{n}\mathrm{Tr}\,(\tau\sigma^k)\psi_1(\tau\sigma^k)\right],$$

i.e.,

$$\dim H_d(D_n,\chi_1)=\frac{1}{2n}\left[\sum_{k=1}^{n}\mathrm{Tr}\,(\sigma^k)+\sum_{k=1}^{n}\mathrm{Tr}\,(\tau\sigma^k)\right].$$

When n is odd, all the reflections of D_n falls into a single conjugacy class. Hence the above summation becomes

$$\dim H_d(D_n, \chi_1) = \frac{1}{2n}\left[n\,\mathrm{Tr}\,(\tau) + \sum_{k=1}^{n} \mathrm{Tr}\,(\sigma^k)\right].$$

Now calculation of $\mathrm{Tr}\,(\sigma^k)$ is the same as in Theorem 1. It remains to find $\mathrm{Tr}\,(\tau)$. Since n is odd, τ is the product of $\frac{n-1}{2}$ transpositions and has one fixed point. Now $\mathrm{Tr}\,(\tau)$ is the count of monomials fixed by τ. We denote the variables by $x_1, x_2, \ldots, x_{(n-1)/2}$, $y_1, y_2, \ldots, y_{(n-1)/2}, z$. Without loss of generality, let us assume that $\tau(z) = z$, $\tau(x_i) = y_i$ and $\tau(y_i) = x_i$ for $1 \le i \le (n-1)/2$. Now a monomial of degree d invariant under τ has to be of the form

$$z^{d_0} x_1^{d_1} y_1^{d_1} x_2^{d_2} y_2^{d_2} x_3^{d_3} y_3^{d_3} \ldots.$$

The number of tuples $(d_0, d_1, \ldots, d_{(n-1)/2})$ such that $d_0 + 2(d_1 + d_2 + \cdots + d_{(n-1)/2}) = d$ is the total number of ordered partitions of $(d - d_0)/2$ into $(n-1)/2$ parts with all d_0 satisfying $d - d_0$ is an even non negative integer. This is easily verified to be $\sum_{l=0}^{\lfloor d/2 \rfloor} \binom{\frac{n-1}{2}+l-1}{l}$. Hence the formula becomes

$$\dim H_d(D_n, \chi_1) = \frac{1}{2n}\left[\sum_{r|n} \binom{r + \frac{d}{n/r} - 1}{r-1} + n \sum_{l=0}^{\lfloor d/2 \rfloor} \binom{\frac{n-1}{2} + l - 1}{l} \right].$$

Using the same arguments as above, we have

$$\dim H_d(D_n, \chi_2) = \frac{1}{2n}\left[\sum_{r|n} \binom{r + \frac{d}{n/r} - 1}{r-1} - n \sum_{l=0}^{\lfloor d/2 \rfloor} \binom{\frac{n-1}{2} + l - 1}{l} \right].$$

Case (*ii*). By definition

$$\dim H_d(D_n, \chi_1) = \frac{1}{2n}\left[\sum_{k=1}^{n} \mathrm{Tr}\,(\sigma^k)\chi_1(\sigma^k) + \sum_{k=1}^{n} \mathrm{Tr}\,(\tau\sigma^k)\chi_1(\tau\sigma^k) \right],$$

i.e.,

$$\dim H_d(D_n, \chi_1) = \frac{1}{2n}\left[\sum_{k=1}^{n} \mathrm{Tr}\,(\sigma^k) + \sum_{k=1}^{n} \mathrm{Tr}\,(\tau\sigma^k) \right].$$

Now $\mathrm{Tr}\,(\sigma^k)$ is the same as in Theorem 1. It remains to find $\mathrm{Tr}\,(\tau\sigma^k)$. Since n is even, all the reflections $\tau\sigma^k$ fall into two conjugacy classes according as k is even or odd. Hence the formula becomes

$$\dim H_d(D_n, \chi_1) = \frac{1}{2n}\left[\sum_{k=1}^{n} \mathrm{Tr}\,(\sigma^k) + \frac{n}{2}\mathrm{Tr}\,(\tau\sigma) + \frac{n}{2}\mathrm{Tr}\,(\tau) \right].$$

Using the same argument as we did in Case (i) of this theorem, we can find the trace of the reflections in both the cases (when n is odd or even). Hence the formula becomes

$$\dim H_d(D_n,\chi_1) = \frac{1}{2n}\left\{\sum_{r|n}\binom{r+\frac{d}{n/r}-1}{r-1}\phi\left(\frac{n}{r}\right) + \frac{n}{2}\binom{\frac{n}{2}+\frac{d}{2}-1}{\frac{d}{2}} + \sum_{l=0}^{d/2}\binom{\frac{n-2}{2}+l-1}{l}(d-2l+1)\right\}.$$

In the same way,

$$\dim H_d(D_n,\chi_2) = \frac{1}{2n}\left\{\sum_{r|n}\binom{r+\frac{d}{n/r}-1}{r-1}\phi\left(\frac{n}{r}\right) - \frac{n}{2}\binom{\frac{n}{2}+\frac{d}{2}-1}{\frac{d}{2}} + \sum_{l=0}^{d/2}\binom{\frac{n-2}{2}+l-1}{l}(d-2l+1)\right\}.$$

Now again using the definition, we have

$$\dim H_d(D_n,\chi_3) = \frac{1}{2n}\left[\sum_{k=1}^{n}\mathrm{Tr}\,(\sigma^k)\chi_3(\sigma^k) + \sum_{k=1}^{n}\mathrm{Tr}\,(\tau\sigma^k)\chi_3(\tau\sigma^k)\right],$$

i.e.,

$$\dim H_d(D_n,\chi_3) = \frac{1}{2n}\left[\sum_{k=1}^{n}(-1)^k)\,\mathrm{Tr}\,(\sigma^k) + \sum_{k=1}^{n}(-1)^k\,\mathrm{Tr}\,(\tau\sigma^k)\right].$$

Using the same result for $\mathrm{Tr}\,(\sigma^k)$ from Theorem 1, we have

$$\dim H_d(D_n,\chi_3) = \frac{1}{2n}\left\{\sum_{r|n}\left[\left(\sum_{k\in S_r(n)}(-1)^k\right)\binom{r+\frac{d}{n/r}-1}{r-1}\right] + \sum_{k=1}^{n}(-1)^k\,\mathrm{Tr}\,(\tau\sigma^k)\right\}.$$

Since n is even, all k's in the inner summation are even or odd according as r is even or odd and it sums up to $(-1)^r\phi(\frac{n}{r})$. Hence the above equation becomes

$$\dim H_d(D_n,\chi_3) = \frac{1}{2n}\left\{\sum_{r|n}\left[(-1)^r\phi\left(\frac{n}{r}\right)\binom{r+\frac{d}{n/r}-1}{r-1}\right] + \sum_{k=1}^{n}(-1)^k\,\mathrm{Tr}\,(\tau\sigma^k)\right\}.$$

It remains to find $\mathrm{Tr}\,(\tau\sigma^k)$. Using the same argument as we did in Case (i), we can calculate it. Hence the above equation reduces to

$$\dim H_d(D_n,\chi_3)=\frac{1}{2n}\left\{\sum_{r|n}(-1)^r\phi\left(\frac{n}{r}\right)\binom{r+\frac{d}{n/r}-1}{r-1}+\frac{n}{2}\left[\binom{\frac{n}{2}+\frac{d}{2}-1}{\frac{d}{2}}-\sum_{l=0}^{d/2}\binom{\frac{n-2}{2}+l-1}{l}(d-2l+1)\right]\right\}.$$

In the same way, we have

$$\dim H_d(D_n,\chi_4)=\frac{1}{2n}\left\{\sum_{r|n}(-1)^r\phi\left(\frac{n}{r}\right)\binom{r+\frac{d}{n/r}-1}{r-1}-\frac{n}{2}\left[\binom{\frac{n}{2}+\frac{d}{2}-1}{l}-\sum_{l=0}^{\lfloor d/2\rfloor}\binom{\frac{n-2}{2}+l-1}{l}(d-2l+1)\right]\right\}.$$

To find the generating functions for $\dim H_d(D_n,\chi_1)$, $\dim H_d(D_n,\chi_2)$, $\dim H_d(D_n,\chi_3)$ *and* $\dim H_d(D_n,\chi_4)$: Let $G_1(t)$ be the generating function for $\sum_{r|n}\phi\left(\frac{n}{r}\right)\binom{r+\frac{d}{n/r}-1}{r-1}$ which is easily seen to be $\sum_{r|n}(-1)^r\phi\left(\frac{n}{r}\right)(1-t^{\frac{n}{r}})^{-r}$. Let $G_2(t)$ and $G_3(t)$ be the generating functions corresponding to $\binom{\frac{n}{2}+\frac{d}{2}-1}{l}$ and $\sum_{l=0}^{d/2}\binom{\frac{n-2}{2}+l-1}{l}(d-2l+1)$ respectively. Now $G_2(t)$ is very easily verified to be $(1-t^2)^{\frac{n}{2}}$. Now the generating functions for $\dim(H_d,\chi_1)$, $\dim(H_d,\chi_2)$, $\dim(H_d,\chi_3)$ and $\dim(H_d,\chi_4)$ are given by

$$\sum_{d=0}^{\infty}\dim H_d(D_n,\chi_1)t^d=\frac{1}{2n}\left\{G_1(t)+\frac{n}{2}\left[G_2(t)+G_3(t)\right]\right\},$$
$$\sum_{d=0}^{\infty}\dim H_d(D_n,\chi_2)t^d=\frac{1}{2n}\left\{G_1(t)-\frac{n}{2}\left[G_2(t)+G_3(t)\right]\right\},$$
$$\sum_{d=0}^{\infty}\dim H_d(D_n,\chi_3)t^d=\frac{1}{2n}\left\{G_1(t)+\frac{n}{2}\left[G_2(t)-G_3(t)\right]\right\},$$
$$\sum_{d=0}^{\infty}\dim H_d(D_n,\chi_3)t^d=\frac{1}{2n}\left\{G_1(t)-\frac{n}{2}\left[G_2(t)-G_3(t)\right]\right\}.$$

Now evaluation of $G_3(t)$ is given in the following lemma. □

Lemma 5.3. *For*

$$\sum_{l=0}^{\lfloor d/2\rfloor}\binom{\frac{n-2}{2}+l-1}{l}(d-2l+1),$$

the generating function is

$$G_3(t)=(1-t^2)^{-\frac{n+2}{2}}(1+2t+t^2+t^3).$$

Proof of lemma. First consider

$$\sum_{l=0}^{\lfloor d/2 \rfloor} \binom{\frac{n-2}{2}+l-1}{l}(d-2l+1).$$

Let

$$b_d = \sum_{l=0}^{\lfloor d/2 \rfloor} \binom{m+l-1}{l}(d-2l+1)$$

and

$$G_4(t) = \sum_{t=0}^{\infty} b_d t^d.$$

One easily verifies

$$b_{2d+1} = b_{2d} + \sum_{l=0}^{d} \binom{m+1-1}{l}.$$

Now again, by induction we get

$$\sum_{l=0}^{d} \binom{m+1-1}{l} = \binom{m+d}{d}.$$

Therefore,

$$b_{2d+1} = b_{2d} + \binom{m+d}{d}.$$

Let

$$\sum_{d=0}^{\infty} b_{2d} t^{2d} = G_5(t).$$

Now

$$\sum_{t=0}^{\infty} b_{2d+1} t^{2d+1} = t \sum_{t=0}^{\infty} \left[b_{2d} t^{2d} + \binom{m+d}{d} t^{2d} \right] = tG_5(t) + t(1-t^2)^{-(m+1)}.$$

Hence

$$G_4(t) = \sum_{t=0}^{\infty} b_d t^d = \sum_{t \text{ even}} + \sum_{t, \text{odd}} = G_5(t) + tG_5(t) + t(1-t^2)^{-(m+1)}.$$

To evaluate $G_5(t)$,

$$
\begin{aligned}
G_5(t) &= \sum_{d=0}^{\infty}\sum_{l=0}^{d}\binom{m+l-1}{l}(2d-2l+1)t^{2d} \\
&= \sum_{d=0}^{\infty}\sum_{l=0}^{d}\binom{m+l-1}{l}(2d+1)t^{2d} - 2\sum_{l=0}^{d} l\binom{m+l-1}{l}t^{2d}.
\end{aligned}
$$

Now using induction one can show that

$$
\begin{aligned}
\sum_{l=0}^{d} l\binom{m+l-1}{l} &= m\sum_{l=0}^{d}\frac{l+m-m}{m}\binom{m+l-1}{l} \\
&= m\sum_{l=0}^{d}\frac{m+l}{m}\binom{m+l-1}{l} - m\sum_{l=0}^{d}\binom{m+l-1}{l},
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\sum_{l=0}^{d} l\binom{m+l-1}{l} &= m\sum_{l=0}^{d}\frac{m+l}{m}\frac{(m+l-1)!}{l!(m-1)!} - m\sum_{l=0}^{d}\binom{m+l-1}{l} \\
&= m\sum_{l=0}^{d}\binom{m+l}{l} - m\sum_{l=0}^{d}\binom{m+l-1}{l}.
\end{aligned}
$$

After simplification, we have

$$
\sum_{l=0}^{d} l\binom{m+l-1}{l} = m\binom{m+d}{d-1}.
$$

Hence

$$
\begin{aligned}
G_5(t) &= \sum_{d=0}^{\infty}\binom{m+d}{d}(2d+1)t^{2d} - 2\sum_{d=0}^{\infty} m\binom{m+d}{d-1}t^{2d} \\
&= \sum_{d=0}^{\infty}\binom{m+d}{d}2dt^{2d} + \sum_{d=0}^{\infty}\binom{m+d}{d}t^{2d} - 2\sum_{d=0}^{\infty} m\binom{m+d}{d-1}t^{2d}.
\end{aligned}
$$

Now using

$$
\sum_{d=0}^{\infty}\binom{m+d}{d}2dt^{2d} = t\frac{d}{dt}[(1-t^2)^{-(m+1)}]
$$

and

$$
\sum_{d=0}^{\infty} m\binom{m+d}{d}t^{2d} = 2mt^2(1-t^2)^{-(m+2)}
$$

and simplifying

$$G_5(t) = (1 - t^2)^{-(m+2)}(2t^2 + 1),$$

we get

$$G_4(t) = (1 - t^2)^{-(m+2)}(1 + 2t + 2t^2 + t^3).$$

Replacing m by $\frac{n-2}{2}$ in $G_4(t)$, we have

$$G_3(t) = (1 - t^2)^{-(n+2)/2}(1 + 2t + 2t^2 + t^3).$$ □

6. Applications

6.1 *Construction of a supercharacter theory*

First we focus on two-dimensional irreducible characters ψ_h of D_n for $1 \le h < n/2$. From the dimensional formulæ in Theorem 5.1 it is evident that

$$\dim H_d(D_n, \psi_{h_1}) = \dim H_d(D_n, \psi_{h_2}) \quad \text{if} \quad \gcd(h_1, n) = \gcd(h_2, n)$$

Calling two irreducible characters of D_n equivalent when the dimension of the space of relative invariants corresponding to them are equal, we studied the partition on the irreducible characters of D_n that are equivalence classes for this relation.

This partition turned out to satisfy the requirements of a supercharacter theory with a suitably defined partition on the elements of D_n.

So our supercharacter theory consists of superclasses S_0, S_r, and supercharacters φ_0, φ_r for each r dividing n which are defined as in the tables below. (Here C_n denotes the cyclic subgroup consisting of all rotations in D_n.)

S_0	S_r for $r \mid n$
$D_n \setminus C_n$	$g \in C_n \subset D_n$ with ord $(g) = r$

φ_0	$\varphi_{n/2}$ for n even	φ_r, for $r \mid n, r < n/2$	φ_n
χ_2	$\chi_3 + \chi_4$	$\sum_h{}' 2\psi_h,\ h < n/2,\ \gcd(h, n) = r$	χ_1

What remains is to prove that the characters φ_0, φ_r are constant on S_0 and S_r. For this, note that our definition of S_r is actually the whole orbit for the action of Aut(C_n) on D_n, and φ_r is the sum of characters in the associated orbit on the set of irreducible characters of D_n. So this property follows.

Theorem 6.1. *The supercharacter theory of D_n constructed above is the $*$-product of a D_n-invariant supercharacter theory of C_n with the unique supercharacter theory of D_n/C_n.*

Proof. Our proof is constructive. We specify the supercharacter theories for C_n and D_n/C_n whose $*$-product gives the above supercharacter theory of D_n.

For C_n, it is $(\mathcal{K}, \mathcal{X})$, with partition $\mathcal{K}$ for C_n defined with one part K_r for each divisor r consisting of elements of order r.

Let σ be a generator for C_n. We use the notation α_j for characters of C_n with $\alpha_j(\sigma) = e^{2\pi ij/n}$, for $j = 1, 2, \ldots, n$. Note that α_n will be the trivial character.

For each divisor r of n, we define $X_r = \{\alpha_j \mid \gcd(j, n) = r\}$. These X_r's define a partition $\mathcal{X}$ of the set of irreducible characters of C_n; this along with the partitions K_r is easily verified to be a supercharacter theory of C_n.

This, in fact, is the supercharacter theory of C_n studied in depth by Fowler *et al.* [3, section 2.2]. Their aim was to connect it with Ramanujan sums in number theory. □

The quotient group in our case being D_n/C_n is of order 2, and has an unique supercharacter theory, viz. the ordinary character theory. That our construction for D_n is a $*$-product of these is a simple consequence of the definitions and the following fact about dihedral group:

PROPOSITION

For n odd and α_j $(j \neq n)$ (i.e., a non-trivial irreducible character of C_n), the induced character $\mathrm{Ind}_{C_n}^{D_n}(\alpha_j)$ *is a degree 2 irreducible character of D_n and coincides with* $\mathrm{Ind}_{C_n}^{D_n}(\alpha_{n-j})$.

For n even, $\mathrm{Ind}_{C_n}^{D_n}(\alpha_{n/2})$ *is the sum of the two 1-dimensional characters χ_3 and χ_4 as specified in section 3.1.*

6.2 *A congruence for some special binomial coefficients*

Now we give another consequence of our main formula for dimension of relative invariants below:

COROLLARY 6.1.1 (Corollary to Theorem 5.1)

For any two odd prime numbers p and q, we have

$$\binom{pq+p-1}{p} \equiv q \pmod{pq}.$$

Proof. Consider the formula in Equation (5.1) for the case $n = pq$ and $d = p$. The summation giving the dimension has just two terms corresponding to $r = q$ and $r = pq$.

Now the dimension of relative invariants is

$$\frac{2}{pq}\left[\binom{q}{q-1}\mu(p) + \binom{pq+p-1}{pq-1}\mu(1)\right].$$

Being a vector space dimension, the above rational number is a positive integer and so pq divides the term inside the square bracket (by assumption, pq is odd). So $-q + \binom{pq+p-1}{pq-1}$ is a multiple of pq, thereby proving the corollary. □

7. Examples

7.1 *The dihedral group D_{10}*

Consider the group D_{10}. It has 4 one-dimensional irreducible representations and 4 two-dimensional irreducible representations. Here we give the generating functions for all the eight irreducible representations.

Two-dimensional representations: As $0 < h < \frac{n}{2}$, h can assume 1, 2, 3 and 4.

Case (*i*). *When h is coprime with n*, *i.e.* $h = 1, 3$. The generating function is

$$\frac{1}{5}\left[\frac{1}{(1-t^{10})} - \frac{1}{(1-t^5)^2} - \frac{1}{(1-t^2)^5} + \frac{1}{(1-t)^{10}}\right].$$

Case (*ii*). *When h is not coprime with n*, *i.e.* $h = 2, 4$. The generating function is

$$\frac{1}{5}\left[-\frac{1}{(1-t^{10})} - \frac{1}{(1-t^5)^2} + \frac{1}{(1-t^2)^5} + \frac{1}{(1-t)^{10}}\right].$$

One-dimensional representations: The generating functions for χ_1 and χ_2 are given by

$$\frac{1}{20}\left\{\left[\frac{1}{(1-t^{10})}+\frac{4}{(1-t^5)^2}+\frac{1}{(1-t^2)^5}+\frac{1}{(1-t)^{10}}\right]\pm 5\left[\frac{(2+t^2)(1+t)}{(1-t^2)^6}\right]\right\}.$$

The generating functions for χ_3 and χ_4 are given by

$$\frac{1}{20}\left\{\left[-\frac{1}{(1-t^{10})}+\frac{4}{(1-t^5)^2}-\frac{1}{(1-t^2)^5}+\frac{1}{(1-t)^{10}}\right]\mp 5\left[\frac{(1+t+t^2)}{(1-t^2)^6}\right]\right\}.$$

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