Proc. Indian Acad. Sci. (Math. Sci.) (2020) 130:15
https://doi.org/10.1007/s12044-019-0532-6

Examples of blown up varieties having projective bundle structures

NABANITA RAY[1,2]

[1]The Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600 113, India
[2]Homi Bhabha National Institute, Training School Complex, Anushakti Nagar, Mumbai 400 094, India
Email: nabanitar@imsc.res.in

MS received 1 May 2019; revised 29 May 2019; accepted 30 May 2019

Abstract. We give some examples of blow up of projective space along some projective subvariety, such that these blown up spaces are isomorphic to a projective bundle over some projective space.

Keywords. Blow up; projective bundle; nef cone; chow ring.

2000 Mathematics Subject Classification. Primary: 14A10; Secondary: 14C20, 14C22.

1. Introduction

It is always interesting to ask, under which criterion, blow up of a projective variety along a projective subvariety is isomorphic to the projective bundle over some projective variety. In general, blow up of a projective space along a projective subvariety is not isomorphic to the projective bundle over some projective space. But we know some examples, where it happens. Let $Z = \tilde{P}^n_\Lambda$ be the blow up of a projective space $\mathbb{P}^n = \mathbb{P}V$ along a linear subspace $\Lambda \simeq \mathbb{P}^{r-1}$. It is well known from Section 9.3.2 of [2] that Z is a total space of projective bundle, i.e. $Z \simeq \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^{n-r}}(1) \oplus \mathcal{O}^r_{\mathbb{P}^{n-r}}$ is a locally free sheaf of rank $r+1$ on $\mathbb{P}^{n-r}$.

Motivated by this result, we produce here some non-linear examples, where blow up of a projective space along some non-linear subvariety will be isomorphic to a projective bundle over a projective variety. Also, we have calculated the nef cone of those varieties.

We take the three-fold $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$ by Segre embedding. This is degree three three-fold in $\mathbb{P}^5$, say X_0. Now, we blow up $\mathbb{P}^5$ along the subvariety X_0 and we get that $\tilde{\mathbb{P}}^5_{X_0}$ is $\mathbb{P}^3$ bundle over $\mathbb{P}^2$ (see Theorem 3.1). Also, we describe explicitly the rank four vector bundle E over $\mathbb{P}^2$, such that $\tilde{\mathbb{P}}^5_{X_0} \simeq \mathbb{P}(E)$ (see Theorem 3.2).

Take a generic hyperplane H in $\mathbb{P}^5$, such that $X_1 = X_0 \cap H$ is a non-singular degree three surface in $\mathbb{P}^4$. We get $\tilde{\mathbb{P}}^4_{X_1} \simeq \mathbb{P}(E_1)$, where E_1 is a rank three bundle over $\mathbb{P}^2$ which is a quotient of E (see Theorem 3.4). Similarly, take the generic hyperplane H_1 in $\mathbb{P}^4$ such

Published online: 22 January 2020

that $X_2 = X_1 \cap H_1$ is a twisted cubic in $\mathbb{P}^3$. We prove $\tilde{\mathbb{P}}^3_{X_2} \simeq \mathbb{P}(E_2)$, where E_2 is the rank two bundle over $\mathbb{P}^2$ which is a quotient of E_1.

Conversely, we prove that if C is a non-linear subvariety of $\mathbb{P}^3$ (i.e. C is not a single point or a line in $\mathbb{P}^3$) and $\tilde{\mathbb{P}}^3_C$ has a projective bundle structure, then C has to be twisted cubic.

2. Notations and definitions

We denote by $\mathbb{P}^n$ the projective space over the field $\mathbb{C}$ of complex numbers. Let Δ be a non-singular sub-variety of $\mathbb{P}^n$ and $\tilde{P}^n_\Delta$ is denoted as $\mathbb{P}^n$ blown up along Δ. Here, $\pi : \tilde{P}^n_\Delta \to \mathbb{P}^n$ is the canonical blowing up map, and E_Δ is the corresponding exceptional divisor. The Picard group of $\tilde{P}^n_\Delta$ is generated by $\pi^*(\mathcal{O}_{\mathbb{P}^n}(1))$ and E_Δ.

Let X be a smooth projective variety, E a holomorphic vector bundle on it, and by $\mathbb{P}(E)$ the projectivization of E, defined as $\mathbb{P}(E) = \mathrm{Proj(Sym(E))}$, where Sym(E) is the symmetric algebra of the sheaf of the section of E. Also, $\mathbb{P}(E)$ can be described as a projective bundle of the one-dimensional quotient of E. We follow here the definition of [4] to describe $\mathbb{P}(E)$.

3. Main results

Let $i : \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ be the Segre embedding defined by sending $[x_0, x_1] \times [y_0, y_1, y_2]$ to $[x_0y_0, x_0y_1, x_0y_2, x_1y_0, x_1y_1, x_1y_2]$. Let $\{z_i \mid i = 0, 1, \ldots, 5\}$ be the homogeneous co-ordinate of $\mathbb{P}^5$. If $i(\mathbb{P}^1 \times \mathbb{P}^2) = X_0$, then X_0 is a three-fold in $\mathbb{P}^5$ which is defined by the equations $f_0 = z_0z_4 - z_1z_3$, $f_1 = z_0z_5 - z_2z_3$, $f_2 = z_2z_4 - z_1z_5$. The morphism i is defined by the complete linear system $(1, 1)$ of $\mathbb{P}^1 \times \mathbb{P}^2$, where the Picard group of $\mathbb{P}^1 \times \mathbb{P}^2$ is $\mathbb{Z} \oplus \mathbb{Z}$. Hence $\deg(X_0) = (1, 1) \cdot (1, 1) \cdot (1, 1) = 3$ in $\mathbb{P}^5$.

Theorem 3.1. *$\mathbb{P}^5$ blown up along the closed subscheme X_0 is a $\mathbb{P}^3$ bundle over $\mathbb{P}^2$. Here the $\mathbb{P}^3$ bundle map $\tilde{\phi}_0 : \tilde{\mathbb{P}}^5_{X_0} \to \mathbb{P}^2$ is defined by the divisor $2\pi^*H - E_{X_0}$.*

Proof. Let us consider the linear system $|\mathcal{O}_{\mathbb{P}^5}(2) \otimes \mathcal{I}_{X_0}|$, which is defined by set of all degree two hypersurfaces of $\mathbb{P}^5$, which contain X_0. Here, $\mathcal{I}_{X_0}$ is the ideal sheaf corresponding to the closed subscheme X_0. The vector space $H^0(\mathcal{O}_{\mathbb{P}^5}(2) \otimes \mathcal{I}_{X_0})$ is generated by the basis, f_0, f_1, f_2. Hence the linear system $|\mathcal{O}_{\mathbb{P}^5}(2) \otimes \mathcal{I}_{X_0}|$ is isomorphic to $\mathbb{P}^2$. So the linear system defines a morphism $\phi_0 : \mathbb{P}^5 \backslash X_0 \to \mathbb{P}^2$ and we can extend this map as $\tilde{\phi}_0 : \tilde{\mathbb{P}}^5_{X_0} \to \mathbb{P}^2$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{\mathbb{P}}^5_{X_0} & \xrightarrow{\pi} & \mathbb{P}^5 \\ & \searrow^{\tilde{\phi}_0} & \downarrow{\scriptstyle \phi_0} \\ & & \mathbb{P}^2 \end{array}$$

and the $\tilde{\phi}_0$ map is defined by the linear system $|2\pi^*H - E_{X_0}|$ of $\tilde{\mathbb{P}}^5_{X_0}$.

Now, our claim is that each fiber of $\tilde{\phi}_0$ is isomorphic to $\mathbb{P}^3$. First, define the ϕ_0 map coordinate-wise, i.e. $\phi_0([z_0, z_1, \ldots, z_5]) = [z_0z_4 - z_1z_3, z_0z_5 - z_2z_3, z_2z_4 - z_1z_5] = [f_0, f_1, f_2]$. Now $\phi_0^{-1}[1, 0, 0] = V(f_1, f_2)$ and $X_0 \subseteq V(f_1, f_2)$. $\deg(V(f_1, f_2)) = 4$. As

$\deg(X_0) = 3$, $V(f_1, f_2) = X_0 \cup L$ such that $\deg L = 1$, i.e. $L \simeq \mathbb{P}^3$. Hence $\tilde{\phi}_0^{-1}[1, 0, 0]$ is isomorphic to the strict transformation of L. Similarly, when $a_0 \neq 0$, then $\tilde{\phi}_0^{-1}[a_0, a_1, a_2]$ is a strict transformation of $\overline{V(a_0 f_1 - a_1 f_0, a_0 f_2 - a_2 f_0)\backslash X_0}$. When $a_1 \neq 0$, then $\tilde{\phi}_0^{-1}[a_0, a_1, a_2]$ is a strict transformation of $\overline{V(a_1 f_0 - a_0 f_1, a_2 f_1 - a_1 f_2)\backslash X_0}$ and when $a_2 \neq 0$, $\tilde{\phi}_0^{-1}[a_0, a_1, a_2]$ is a strict transformation of $\overline{V(a_0 f_2 - a_2 f_0, a_1 f_2 - a_2 f_1)\backslash X_0}$.

Now, we need to check if the $\tilde{\phi}_0$ map satisfies the co-cycle condition over an affine cover of $\mathbb{P}^2$ ([4], Chapter II.7, Exercise 10). Let $\mathbb{P}^2 = \cup_{i=0}^2 U_i$, where $U_i = \{a_i \neq 0\}$. Let $a_0 \neq 0$, $V(a_0 f_1 - a_1 f_0, a_0 f_2 - a_2 f_0) = X_0 \cup \mathbb{P}^3$, where $\mathbb{P}^3 = V(g_1, g_2)$, and g_i are the hyperplanes in $\mathbb{P}^5$. Then, $a_0 f_1 - a_1 f_0 = h_{11} g_1 + h_{12} g_2$ and $a_0 f_2 - a_2 f_0 = h_{21} g_1 + h_{22} g_2$, where $\{h_{ij}\}$ are degree one and linearly independent set. Define the morphism $\psi_0 : \tilde{\phi}_0^{-1}(U_0) \to U_0 \times \mathbb{P}^3$. Let $x \in \tilde{\phi}_0^{-1}(U_0)$, where $\tilde{\phi}_0(x) = [a_0, a_1, a_2]$ and x corresponds to the point b in $\mathbb{P}^3$ over $[a_0, a_1, a_2]$. Then we define the map $\psi_0(x) = [a_0, a_1, a_2] \times b$. Now define $\lambda_0 : U_0 \times \mathbb{P}^3 \to \tilde{\phi}_0^{-1}(U_0)$. Let $[a_0, a_1, a_2] \times [b_0, b_1, b_2, b_3] \in U_0 \times \mathbb{P}^3$. Then we can find g_1 and g_2 such that $V(g_1, g_2) \subset V(a_0 f_1 - a_1 f_0, a_0 f_2 - a_2 f_0)$, where f_i and g_i are the same as defined above. Here $[b_0, b_1, b_2, b_3] \in \mathbb{P}^3 \simeq V(g_1, g_2) = L \simeq \tilde{L}$. Then $[b_0, b_1, b_2, b_3]$ corresponds to a point in $\tilde{\phi}_0^{-1}(U_0)$ on the fiber of $[a_0, a_1, a_2]$. Hence, clearly $\psi_0 \circ \lambda_0 = id$ and $\lambda_0 \circ \psi_0 = id$. Similarly, ψ_1 and ψ_2 are defined.

Now, consider $a = [a_0, a_1, a_2] \in U_0 \cap U_1$. We need to show the map $\psi_1 \circ \psi_0^{-1} : a \times \mathbb{P}^3 \to a \times \mathbb{P}^3$ is a linear automorphism. But, this will clearly follow from the fact that, if $a_0 f_1 - a_1 f_0 = h_{11} g_1 + h_{12} g_2$ and $a_0 f_2 - a_2 f_0 = h_{21} g_1 + h_{22} g_2$, then $a_2 f_1 - a_1 f_2 = \frac{a_2}{a_0}(a_0 f_1 - a_1 f_0) + \frac{a_1}{a_0}(a_0 f_2 - a_2 f_0) = (\frac{a_2}{a_0} h_{11} + \frac{a_1}{a_0} h_{21}) g_1 + (\frac{a_2}{a_0} h_{12} + \frac{a_1}{a_0} h_{22}) g_2$. Hence the result is proved. □

Now, $\tilde{\mathbb{P}}^5_{X_0} = \mathbb{P}(E)$, where E is a rank four vector bundle over $\mathbb{P}^2$. Then $\tilde{\phi}_0^* H' = 2\pi^* H - E_{X_0}$, where H' is the hyperplane section of $\mathbb{P}^2$. Picard group as well as Neron Severi group of $\mathbb{P}(E)$ is generated by $\tilde{\phi}_0^* H'$ and $\mathcal{O}_{\mathbb{P}(E)}(1)$. So $\pi^*(H) = \mathcal{O}_{\mathbb{P}(E)}(n_1) \otimes \tilde{\phi}_0^*(n_2 H')$. Note that $1 = \pi^*(H)^3 (2\pi^*(H) - E_{X_0})^2 = (\mathcal{O}_{\mathbb{P}(E)}(n_1) + \tilde{\phi}_0^*(n_2 H'))^3 \tilde{\phi}_0^*(H')^2$. Hence $n_1 = 1$ which implies $\pi^*(H) = \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \tilde{\phi}_0^*(n_2 H')$. If we take $E' = E \otimes n_2 H'$, then $\mathbb{P}(E) \simeq \mathbb{P}(E')$ and $\mathcal{O}_{\mathbb{P}(E')}(n_1) = \mathcal{O}_{\mathbb{P}(E)}(n_1) \otimes \tilde{\phi}_0^*(n_2 H')$. So w.l.o.g., we can consider $\tilde{\mathbb{P}}^5_{X_0} = \mathbb{P}(E)$ such that $\pi^*(H) = \mathcal{O}_{\mathbb{P}(E)}(n)$, $n > 0$. As π is a generically finite map, $\deg(\pi) = 1$ implies that $(\pi^*(H))^5 = 1 \Rightarrow n^5 \cdot \mathcal{O}_{\mathbb{P}(E)}(1)^5 = 1 \Rightarrow n = 1$, i.e. $\pi^*(H) \sim \mathcal{O}_{\mathbb{P}(E)}(1)$.

Theorem 3.2. *If $\mathbb{P}(E)$ is as above, then E is the cokernel of an injective homomorphism $\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^6$, i.e. we have an exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^6 \longrightarrow E \longrightarrow 0. \tag{1}$$

Proof. As $\pi^*(H) \sim \mathcal{O}_{\mathbb{P}(E)}(1)$, the map π is given by the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ and then we have $6 = \dim H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1)) = \dim H^0(\mathbb{P}^2, \pi_*(\mathcal{O}_{\mathbb{P}(E)}(1))) = \dim H^0(\mathbb{P}^2, E)$. As $\mathcal{O}_{\mathbb{P}(E)}(1)$ is globally generated, E is also a globally generated vector bundle of $\mathbb{P}^2$. Hence, we get a surjective morphism from $\mathcal{O}_{\mathbb{P}^2}^6$ to E, where $\mathcal{F}$ is the kernel, i.e.,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}^2}^6 \longrightarrow E \longrightarrow 0. \tag{2}$$

So, clearly $H^0(\mathbb{P}^2, \mathcal{F}) = 0$ and $H^1(\mathbb{P}^2, \mathcal{F}) = 0$. Now, consider another exact sequence of vector bundle of $\mathbb{P}^2$:

$$0 \longrightarrow \mathcal{G} \longrightarrow S^2(\mathcal{O}^6_{\mathbb{P}^2}) \longrightarrow S^2(E) \longrightarrow 0. \tag{3}$$

Here $S^2(F)$ is the second symmetric power of any locally free sheaf F. Corresponding to the above exact sequence (2), we can get a filtration of $S^2(\mathcal{O}^6_{\mathbb{P}^2})$ ([4], Chapter II.5, Exercise 16),

$$S^2(\mathcal{O}^6_{\mathbb{P}^2}) = \mathcal{M}_0 \supseteq \mathcal{M}_1 \supseteq \mathcal{M}_2 \supseteq \mathcal{M}_3 = 0 \tag{4}$$

such that $S^2(E) \simeq \mathcal{M}_0/\mathcal{M}_1 \simeq S^2(\mathcal{O}^6_{\mathbb{P}^2})/\mathcal{M}_1$, then $\mathcal{M}_1 \simeq \mathcal{G}$. Also, $\mathcal{F} \otimes E \simeq \mathcal{M}_1/\mathcal{M}_2$, $S^2(\mathcal{F}) \simeq \mathcal{M}_2$. Using the above isomorphism, we have

$$0 \longrightarrow S^2(\mathcal{F}) \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \otimes E \longrightarrow 0. \tag{5}$$

Tensor the exact sequence (2) by the locally free sheaf $\mathcal{F}$,

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{F} \longrightarrow \mathcal{O}^6_{\mathbb{P}^2} \otimes \mathcal{F} \longrightarrow E \otimes \mathcal{F} \longrightarrow 0. \tag{6}$$

We have the surjective map $\mathcal{F} \otimes \mathcal{F} \to S^2(\mathcal{F})$ with kernel $\wedge^2(\mathcal{F})$, the second exterior power of $\mathcal{F}$ and the identity map $E \otimes \mathcal{F} \to E \otimes \mathcal{F}$. This induces a map $\mathcal{O}^6_{\mathbb{P}^2} \otimes \mathcal{F} \to \mathcal{G}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} \otimes \mathcal{F} & \longrightarrow & \mathcal{O}^6_{\mathbb{P}^2} \otimes \mathcal{F} & \longrightarrow & E \otimes \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S^2(\mathcal{F}) & \longrightarrow & \mathcal{G} & \longrightarrow & E \otimes \mathcal{F} & \longrightarrow & 0. \end{array} \tag{7}$$

By the construction, the middle column map $\mathcal{O}^6_{\mathbb{P}^2} \otimes \mathcal{F} \to \mathcal{G}$ is surjective and $\wedge^2(\mathcal{F})$ is contained in the kernel. As the rank of the kernel is one, the kernel is exactly $\wedge^2(\mathcal{F})$. Hence, we have the exact sequence

$$0 \longrightarrow \wedge^2(\mathcal{F}) \longrightarrow \mathcal{O}^6_{\mathbb{P}^2} \otimes \mathcal{F} \to \mathcal{G} \longrightarrow 0. \tag{8}$$

We have $\tilde{\phi}_0^* H' = 2\pi^* H - E_{X_0}$, which implies that $E_{X_0} = 2\pi^* H - \tilde{\phi}_0^* H' \sim \mathcal{O}_{\mathbb{P}(E)}(2) \otimes \tilde{\phi}_0^* \mathcal{O}_{\mathbb{P}^2}(-1)$. The dimension of the global section of an exceptional divisor is one. Hence, $1 = \dim H^0(\tilde{\mathbb{P}}^5_{X_0}, E_{X_0}) = \dim H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(2) \otimes \tilde{\phi}_0^* \mathcal{O}_{\mathbb{P}^2}(-1))$. Using the projection formulas (3) and (8) from above, we have $1 = \dim H^0(\mathbb{P}^2, S^2(E) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) = \dim H^1(\mathbb{P}^2, \mathcal{G}(-1)) = \dim H^2(\mathbb{P}^2, \wedge^2(\mathcal{F})(-1))$. Hence, the only possibility of $\wedge^2(\mathcal{F})$ is $\mathcal{O}_{\mathbb{P}^2}(-2)$. We know any rank two bundle of $\mathbb{P}^2$ can be written as

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(m) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(n) \otimes \mathcal{I}_Z \longrightarrow 0, \tag{9}$$

where Z is the dimension zero closed subscheme of $\mathbb{P}^2$ and $\mathcal{I}_Z$ is the corresponding ideal sheaf ([3], Chapter 2).

Here in our case,

$$-2H' = c_1(\mathcal{F}) = (m+n)H', c_2(\mathcal{F}) = mn + l(Z). \tag{10}$$

We claim that $l(Z) = 0$ and $m = n = -1$. From (2), $\wedge^4(E) = \mathcal{O}_{\mathbb{P}^2}(2)$, i.e. $c_1(E) = 2H'$. Let $\zeta = \mathcal{O}_{\mathbb{P}(E)}(1)$. Hence, we have the equation

$$\zeta^4 = c_1(E)\zeta^3 - c_2(E)\zeta^2. \tag{11}$$

We know $\zeta^5 = 1$, then

$$1 = c_1(E)\zeta^4 - c_2(E)\zeta^3. \tag{12}$$

Substituting ζ^4 in (12), we get

$$1 = (4H'^2 - c_2(E))\zeta^3 \Rightarrow c_2(E) = 3. \tag{13}$$

Now, from (2), we have $c(E)\cdot c(\mathcal{F}) = c(\mathcal{O}_{\mathbb{P}^2}) \Rightarrow c(\mathcal{F}) = c(E)^{-1} = (1+2H'+3H'^2)^{-1} = 1 - 2H' + H'^2 \Rightarrow c_2(\mathcal{F}) = 1$. So from (10), it is clear that either $n = m = -1$, then we are done or $m < -1; n \geq 0$.

Now, assume $m < -1$ and $n \geq 0$. Tensoring (10) by $\mathcal{O}_{\mathbb{P}}(1)$, we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(m+1) \longrightarrow \mathcal{F}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(n+1) \otimes \mathcal{I}_Z \longrightarrow 0. \tag{14}$$

Here $c_1(\mathcal{F}(1)) = 0$ and $c_2(\mathcal{F}(1)) = 0$. Then we can apply Theorem 4.14.(iv) of [3] which states that $\mathcal{E}$ is a rank two bundle of $\mathbb{P}^2$ with $c_1(\mathcal{E}) = 0$. If $\mathcal{E}$ is stable, then $c_2(\mathcal{E}) \geq 2$. So, in our case, $\mathcal{F}(1)$ is not stable. Then from Theorem 4.14.(i) of [3], we have $m + 1 \geq 0$, hence $m \geq -1$. This contradicts our assumption.

Finally, we get $\mathcal{F} = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ as there are no non-linear extension of the vector bundle $\mathcal{F}$ by the line bundle over $\mathbb{P}^2$. Hence the result follows. □

Remark 3.3. We know from Theorem 9.6 of [2] that the Chow ring of $\mathbb{P}(E)$ is

$$A(\mathbb{P}(E)) = \frac{A(\mathbb{P}^2)[\zeta]}{\langle \zeta^4 + c_1(E^*)\zeta^3 + c_2(E^*)\zeta^2 \rangle}, \tag{15}$$

where $\zeta \sim \pi^*(H) \sim \mathcal{O}_{\mathbb{P}(E)}(1)$. Also, we have $A(\mathbb{P}^2) = \frac{\mathbb{Z}[\alpha]}{\alpha^3}$, where $\alpha \sim H'$. Using the following short exact sequence,

$$0 \longrightarrow E^* \longrightarrow \mathcal{O}^6_{\mathbb{P}^2} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) = F \longrightarrow 0 \tag{16}$$

and the Whitney sum formula, we get

$$c_t(E^*) = c_t(\mathcal{O}^6_{\mathbb{P}^2}).c_t(F)^{-1} = \frac{1}{(1+H't)^2} = 1 - 2H't + 3H'^2t^2. \tag{17}$$

Hence, the Chow ring of $\tilde{\mathbb{P}}^5_{X_0}$ or $\mathbb{P}(E)$ is $\frac{\mathbb{Z}[\alpha,\zeta]}{\langle \alpha^3, \zeta^4-2\alpha\zeta^3+3\alpha^2\zeta^2\rangle}$, $\zeta, \alpha \in A^1(\tilde{\mathbb{P}}^5_{X_0})$ and $E_{X_0} \sim 2\zeta - \alpha$.

The bundle $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$ has six independent sections and it can be generated by four sections. Let us take five independent global sections of $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, which generate this rank two bundle. Then we get a surjection $\mathcal{O}^5_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, where E_1^* is the kernel of the map. Hence we have the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow \mathcal{O}^5_{\mathbb{P}^2} \longrightarrow E_1 \longrightarrow 0. \tag{18}$$

Theorem 3.4. *Let E_1 be as above. Then $\mathbb{P}(E_1) \simeq \tilde{\mathbb{P}}^4_{X_1}$, where X_1 is the hyperplane section of X_0 in $\mathbb{P}^5$, i.e. X_1 is a cubic surface in $\mathbb{P}^4$. If $\pi_1 : \tilde{\mathbb{P}}^4_{X_1} \to \mathbb{P}^4$ is the blown up map and $\tilde{\phi}_1 : \mathbb{P}(E_1) \to \mathbb{P}^2$ is the projectivization map, then $\tilde{\phi}_1^*(\mathcal{O}_{\mathbb{P}^2}(1)) \sim 2\pi_1^*\mathcal{O}_{\mathbb{P}^4}(1) - E_{X_1}$.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1^* & \longrightarrow & \mathcal{O}^5_{\mathbb{P}^2} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E^* & \longrightarrow & \mathcal{O}^6_{\mathbb{P}^2} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) & \longrightarrow & 0. \end{array} \tag{19}$$

As the right and the middle arrows are identity and injective respectively, the left arrow also exists and is injective, which implies that $E \to E_1$ is surjective. This corresponds to the inclusion $i : \mathbb{P}(E_1) \hookrightarrow \mathbb{P}(E)$ such that $\mathcal{O}_{\mathbb{P}(E_1)}(1) \simeq i_1^*\mathcal{O}_{\mathbb{P}(E)}(1)$.

Let $\tilde{\phi}_1 : \mathbb{P}(E_1) \to \mathbb{P}^2$ be the projection morphism. From (18), it is clear that E_1 is globally generated and $h^0(\mathbb{P}^2, E_1) = 5$. Then, we have a morphism $\pi_1 : \mathbb{P}(E_1) \to \mathbb{P}^4$ by the line bundle $\mathcal{O}_{\mathbb{P}(E_1)}(1)$, such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{P}(E_1) & \xrightarrow{i_1} & \mathbb{P}(E) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}^4 & \xrightarrow{i} & \mathbb{P}^5 \end{array}$$

Here i is the canonical inclusion map, $i^*(\mathcal{O}_{\mathbb{P}^5}(1)) = \mathcal{O}_{\mathbb{P}^4}(1)$. In Proposition 3.1, we have defined a rational map $\phi_0 : \mathbb{P}^5 \dashrightarrow \mathbb{P}^2$, given by $\phi_0([z_0, z_1, \ldots, z_5]) = [z_0z_4 - z_1z_3, z_0z_5 - z_2z_3, z_2z_4 - z_1z_5]$. Now, consider $\mathbb{P}^4$ as a hyperplane of $\mathbb{P}^5$, given by the equation $z_0 = z_4$. Then we get a rational map $\phi_1 : \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$, which is $\phi_0([z_0, z_1, z_2, z_3, z_5]) = [z_0z_4 - z_1z_3, z_0^2 - z_2z_3, z_2z_4 - z_1z_0]$. The map ϕ_1 is not defined on the cubic surface X_1 of $\mathbb{P}^4$ given by the equations $g_0 = z_0z_4 - z_1z_3$, $g_1 = z_0^2 - z_2z_3$, $g_2 = z_2z_4 - z_1z_0$. Let $\mathcal{I}_{X_1}$ be the ideal sheaf of X_1 in $\mathbb{P}^4$. $\pi_1^{-1}\mathcal{I}_{X_1} \cdot \mathcal{O}_{\mathbb{P}(E_1)}$ is an invertible sheaf of ideal on $\mathbb{P}(E_1)$, because $\pi_1^{-1}\mathcal{I}_{X_1} \cdot \mathcal{O}_{\mathbb{P}(E_1)} = i_1^{-1}(\pi^{-1}\mathcal{I}_{X_0} \cdot \mathcal{O}_{\mathbb{P}(E)})$ and $\pi^{-1}\mathcal{I}_{X_0} \cdot \mathcal{O}_{\mathbb{P}(E)}$ is an invertible sheaf of ideal on $\mathbb{P}(E) = \tilde{\mathbb{P}}^5_{X_0}$. By the Universal Property of Blowing Up ([4], Chapter II.7), we have the unique morphism $\mathbb{P}(E_1) \to \tilde{\mathbb{P}}^4_{X_1}$ over $\mathbb{P}^4$.

Now, ϕ_1 can be extended as $\tilde{\phi_1} : \tilde{\mathbb{P}}^4_{X_1} \to \mathbb{P}^2$ and defined by the linear system $2\pi_1^*\mathcal{O}_{\mathbb{P}^4}(1) - E_{X_1}$. We have the natural blowing up map $\pi_1 : \tilde{\mathbb{P}}^4_{X_1} \to \mathbb{P}^4$. Here, clearly $\tilde{\phi_1}^* E \to \tilde{\phi_1}^* E_1 \to \pi_1^*(\mathcal{O}_{\mathbb{P}^4}(1))$ is a surjective map. This corresponds to a morphism $\tilde{\mathbb{P}}^4_{X_1} \to \mathbb{P}(E_1)$ over $\mathbb{P}^2$ ([4], Chapter II.7). Hence we have the commutative diagram

$$\begin{array}{ccccc} \mathbb{P}(E_1) & \longrightarrow & \tilde{\mathbb{P}}^4_{X_1} & \longrightarrow & \mathbb{P}(E_1) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}(E) & \longrightarrow & \tilde{\mathbb{P}}^5_{X_0} & \longrightarrow & \mathbb{P}(E). \end{array} \tag{20}$$

The composition of the lower horizontal arrows is identity and the vertical maps are inclusion. Then $\mathbb{P}(E_1) \to \tilde{\mathbb{P}}^4_{X_1} \to \mathbb{P}(E_1)$ is also an identity. Similarly, $\tilde{\mathbb{P}}^4_{X_1} \to \mathbb{P}(E_1) \to \tilde{\mathbb{P}}^4_{X_1}$ is also an identity. Hence we have $\tilde{\mathbb{P}}^4_{X_1} \simeq \mathbb{P}(E_1)$. □

Remark 3.5. $A(\tilde{\mathbb{P}}^4_{X_1}) = A(\mathbb{P}(E_1)) = \frac{\mathbb{Z}[\alpha,\zeta]}{\langle \alpha^3, \zeta^3 - 2\alpha\zeta^2 + 3\alpha^2\zeta \rangle}$, where $\zeta \sim \mathcal{O}_{\mathbb{P}(E_1)}(1)$ and $\alpha \sim \tilde{\phi_1}^*(H')$ $\zeta, \alpha \in A^1(\mathbb{P}(E_1))$ and $E_{X_1} \sim 2\zeta - \alpha$.

COROLLARY 3.6

If we have the following short exact sequence:

$$0 \longrightarrow E_2^* \longrightarrow \mathcal{O}^4_{\mathbb{P}^2} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow 0, \tag{21}$$

then $\tilde{\mathbb{P}}^3_{X_2} \simeq \mathbb{P}(E_2)$, *where* X_2 *is a twisted cubic in* $\mathbb{P}^3$, *which can be obtained by cutting down the cubic surface in* $\mathbb{P}^4$ *by a hyperplane.*

Proof. The proof will follow similarly as we have done in Theorem 3.4. □

Remark 3.7. The Chow ring of $\mathbb{P}^3$ blown up along the twisted cubic is $A(\tilde{\mathbb{P}}^3_{X_2}) = A(\mathbb{P}(E_2)) = \frac{\mathbb{Z}[\alpha,\zeta]}{\langle \alpha^3, \zeta^2 - 2\alpha\zeta + 3\alpha^2 \rangle}$, where $\zeta \sim \mathcal{O}_{\mathbb{P}(E_2)}(1) \sim \pi_2^*(\mathcal{O}_{\mathbb{P}^3}(1))$ and $\alpha \sim \tilde{\phi_2}^*(H')$ $\zeta, \alpha \in A^1(\mathbb{P}(E_2))$ and $E_{X_2} \sim 2\zeta - \alpha$. Here, $\tilde{\phi_2} : \mathbb{P}(E_2) \to \mathbb{P}^2$ is a natural projective bundle map and $\pi_2 : \tilde{\mathbb{P}}^3_{X_2} \to \mathbb{P}^3$ is the blowing up map.

Theorem 3.8. *Let* C *be an irreducible subvariety of* $\mathbb{P}^3$ *other than the linear subspaces of* $\mathbb{P}^3$. $\tilde{\mathbb{P}}^3_C$ *has a projective bundle structure if* $C = V(f_0, f_1, f_2)$, *where* f_i *are irreducible homogeneous polynomials*, $\deg(f_i) = \deg(f_j) = d$ *and* $d_1 = \deg(C) = d^2 - 1$ *in* $\mathbb{P}^3$.

Proof. Let $\tilde{\mathbb{P}}^3_C$ has projective bundle structure, i.e. $\tilde{\mathbb{P}}^3_C \simeq \mathbb{P}(E) \to \mathbb{P}^n$, where π is the projectivization map and $n \leq 2$.

Let $C = V(\{g_i \mid i = 0, 1, \ldots, r\})$. If $\deg(g_i) \neq \deg(g_j)$ for $i \neq j$, then we can construct f_i such that $C = V(\{f_i \mid i = 0, 1, \ldots, r\})$ and $f_i = g_i^{n_i}$ for some positive integer n_i such that the degree of each f_i is the same.

Consider the following diagram:

$$\begin{array}{ccc} \tilde{\mathbb{P}}^3_C & \xrightarrow{I} & \mathbb{P}(E) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}^3 & & \mathbb{P}^n \end{array}$$

Here I is the isomorphism and ϕ is the blow up map. $\psi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^n$ is the corresponding rational map which is defined on $\mathbb{P}^3 \backslash C$, i.e. ψ is given by the linear system $|\mathcal{O}_{\mathbb{P}^3}(d) \otimes \mathcal{I}_C|$, where $d = \deg(f_i) > 1$ and $\mathcal{I}_C$ is the ideal sheaf corresponding to the subvariety C.

Case I (n=1). $\mathbb{P}(E) \xrightarrow{\pi} \mathbb{P}^1$, rank$(E) = 3$ and $C = V(f_0, f_1)$. Clearly, $\psi^{-1}([a_0, a_1]) = V(a_0 f_1 - a_1 f_0)$ and then $\phi^{-1}(\psi^{-1}([a_0, a_1])) = \overline{V(a_0 f_1 - a_1 f_0)\backslash C} \cup E_C$, where $\overline{V(a_0 f_1 - a_1 f_0)\backslash C}$ is the strict transformation of $\overline{V(a_0 f_1 - a_1 f_0)\backslash C}$ and E_C is the exceptional divisor corresponding to the blow-up map ϕ. This gives $\pi^{-1}(a) = \overline{V(a_0 f_1 - a_1 f_0)\backslash C} \simeq S_d$ which is degree d hypersurface in $\mathbb{P}^3$ which is not isomorphic to $\mathbb{P}^2$. Hence we get a contradiction. This implies C will never be a complete intersection curve in $\mathbb{P}^3$.

Case II (n=2). $\mathbb{P}(E) \xrightarrow{\pi} \mathbb{P}^2$, rank$(E) = 2$ and $C = V(f_0, f_1, f_2)$. Clearly, $\psi^{-1}([a_0, a_1, a_2]) = V(a_0 f_1 - a_1 f_0, a_0 f_2 - a_2 f_0)$. As C is an irreducible curve, C becomes an irreducible component of $\psi^{-1}(a)$, where $a = [a_0, a_1, a_2]$. Then $\psi^{-1}(a) = C \cup C_a$. Hence $\phi^{-1}(\psi^{-1}(a)) = E_C \cup \widetilde{C_a}$, where E_C is the exceptional divisor corresponding to the blowing up map ϕ and $\widetilde{C_a}$ is the strict transformation of C_a. This gives $\pi^{-1}(a) = \widetilde{C_a}$ and this will be isomorphic to $\mathbb{P}^1$ only when $\deg(C_a) = 1$ in $\mathbb{P}^3$. Hence we get that each f_i is a reduced polynomial (otherwise C_a will become a non-reduced curve for some $a \in \mathbb{P}^2$) and $\deg(C) = d^2 - 1$ (as $\deg(V(a_0 f_1 - a_1 f_0, a_0 f_2 - a_2 f_0)) = d^2$.

Hence the theorem is proved. □

PROPOSITION 3.9

Considering the same notations of Theorem 3.8, $\tilde{\mathbb{P}}^3_C$ has a projective bundle structure only when C is a genus zero curve.

Proof. Let $\tilde{\mathbb{P}}^3_C \simeq \mathbb{P}(E)$, where E is a rank 2 vector bundle over $\mathbb{P}^2$. Let $A^i(X)$ be the rational equivalence class of codimension i cycle of the scheme X. Here,

$$A^1(\mathbb{P}(E)) = \mathbb{Z}\mathcal{O}_{\mathbb{P}(E)}(1) \oplus \mathbb{Z}\pi^*\mathcal{O}_{\mathbb{P}^2}(1) \simeq \mathbb{Z} \oplus \mathbb{Z},$$
$$A^2(\mathbb{P}(E)) = \mathbb{Z}\mathcal{O}_{\mathbb{P}(E)}(1) \cdot \pi^*\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathbb{Z}(\pi^*\mathcal{O}_{\mathbb{P}^2}(1) \cdot \pi^*\mathcal{O}_{\mathbb{P}^2}(1)) \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

Now consider the following commutative diagram:

$$\begin{array}{ccc} \mathbb{P}(N) & \xrightarrow{j} & \tilde{\mathbb{P}}^3_C \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ C & \xrightarrow{i} & \mathbb{P}^3 \end{array}$$

N is a normal bundle of C in $\mathbb{P}^3$, $\mathbb{P}(N) = E_C$, the exceptional divisor corresponding to the blowing up map ϕ, i and j are inclusions. Let $\tilde{h} = \phi^*\mathcal{O}_{\mathbb{P}^3}(1) \in A^1(\tilde{\mathbb{P}}^3_C)$ and $e = [E_C] \in A^1(\tilde{\mathbb{P}}^3_C)$.

From Proposition 13.13 of [2], we have $A^1(\tilde{\mathbb{P}}^3_C) = \mathbb{Z}e \oplus \mathbb{Z}\tilde{h}$, $A^2(\tilde{\mathbb{P}}^3_C)$ is generated by $e^2 = -j_*(\mathcal{O}_{\mathbb{P}(N)}(1))$, $\tilde{h}^2$ and $j_*(F_D)$ for $D \in A^1(C)$ and $F_D = \pi^*(D)$. $A^2(\tilde{\mathbb{P}}^3_C) \simeq \mathbb{Z} \oplus \mathbb{Z}$ if and only if any two point on the curve C is rationally equivalent, i.e. $\mathrm{Pic}(C) = \mathbb{Z}$. Hence C is a genus zero curve. □

Theorem 3.10. *$\tilde{\mathbb{P}}^3_C$ has a projective bundle structure if and only if C is a twisted cubic in $\mathbb{P}^3$.*

Proof. Using the same notations as in Proposition 3.9, we have $\mathrm{Pic}(\tilde{\mathbb{P}}^3_C)=\mathbb{Z}e \oplus \mathbb{Z}\tilde{h}$, $\mathrm{Pic}(\mathbb{P}(E)) = \mathbb{Z}\mathcal{O}_{\mathbb{P}(E)}(1) \oplus \mathbb{Z}\pi^*\mathcal{O}_{\mathbb{P}^2}(1) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Rename $\zeta_E = \mathcal{O}_{\mathbb{P}(E)}(1)$ and $H = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$, $H \sim d\tilde{h} - e$ and $\zeta_E \sim \tilde{h}$ in the Picard group.

We know $\zeta_E^2 \cdot H = c_1(E)$. From the above equivalence, we get $\zeta_E^2 \cdot H = \tilde{h}^2(d\tilde{h} - e)$. This implies

$$d = c_1(E), \tag{22}$$

using the intersection products from Proposition 13.13 of [2]. Also, $e^3 = -4d_1 - 2g + 2$, where from Theorem 3.8, the degree of C in $\mathbb{P}^3$, $d_1 = d^2 - 1$ and from Proposition 3.9, genus of $C = g = 0$. Then

$$\begin{aligned}
&(d\zeta_E - H)^3 = e^3 = -4(d^2 - 1) + 2\\
&\Rightarrow d^3\zeta_E^3 - 3d^2\zeta_E^2 H + 3d\zeta_E H^2 - H^3 = -4d^2 + 6\\
&\Rightarrow -2d^3 + 3d = -4d^2 + 6 (As\, \zeta_E^2 H = c_1(E) = d,\ \zeta_E H^2 = 1 \text{ and } H^3 = 0)\\
&\Rightarrow 2d^3 - 4d^2 - 3d + 6 = 0 \Rightarrow d = 2 or \pm \frac{3}{2}.
\end{aligned}$$

Hence $d = 2$. So C is degree three curve in $\mathbb{P}^3$ which is either a cubic in $\mathbb{P}^2$ or a twisted cubic in $\mathbb{P}^3$. As C is not a complete intersection curve, C is a twisted cubic.

Clearly, $f_0 = Z_0Z_2 - Z_1^2$, $f_1 = Z_1Z_3 - Z_2^2$, $f_3 = Z_0Z_3 - Z_1Z_2$ and $C = V(f_0, f_1, f_2)$ is a twisted cubic in $\mathbb{P}^3$. □

Remark 3.11. We know that linear subspace is always a degree one complete intersection variety. But in the above three examples, we have seen that none of X_0, X_1, X_2 are complete intersection variety. So if blow up of projective space along a projective variety is isomorphic to a projective bundle, then this projective variety need not be a complete intersection.

Now, it is interesting to know what are the nef cones of $\mathbb{P}(E)$, $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$.

X_2 is a twisted cubic in $\mathbb{P}^3$. X_2 is the image of $\mathbb{P}^1$ into $\mathbb{P}^3$ by the map $[u, v] \to [u^3, u^2v, uv^2, v^3]$, where u, v are homogeneous co-ordinates of $\mathbb{P}^1$. Also, we can consider the map via $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. $\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, where $\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ is given by $[u, v] \to [u^2, v^2] \times [u, v]$ and $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is the Segre embedding. $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)=\mathbb{Z} \times \mathbb{Z}$, then $X_2 \sim (2, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. We have the blowing up map $\pi_2 : \tilde{\mathbb{P}}^3_{X_2} \to \mathbb{P}^3$. The Neron

Severi group, $N^1(\tilde{\mathbb{P}}^3_{X_2})$ is generated by $H = \pi_2^*\mathcal{O}_{\mathbb{P}^3}(1)$ and the exceptional divisor E_{X_2}. The numerical equivalence class of one cycle, $N_1(\tilde{\mathbb{P}}^3_{X_2})$ is generated by the pullback of the general line in $\mathbb{P}^3$, $l = \pi_2^* l_{\mathbb{P}^3}$ and an exceptional curve e, as described in [1]. Then

$$l \cdot H = 1, l \cdot E_{X_2} = 0, e \cdot H = 0, e \cdot E_{X_2} = -1. \tag{23}$$

The effective cone of curves $\overline{NE}(\tilde{\mathbb{P}}^3_{X_2}) \subset N_1(\tilde{\mathbb{P}}^3_{X_2})$ is generated by e and $\tilde{C} \sim al - be$, where $\tilde{C}$ is a strict transformation of degree a curve in $\mathbb{P}^3$ which intersect X_2 in b points with b/a maximum. Claim $\tilde{C} \sim l - 2e$. If the claim is not true, then $\frac{b}{a} > 2$. Let S be the image of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ containing the twisted cubic, then $S \sim \mathcal{O}_{\mathbb{P}^3}(2)$. So the strict transformation of S is $\tilde{S} \sim 2H - E_{X_2}$ in $\tilde{\mathbb{P}}^3_{X_2}$. Then $\tilde{C} \cdot \tilde{S} = 2a - b < 0$, which implies $C \subset S$. Let $C \sim (\alpha, \beta)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. $\deg C = \alpha + \beta = a$ in $\tilde{\mathbb{P}}^3_{X_2}$ and $C \cdot X_2 = 2\alpha + \beta = b$. $b = 2\alpha + \beta < 2(\alpha + \beta) = 2a < b$, hence the contradiction. So our claim is true, i.e. $\overline{NE}(\tilde{\mathbb{P}}^3_{X_2})$ is generated by e and $\tilde{C} \sim l - 2e$. For a more general calculation, see [1].

COROLLARY 3.12

Nef cone of $\tilde{\mathbb{P}}^3_{X_2}$ *is generated by* $\pi_2^*\mathcal{O}_{\mathbb{P}^3}(1)$ *and* $2\pi_2^*\mathcal{O}_{\mathbb{P}^3}(1) - E_{X_2}$.

Proof. As we know, the nef cone is a dual of the closure of the effective cone of curves, and hence the result will follow from the above discussion. □

COROLLARY 3.13

Nef cone of $\tilde{\mathbb{P}}^4_{X_1}$ *is generated by* $\pi_1^*\mathcal{O}_{\mathbb{P}^4}(1)$ *and* $2\pi_1^*\mathcal{O}_{\mathbb{P}^4}(1) - E_{X_1}$.

Proof. We know $\tilde{\mathbb{P}}^4_{X_2} \simeq \mathbb{P}(E_1) \xrightarrow{\tilde{\phi}_1} \mathbb{P}^2$ is defined by the linear system $2\pi_1^*\mathcal{O}_{\mathbb{P}^4}(1) - E_{X_1} \sim \tilde{\phi_1}^* \mathcal{O}_{\mathbb{P}^2}(1)$ which is nef because pullback of ample is nef. Here our claim is that this is a boundary of the nef cone. This is an effective divisor but not big, as the highest power self-intersection is zero. So our claim is proved.

Now, another generator is $\pi_1^*\mathcal{O}_{\mathbb{P}^4}(1) \sim \mathcal{O}_{\mathbb{P}(E_1)}(1)$. We have the short exact sequence in $\mathbb{P}^2$:

$$0 \longrightarrow \mathcal{L} \longrightarrow E_1 \longrightarrow E_2 \longrightarrow 0. \tag{24}$$

As $\deg(E_1) = \deg(E_2)$, $\deg(\mathcal{L}) = 0$, i.e. $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}$. As the extension of the two nef bundle is nef, E_1 is a nef vector bundle. Also, $\mathcal{O}_{\mathbb{P}(E_1)}(1)$ is a nef. This is not ample because the quotient of ample is ample, but E_2 is not ample. Hence the result follows. □

COROLLARY 3.14

Nef cone of $\tilde{\mathbb{P}}^5_{X_0}$ *is generated by* $\pi^*\mathcal{O}_{\mathbb{P}^5}(1)$ *and* $2\pi^*\mathcal{O}_{\mathbb{P}^5}(1) - E_{X_0}$.

Proof. Same as Corollary 3.13. □

Acknowledgements

The author would like to thank Prof. D. S. Nagaraj, IMSc, Chennai for suggesting this problem and for valuable suggestions while the author was working on this project. This work is financially supported by a fellowship from IMSc, Chennai (HBNI), DAE, Government of India.

References

[1] Blanc J and Lamy S, Weak Fano threefolds obtained by blowing-up a space curve and construction of Sarkisov links, *Proc. Lond. Math. Soc. (3)* **105(5)** (2012) 1047–1075
[2] Eisenbud D and Harris J, 3264 and All That—A Second Course in Algebraic Geometry (2016) (Cambridge: Cambridge University Press)
[3] Friedman R, Algebraic Surface and Holomorphic Vector Bundles (1998) (New York: Springer)
[4] Hartshorne R, Algebraic Geometry (1977) (NewYork: Springer-Verlag)

Communicating Editor: Parameswaran Sankaran