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Ring endomorphisms satisfying the central reversible property

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Abstract. A ring R is called *reversible* if for $a, b \in R$, $ab = 0$ implies $ba = 0$. These rings play an important role in the study of noncommutative ring theory. Kafkas *et al.* (*Algebra Discrete Math.* **12** (2011) 72–84) generalized the notion of reversible rings to central reversible rings. In this paper, we extend the notion of central reversibility of rings to ring endomorphisms. We investigate various properties of these rings and answer relevant questions that arise naturally in the process of development of these rings, and as a consequence many new results related to central reversible rings are also obtained as corollaries to our results.

Keywords. Reversible ring; central reversible ring; α-skew (central) reversible ring; matrix ring; polynomial ring; Dorroh extension; Jordan extension.

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1. Introduction

Throughout this article, all rings are associative with unity unless explicitly mentioned and all ring endomorphisms are nonzero. Given a ring R, the polynomial ring with an indeterminate x over R is denoted by $R[x]$, the center of R by $Z(R)$, the set of nilpotent elements of R by $N(R)$, the n by n full (resp., the upper triangular) matrix ring over R by $M_n(R)$ (resp., $U_n(R)$), E_{ij} denotes the matrix with (i, j)-entry 1 and other entries 0, and $\mathbb{Z}_n$ denotes the ring of integers modulo n.

Following Cohn [9], a ring R is called *reversible* if for $a, b \in R$, $ab = 0$ implies $ba = 0$. Anderson and Camillo [3] used the term ZC_2 for the reversible property. Kafkas *et al.* [17] generalized the notion of reversible rings to central reversible rings and these rings were further studied in [16,20]. A ring R is called *central reversible* [17] if for $a, b \in R$, $ab = 0$ implies $ba \in Z(R)$. Following the literature, a ring R is called *reduced* if it has no nonzero nilpotent elements. Due to Bell [6], a ring R is said to satisfy the *insertion-of-factors property* or is simply called an IFP ring if for $a, b \in R$, $ab = 0$ implies $aRb = 0$. A ring R is called *abelian* if each idempotent of R is central. Agayev *et al.* [1] generalized the notion of reduced rings to central reduced rings and these rings were further studied in [24]. A ring R is called *central reduced* [1] if each nilpotent element of R is central, i.e., $N(R) \subseteq Z(R)$. The relations among the classes of rings mentioned above are given as follows:

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$$\begin{array}{ccccc}
\text{Reduced} & \Rightarrow & \text{Reversible} & \Rightarrow & \text{IFP} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{Central reduced} & \Rightarrow & \text{Central reversible} & \Rightarrow & \text{Abelian.}
\end{array}$$

Due to Krempa [21], an endomorphism α of a ring R is called *rigid* if for $a \in R$, $a\alpha(a) = 0$ implies $a = 0$ and a ring R is called α*-rigid* if there exists a rigid endomorphism α of R. Following [13, pp. 218], any rigid endomorphism is injective and α-rigid rings are reduced. Also from [2, Lemma 2.1(iii)], a ring R is α-rigid if and only if for $a \in R$, $\alpha(a)a = 0$ implies $a = 0$. Following [12], an endomorphism α of a ring R is called *compatible* if for $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$ and a ring R is called α*-compatible* if there exists a compatible endomorphism α of R. Note that a compatible endomorphism is injective by [2, Lemma 2.1(i)]. Due to Başer *et al.* [5], an endomorphism α of a ring R is called *right* (resp., *left*) *skew reversible* if for a, b in R, $ab = 0$ implies $b\alpha(a) = 0$ (resp., $\alpha(b)a = 0$) and a ring R is called *right* (resp., *left*) α*-skew reversible* if there exists a right (resp., left) skew reversible endomorphism α of R. A ring R is called α*-skew reversible* [5] if it is both right and left α-skew reversible. We change over from 'an α-reversible ring' in [5] to 'an α-skew reversible ring' to cohere with other related definitions. Note that α-rigid rings are α-skew reversible by the proof of [5, Proposition 2.5(iii)].

Motivated by the above, we extend the notion of central reversible rings to ring endomorphisms and introduce α-skew central reversible rings as a generalization of α-skew reversible rings. We begin with the following definitions.

DEFINITION 1.1

(1) An endomorphism α of a ring R is called *right* (resp., *left*) *skew central reversible* if for $a, b \in R$, $ab = 0$ implies $b\alpha(a) \in Z(R)$ (resp., $\alpha(b)a \in Z(R)$).
(2) A ring R is called *right* (resp., *left*) α*-skew central reversible* if there exists a right (resp., left) skew central reversible endomorphism α of R.
(3) A ring R is called α*-skew central reversible* if it is both right and left α-skew central reversible.

Remark 1.2.

(1) A ring R is central reversible if it is right (left) 1_R-skew central reversible, where 1_R denotes the identity endomorphism of R.
(2) Commutative rings and domains are α-skew central reversible for any endomorphism α.
(3) Every right (resp., left) α-skew reversible ring is right (resp., left) α-skew central reversible, however, the converse is not true in general by means of [5, Example 2.3] and (2) above.

We provide non-trivial examples of α-skew central reversible rings as follows.

PROPOSITION 1.3

Let R be a central reduced ring. Then

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in R \right\}$$

is α-skew central reversible, where $\alpha : S \to S$ is defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

Proof. Suppose $A, B \in S$ such that $AB = 0$, where

$$A = \begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix}.$$

Then $a_1a_2 = 0$ and $a_1b_2 + b_1a_2 = 0$. From $a_1a_2 = 0$, we have $a_2a_1 \in N(R) \subseteq Z(R)$ as R is central reduced. Multiplying $a_1b_2 + b_1a_2 = 0$ by a_2 from left, we get

$$a_2a_1b_2 + a_2b_1a_2 = 0. \tag{1}$$

Multiplying (1) by b_1 from right and using commutativity of a_2a_1, we get

$$b_2b_1a_2a_1 + a_2b_1a_2b_1 = 0. \tag{2}$$

Multiplying (2) by a_2 from right and using $a_1a_2 = 0$, we get $a_2b_1a_2b_1a_2 = 0$, entailing $a_2b_1 \in N(R) \subseteq Z(R)$ as R is central reduced. By similar arguments, we can show that $b_2a_1 \in Z(R)$. Thus we have

$$B\alpha(A) = \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} a_1 & -b_1 \\ 0 & a_1 \end{pmatrix} = \begin{pmatrix} a_2a_1 & -a_2b_1 + b_2a_1 \\ 0 & a_2a_1 \end{pmatrix} \in Z(S),$$
$$\alpha(B)A = \begin{pmatrix} a_2 & -b_2 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} = \begin{pmatrix} a_2a_1 & a_2b_1 - b_2a_1 \\ 0 & a_2a_1 \end{pmatrix} \in Z(S).$$

Hence S is α-skew central reversible. □

The condition 'R is a central reduced ring' in Proposition 1.3 cannot be replaced by 'R is a central reversible ring' by the following example.

Example 1.4. We refer to the ring in [16, Example 2.12]. Let $A = \mathbb{Z}_2[a_0, a_1, a_2, b_0, b_1, b_2, c]$ be the free algebra with zero constant terms in noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Note that A is a ring without unity and consider an ideal I of the ring $\mathbb{Z}_2 + A$, generated by

$$\begin{aligned} &a_0b_0,\ a_0b_1 + a_1b_0,\ a_0b_2 + a_1b_1 + a_2b_0,\\ &a_1b_2 + a_2b_1,\ a_2b_2,\ a_0rb_0,\ a_2rb_2,\\ &b_0a_0,\ b_0a_1 + b_1a_0,\ b_0a_2 + b_1a_1 + b_2a_0,\\ &b_1a_2 + b_2a_1,\ b_2a_2,\ b_0ra_0,\ b_2ra_2,\\ &(a_0 + a_1 + a_2)r(b_0 + b_1 + b_2),\ (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2),\\ &\text{and } r_1r_2r_3r_4r_5, \end{aligned}$$

where $r, r_1, r_2, r_3, r_4, r_5 \in A$. Then clearly $A^5 \subseteq I$. Let $R = (\mathbb{Z}_2 + A)/I$. Then R is central reversible by [16, Example 2.12]. We call each product of the indeterminates

$a_0, a_1, a_2, b_0, b_1, b_2, c$ a monomial and a monomial of degree n means a product of exactly n number of indeterminates. Let H_n be the set of all linear combinations of monomials of degree n over $\mathbb{Z}_2$. Note that any $f \in R$ may be written as $f = k + f_1 + f_2 + f_3 + f_4 + f_5$, where $k \in \mathbb{Z}_2$, $f_i \in H_i$ (for $i = 1, 2, 3, 4$) and $f_5 \in I$ as $H_i \subseteq I$ for $i \geq 5$. It is clear that $Z(R) = \{k + f_4 + f_5 : k \in \mathbb{Z}_2, f_4 \in H_4, f_5 \in I\}$. For simplicity, we identify the elements of $\mathbb{Z}_2 + A$ with their images in R. Let

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in R \right\}$$

and $\alpha : S \to S$ be defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

For

$$A = \begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_0c & b_1c \\ 0 & b_0c \end{pmatrix} \in S,$$

we have $AB = 0$, however,

$$B\alpha(A) = \begin{pmatrix} b_0c & b_1c \\ 0 & b_0c \end{pmatrix}\begin{pmatrix} a_0 & -a_1 \\ 0 & a_0 \end{pmatrix} = \begin{pmatrix} 0 & -b_0ca_1 + b_1ca_0 \\ 0 & 0 \end{pmatrix} \notin Z(S),$$

$$\alpha(B)A = \begin{pmatrix} b_0c & -b_1c \\ 0 & b_0c \end{pmatrix}\begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix} = \begin{pmatrix} 0 & b_0ca_1 - b_1ca_0 \\ 0 & 0 \end{pmatrix} \notin Z(S)$$

as $b_0ca_1 - b_1ca_0 \notin Z(R)$. Therefore, S is neither right nor left α-skew central reversible and so S is not α-skew central reversible.

Next we show that the notion of α-skew central reversible rings is not left–right symmetric.

Example 1.5. The argument here is due to [5, Example 2.2]. Let S be a reversible ring. Consider a ring $R = U_2(S)$. Clearly, R is not abelian and so it is not central reversible by means of [20, Lemma 2.13].

(1) Let $\alpha : R \to R$ be an endomorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Suppose

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in R$$

such that $AB = 0$. Then $aa' = 0$ and so $a'a = 0$ as S is reversible. Thus $B\alpha(A) \in Z(R)$ and so R is right α-skew central reversible. For $A = E_{12}+E_{22}$, $B = E_{11}+E_{12} \in R$, $AB = 0$ but $\alpha(B)A = E_{12} \notin Z(R)$, entailing R is not left α-skew central reversible.

(2) Let $\alpha' : R \to R$ be an endomorphism defined by

$$\alpha'\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}.$$

By arguments similar to (1), we can show that R is left α'-skew central reversible which is not right α'-skew central reversible.

Next we show that there exists an endomorphism α of a reduced ring R for which R is neither right nor left α-skew central reversible.

Example 1.6. Let $\mathbb{H}$ be the ring of real quaternions. Consider $R = \mathbb{H} \oplus \mathbb{H}$ with the usual addition and multiplication. Let $\alpha : R \to R$ be an endomorphism defined by $\alpha((a, b)) = (b, a)$. For $a = (i, 0)$, $b = (0, j) \in R$, we have $ab = 0$, but $b\alpha(a) = (0, -k)$ and $\alpha(b)a = (-k, 0)$, entailing R is neither right nor left α-skew central reversible.

Following Examples 1.5 and 1.6, it is clear that the notions of central reversible and right (left) α-skew central reversible rings are independent of each other. However, we have the following.

PROPOSITION 1.7

For a ring R with a compatible endomorphism α, the following are equivalent:

(1) *R is central reversible.*
(2) *R is right α-skew central reversible.*
(3) *R is left α-skew central reversible.*

Proof.

(1) $\Rightarrow$ (2). Let R be a central reversible ring and let $a, b \in R$ such that $ab = 0$. Since R is α-compatible, therefore, $\alpha(a)b = 0$ by [2, Lemma 2.1(ii)] and so $b\alpha(a) \in Z(R)$, by assumption. Thus R is right α-skew central reversible.
(2) $\Rightarrow$ (1). Let R be right α-skew central reversible and let $a, b \in R$ such that $ab = 0$. Since R is α-compatible, therefore, $a\alpha(b) = 0$ and so $\alpha(ba) = \alpha(b)\alpha(a) \in Z(R)$ as R is right α-skew central reversible. Then for any $s \in R$, $\alpha(ba)\alpha(s) = \alpha(s)\alpha(ba)$ implies $(ba)s = s(ba)$ as α being a compatible endomorphism, is injective. Therefore, $ba \in Z(R)$ and hence R is central reversible.
(1) $\Leftrightarrow$ (3). This can be proved similarly. □

2. Properties and extensions

In this section, we study ring theoretic properties and extensions related to the right version of α-skew central reversible rings.

PROPOSITION 2.1

For a ring R with a monomorphism α, the following are equivalent:

(1) *R is right α-skew central reversible.*
(2) *For $a, b \in R$, $a\alpha(b) = 0$ implies $ba \in Z(R)$.*

Proof.

(1) ⇒ (2). Assume that R is right α-skew central reversible. Suppose $a, b \in R$ such that $a\alpha(b) = 0$. By assumption, $\alpha(ba) = \alpha(b)\alpha(a) \in Z(R)$. Since α is injective, therefore, $ba \in Z(R)$ by the arguments given in the proof of Proposition 1.7.
(2) ⇒ (1). Assume that (2) holds. Suppose $a, b \in R$ such that $ab = 0$. Then $0 = \alpha(ab) = \alpha(a)\alpha(b)$. By assumption, $b\alpha(a) \in Z(R)$. Therefore, R is right α-skew central reversible. □

The condition 'α is injective' in Proposition 2.1 is not superfluous by the following example.

Example 2.2. Consider a ring $R = U_2(\mathbb{Z})$ over the ring $\mathbb{Z}$ of integers and an endomorphism $\alpha : R \to R$ defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly, α is not injective. By Example 1.5(1), it is clear that R is right α-skew central reversible. For $A = E_{12} + E_{22}$ and $B = E_{11} + E_{12} \in R$, by a simple computation, $A\alpha(B) = 0$ but $BA = 2E_{12} \notin Z(R)$.

Following [22], a ring R with an endomorphism α is called *central α-rigid* if for $a \in R$, $a\alpha(a) = 0$ implies $a \in Z(R)$.

PROPOSITION 2.3

For a ring R with an endomorphism α, if R is central α-rigid, then R is right α-skew central reversible.

Proof. Let R be central α-rigid and let $a, b \in R$ be such that $ab = 0$. Then

$$b\alpha(a)\alpha(b\alpha(a)) = b\alpha(ab)\alpha^2(a) = 0.$$

Since R is central α-rigid, therefore, $b\alpha(a) \in Z(R)$ and hence R is right α-skew central reversible. □

Due to Chen [7], a ring R is called 2-*central reduced* if for $a \in R$, $a^2 = 0$ implies $a \in Z(R)$. Note that central reduced rings are 2-central reduced but not conversely [7, Example 3.3].

COROLLARY 2.4

Every 2*-central reduced ring is central reversible.*

The converse of Proposition 2.3 is not true in general by the following example.

Example 2.5. Consider a ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{H} \right\}$$

over the ring $\mathbb{H}$ of real quaternions and an endomorphism $\alpha : R \to R$ defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

By Proposition 1.3, R is right α-skew central reversible. However, for $A = iE_{12} \in R$, $A\alpha(A) = 0$, but $A \notin Z(R)$. Therefore, R is not central α-rigid.

Remark 2.6. The ring R in Example 2.5 is central reversible by [20, Proposition 2.20] which is not 2-central reduced as $A^2 = 0$ for $A = iE_{12}$ but $A \notin Z(R)$. This shows that the class of 2-central reduced rings lies strictly between the classes of central reduced and central reversible rings.

Next, we show that the notion of right α-skew central reversible rings is preserved under isomorphisms.

PROPOSITION 2.7

Let R be a ring with an endomorphism α. Let S be a ring and let $\sigma : R \to S$ be an isomorphism. Then R is right α-skew central reversible if and only if S is right $\bar{\alpha}$-skew central reversible, where $\bar{\alpha} = \sigma \circ \alpha \circ \sigma^{-1}$.

Proof. Let R be right α-skew central reversible and let $x, y \in S$ be such that $xy = 0$. Since σ is bijective, therefore, there exist $a, b \in R$ such that $x = \sigma(a)$, $y = \sigma(b)$ and $ab = 0$. Since R is right α-skew central reversible, therefore, $b\alpha(a) \in Z(R)$. Then

$$y\bar{\alpha}(x) = \sigma(b)(\sigma \circ \alpha \circ \sigma^{-1})(\sigma(a)) = \sigma(b\alpha(a)).$$

For any $s \in S$, there exists $r \in R$ such that $\sigma(r) = s$ and

$$\begin{aligned} \sigma(b\alpha(a))s &= \sigma(b\alpha(a))\sigma(r) = \sigma((b\alpha(a))r) = \sigma(r(b\alpha(a))) \\ &= \sigma(r)\sigma(b\alpha(a)) = s\sigma(b\alpha(a)), \end{aligned}$$

using commutativity of $b\alpha(a)$. Thus $y\bar{\alpha}(x) = \sigma(b\alpha(a)) \in Z(S)$ and hence S is right $\bar{\alpha}$-skew central reversible. The converse can be proved similarly. □

However, the notion of right α-skew central reversible rings is not preserved under homomorphic images. Note that for a ring R with an endomorphism α and an ideal I of R with $\alpha(I) \subseteq I$, the mapping $\bar{\alpha} : R/I \to R/I$ defined by $\bar{\alpha}(a + I) = \alpha(a) + I$ for $a \in R$, induces an endomorphism of R/I.

Example 2.8. We refer to the ring in [19, Example 2.1]. Let $A = \mathbb{Z}_2[a_0, a_1, a_2, b_0, b_1, b_2, c]$ be the free algebra with zero constant terms in noncommuting indeterminates a_0,

$a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Note that A is a ring without unity and consider an ideal I of the ring $\mathbb{Z}_2 + A$, generated by

$$\begin{aligned} &a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0,\\ &\quad a_1b_2 + a_2b_1, a_2b_2, a_0rb_0, a_2rb_2,\\ &\quad b_0a_0, b_0a_1 + b_1a_0, b_0a_2 + b_1a_1 + b_2a_0,\\ &\quad b_1a_2 + b_2a_1, b_2a_2, b_0ra_0, b_2ra_2,\\ &\quad (a_0 + a_1 + a_2)r(b_0 + b_1 + b_2), (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2),\\ &\quad \text{and } r_1r_2r_3r_4, \end{aligned}$$

where $r, r_1, r_2, r_3, r_4 \in A$. Then clearly $A^4 \subseteq I$. Let $R = (\mathbb{Z}_2 + A)/I$. Let α be an endomorphism of $\mathbb{Z}_2 + A$ defined by

$$a_0, a_1, a_2, b_0, b_1, b_2, c \mapsto b_0, b_1, b_2, a_0, a_1, a_2, c.$$

Since $\mathbb{Z}_2 + A$ is a domain, therefore, $\mathbb{Z}_2 + A$ is right α-skew central reversible, by Remark 1.2(1). Clearly, $\alpha(I) \subseteq I$ and so $\bar{\alpha} : R \to R$ is well defined.

Let H_n be the set of all linear combinations of monomials of degree n over $\mathbb{Z}_2$. Note that any $f \in R$ may be written as $f = k + f_1 + f_2 + f_3 + f_4$, where $k \in \mathbb{Z}_2$, $f_i \in H_i$ (for $i = 1, 2, 3$) and $f_4 \in I$ as $H_i \subseteq I$ for $i \geq 4$. Also, $Z(R) = \{k + f_3 + f_4 : k \in \mathbb{Z}_2, f_3 \in H_3, f_4 \in I\}$.

For simplicity, we identify the elements of $\mathbb{Z}_2 + A$ with their images in R. Clearly, $a_0b_0 = 0$ but $b_0\bar{\alpha}(a_0) = b_0^2 \notin Z(R)$, entailing R is not right $\bar{\alpha}$-skew central reversible.

Remark 2.9.

(1) Every subring S of a right α-skew central reversible ring with $\alpha(S) \subseteq S$ is also right α-skew central reversible.

(2) Let $\{R_\lambda : \lambda \in \Lambda\}$ be a class of rings such that for each $\lambda \in \Lambda$, α_λ is an endomorphism of R_λ. For the direct product $\prod_{\lambda\in\Lambda} R_\lambda$, the mapping $\bar{\alpha} : \prod_{\lambda\in\Lambda} R_\lambda \to \prod_{\lambda\in\Lambda} R_\lambda$ defined by $\bar{\alpha}((a_\lambda)) = (\alpha_\lambda(a_\lambda))$ induces an endomorphism of $\prod_{\lambda\in\Lambda} R_\lambda$. It is easy to show that $\prod_{\lambda\in\Lambda} R_\lambda$ is right $\bar{\alpha}$-skew central reversible if and only if each R_λ is right α_λ-skew central reversible.

As a direct consequence of Remark 2.9, we have the following.

PROPOSITION 2.10

Let R be a ring and α an endomorphism of R such that $\alpha(e) = e$ for each $e^2 = e \in R$.

(1) *If R is right α-skew central reversible, then so is eRe for each $e^2 = e \in R$.*

(2) *If e is central idempotent in R, then R is right α-skew central reversible if and only if eR and $(1 - e)R$ are right α-skew central reversible.*

Recall that a ring R is called *Armendariz* [23] if for $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j$ in $R[x]$, $f(x)g(x) = 0$ implies $a_ib_j = 0$ for all i, j. It is well known that reduced rings are Armendariz [4, Lemma 1] and Armendariz rings are abelian [14, Corollary 8], and these implications are strict [23, Proposition 2.1, Example 3.2].

For a ring R with an endomorphism α, the mapping $\bar{\alpha} : R[x] \to R[x]$ defined by

$$\bar{\alpha}\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} \alpha(a_i) x^i$$

induces an endomorphism of $R[x]$.

PROPOSITION 2.11

For an Armendariz ring R and an endomorphism α of R, R is right α-skew central reversible if and only if $R[x]$ is right $\bar{\alpha}$-skew central reversible.

Proof. Let R be right α-skew central reversible and let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x]$ such that $f(x)g(x) = 0$. Since R is Armendariz, therefore, $a_i b_j = 0$ for all i, j and so $b_j \alpha(a_i) \in Z(R)$, entailing $g(x)\bar{\alpha}(f(x)) \in Z(R[x])$. Therefore, $R[x]$ is right $\bar{\alpha}$-skew central reversible. The converse is straightforward by Remark 2.9(1). □

Next, we show that for a right α-skew central reversible ring R, $R[x]$ need not be right $\bar{\alpha}$-skew central reversible.

Example 2.12. The ring construction here is essentially due to [16, Example 2.12] and [18, Example 2.8]. Let $A = \mathbb{Z}_2[a_0, a_1, a_2, b_0, b_1, b_2, c]$ be the free algebra with zero constant terms in noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Note that A is a ring without unity and consider an ideal I of the ring $\mathbb{Z}_2 + A$, generated by

$$\begin{gathered} a_0b_0,\ a_0b_1 + a_1b_0,\ a_0b_2 + a_1b_1 + a_2b_0,\ a_1b_2 + a_2b_1,\ a_2b_2,\ a_0rb_0,\ a_2rb_2, \\ b_0a_0,\ b_0a_1 + b_1a_0,\ b_0a_2 + b_1a_1 + b_2a_0,\ b_1a_2 + b_2a_1,\ b_2a_2,\ b_0ra_0,\ b_2ra_2, \\ (a_0 + a_1 + a_2)r(b_0 + b_1 + b_2),\ (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2), \\ a_0a_0,\ a_2a_2,\ a_0ra_0,\ a_2ra_2,\ b_0b_0,\ b_2b_2,\ b_0rb_0,\ b_2rb_2,\ r_1r_2r_3r_4r_5, \\ a_0a_1 + a_1a_0,\ a_0a_2 + a_1a_1 + a_2a_0,\ a_1a_2 + a_2a_1, \\ b_0b_1 + b_1b_0,\ b_0b_2 + b_1b_1 + b_2b_0,\ b_1b_2 + b_2b_1, \\ (a_0 + a_1 + a_2)r(a_0 + a_1 + a_2),\ (b_0 + b_1 + b_2)r(b_0 + b_1 + b_2), \end{gathered}$$

where $r, r_1, r_2, r_3, r_4, r_5 \in A$. Then clearly $A^5 \subseteq I$. Let $R = (\mathbb{Z}_2 + A)/I$. Let σ be an endomorphism of $\mathbb{Z}_2 + A$ defined by

$$a_0, a_1, a_2, b_0, b_1, b_2, c \mapsto b_0, b_1, b_2, a_0, a_1, a_2, c.$$

Since $\sigma(I) \subseteq I$, we obtain an endomorphism α of R given by $\alpha(s + I) = \sigma(s) + I$ for $s \in \mathbb{Z}_2 + A$. Let H_n be the set of all linear combinations of monomials of degree n over $\mathbb{Z}_2$. Note that any $f \in R$ may be written as $f = k + f_1 + f_2 + f_3 + f_4 + f_5$, where $k \in \mathbb{Z}_2$, $f_i \in H_i$ (for $i = 1, 2, 3, 4$) and $f_5 \in I$ as $H_i \subseteq I$ for $i \geq 5$. It is clear that $Z(R) = \{k + f_4 + f_5 : k \in \mathbb{Z}_2, f_4 \in H_4, f_5 \in I\}$.

Consider $R[x] \cong (\mathbb{Z}_2 + A)[x]/I[x]$. For $f(x) = a_0 + a_1x + a_2x^2$, $g(x) = b_0c + b_1cx + b_2cx^2$ in $(\mathbb{Z}_2 + A)[x]$, we have $f(x)g(x) \in I[x]$, however, $g(x)\bar{\alpha}(f(x)) \notin Z(R[x])$ as $b_0cb_1 + b_1cb_0$ is not central. Therefore, $R[x]$ is not right $\bar{\alpha}$-skew central reversible.

We apply the arguments given in [19, Example 2.1] to show that R is right α-skew central reversible.

Claim 1. If $f_1 g_1 \in I$ for $f_1, g_1 \in H_1$, then $g_1\alpha(f_1) \in I$.

Proof. By the definition of I, we have only the following cases:

$$\begin{aligned}
&(f_1 = a_0, g_1 = b_0), (f_1 = a_0, g_1 = a_0),\\
&\quad (f_1 = b_0, g_1 = a_0), (f_1 = b_0, g_1 = b_0),\\
&(f_1 = a_2, g_1 = b_2), (f_1 = a_2, g_1 = a_2),\\
&\quad (f_1 = b_2, g_1 = a_2), (f_1 = b_2, g_1 = b_2),\\
&(f_1 = a_0 + a_1 + a_2, g_1 = b_0 + b_1 + b_2),\\
&\quad (f_1 = a_0 + a_1 + a_2, g_1 = a_0 + a_1 + a_2),\\
&(f_1 = b_0 + b_1 + b_2, g_1 = a_0 + a_1 + a_2),\\
&\quad (f_1 = b_0 + b_1 + b_2, g_1 = b_0 + b_1 + b_2).
\end{aligned}$$

So we obtain the result using the definitions of I and α.

Claim 2. If $fg \in I$ for $f, g \in A$, then $g\alpha(f) \in Z(R)$.

Proof. We may write $f = f_1 + f_2 + f_3 + f_4 + f_5$, $g = g_1 + g_2 + g_3 + g_4 + g_5$, where $f_i, g_i \in H_i$ for $i = 1, 2, 3, 4$ and $f_5, g_5 \in I$. Then $fg = f_1g_1 + (f_1g_2 + f_2g_1) + (f_1g_3 + f_2g_2 + f_3g_1) + h$ with $h \in I$, so $fg \in I$ implies $f_1g_1 + (f_1g_2 + f_2g_1) + (f_1g_3 + f_2g_2 + f_3g_1) \in I$. Since I is homogeneous, therefore, $f_1g_1 \in I$ and $f_1g_2 + f_2g_1 \in I$. By Claim 1, $g_1\alpha(f_1) \in I$. We show that $g_1\alpha(f_2) + g_2\alpha(f_1) \in I$. From $f_1g_2 + f_2g_1 \in I$, we have the following cases:

$$\begin{aligned}
&(f_1 = a_0, g_1 = b_0), (f_1 = a_0, g_1 = a_0),\\
&\quad (f_1 = b_0, g_1 = a_0), (f_1 = b_0, g_1 = b_0),\\
&(f_1 = a_2, g_1 = b_2), (f_1 = a_2, g_1 = a_2),\\
&\quad (f_1 = b_2, g_1 = a_2), (f_1 = b_2, g_1 = b_2),\\
&(f_1 = a_0 + a_1 + a_2, g_1 = b_0 + b_1 + b_2),\\
&\quad (f_1 = a_0 + a_1 + a_2, g_1 = a_0 + a_1 + a_2),\\
&(f_1 = b_0 + b_1 + b_2, g_1 = a_0 + a_1 + a_2),\\
&\quad (f_1 = b_0 + b_1 + b_2, g_1 = b_0 + b_1 + b_2).
\end{aligned}$$

If f_2, g_2 are in I then we get the result, so we consider other cases of f_2 and g_2.

When $f_1 = a_0, g_1 = b_0$, we may obtain the following cases:

$$\begin{aligned}
&(f_2 \in I, g_2 = b_0t), (f_2 \in I, g_2 = tb_0),\\
&\quad (f_2 \in I, g_2 = a_0t), (f_2 \in I, g_2 = ta_0),\\
&(f_2 = a_0s, g_2 \in I), (f_2 = sa_0, g_2 \in I),\\
&\quad (f_2 = b_0s, g_2 \in I), (f_2 = sb_0, g_2 \in I),\\
&(f_2 = a_0s, g_2 = b_0t), (f_2 = a_0s, g_2 = tb_0),
\end{aligned}$$

$$(f_2 = a_0 s, g_2 = a_0 t), (f_2 = a_0 s, g_2 = t a_0),$$
$$(f_2 = s a_0, g_2 = b_0 t), (f_2 = s a_0, g_2 = t b_0),$$
$$(f_2 = s a_0, g_2 = a_0 t), (f_2 = s a_0, g_2 = t a_0),$$
$$(f_2 = b_0 s, g_2 = b_0 t), (f_2 = b_0 s, g_2 = t b_0),$$
$$(f_2 = b_0 s, g_2 = a_0 t), (f_2 = b_0 s, g_2 = t a_0),$$
$$(f_2 = s b_0, g_2 = b_0 t), (f_2 = s b_0, g_2 = t b_0),$$
$$(f_2 = s b_0, g_2 = a_0 t), (f_2 = s b_0, g_2 = t a_0),$$

where $s, t \in H_1$. Then $g_1\alpha(f_2) + g_2\alpha(f_1) = b_0\alpha(f_2) + g_2 b_0 \in I$. The computations for other cases are similar. Thus we get $g\alpha(f) \in Z(R)$.

Let $g, h \in \mathbb{Z}_2 + A$ such that $gh \in I$. We may write $g = \beta + g'$, $h = \gamma + h'$ for some $\beta, \gamma \in \mathbb{Z}_2$ and some $g', h' \in A$. Since $gh = \beta\gamma + \beta h' + \gamma g' + g'h' \in I$, therefore, $\beta = 0$ or $\gamma = 0$. Assume $\beta = 0$. Then $\gamma g' + g'h' \in I$, so $g' \in I$ and $g'h' \in I$ as I is homogeneous and $\gamma \in \mathbb{Z}_2$; so by Claim 2, $h'\alpha(g') \in Z(R)$ and consequently, $h\alpha(g) \in Z(R)$. For the case $\gamma = 0$, we obtain $h\alpha(g) \in Z(R)$ similarly.

For a ring R and for $n \geq 2$, consider the following rings:

$$D_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} : a, a_{ij} \in R \right\}$$

and

$$V_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} : a_i \in R \right\}$$

which are subrings of $M_n(R)$. Note that $R[x]/(x^n) \cong V_n(R)$, where (x^n) is the ideal of $R[x]$ generated by x^n.

For simplicity, we use $(a_1, a_2, a_3, \ldots, a_n) \in V_n(R)$ to denote

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix}.$$

Lemma 2.13. *Let R be a ring. Then we have*

(1) $$Z(D_n(R)) = \left\{ \begin{pmatrix} a & 0 & 0 & \cdots & a_{1n} \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} : a, a_{1n} \in Z(R) \right\}.$$

(2) $Z(V_n(R)) = \{(a_1, a_2, a_3, \ldots, a_n) : a_i \in Z(R)\}$.

Proof.

(1) By [8, Proposition 1.7].
(2) Let $(a_1, a_2, a_3, \ldots, a_n) \in Z(V_n(R))$. For $b \in R$, $(a_1, a_2, a_3, \ldots, a_n)(b, 0, 0, \ldots, 0) = (b, 0, 0, \ldots, 0)(a_1, a_2, a_3, \ldots, a_n)$ yields $a_i b = b a_i$, where $1 \le i \le n$, entailing $a_i \in Z(R)$ for $1 \le i \le n$. This shows that $Z(V_n(R)) \subseteq \{(a_1, a_2, a_3, \ldots, a_n) : a_i \in Z(R)\}$. It is very much clear that any $(a_1, a_2, a_3, \ldots, a_n)$ with each $a_i \in Z(R)$ is an element of $Z(V_n(R))$, and hence $Z(V_n(R)) = \{(a_1, a_2, a_3, \ldots, a_n) : a_i \in Z(R)\}$. □

For a ring R with an endomorphism α, the correspondence $(a_{ij}) \overset{\bar{\alpha}}{\mapsto} (\alpha(a_{ij}))$ induce endomorphisms of $M_n(R)$, $U_n(R)$, $D_n(R)$ and $V_n(R)$.

PROPOSITION 2.14

Let R be any ring and α an endomorphism of R such that $\alpha(1) = 1$. Then we have the following:

(1) *$M_n(R)$ is not right $\bar{\alpha}$-skew central reversible for $n \ge 2$.*
(2) *$U_n(R)$ is not right $\bar{\alpha}$-skew central reversible for $n \ge 2$.*
(3) *$D_n(R)$ is not right $\bar{\alpha}$-skew central reversible for $n \ge 4$.*

Proof.

(1) Let $n \ge 2$. For $A = E_{12}$, $B = E_{11} \in M_n(R)$, we have $AB = 0$ but $B\bar{\alpha}(A) = A \notin Z(M_n(R))$, entailing $M_n(R)$ is not right $\bar{\alpha}$-skew central reversible.
(2) By arguments similar to (1).
(3) Let $n \ge 4$. For $A = E_{23}$, $B = E_{12} \in D_n(R)$, we have $AB = 0$ but $B\bar{\alpha}(A) = E_{13} \notin Z(D_n(R))$, entailing $D_n(R)$ is not right $\bar{\alpha}$-skew central reversible. □

Remark 2.15. Suppose R is a noncommutative ring with an endomorphism α such that $\alpha(1) = 1$. Then there exists $s \in R$ such that $s \notin Z(R)$. For $A = E_{23}$, $B = sE_{12} \in D_3(R)$, we have $AB = 0$ but $B\bar{\alpha}(A) = sE_{13} \notin Z(D_3(R))$, entailing $D_3(R)$ is not right $\bar{\alpha}$-skew central reversible.

For a right α-skew central reversible ring R, we show that $V_n(R)$ need not be right $\bar{\alpha}$-skew central reversible for $n \ge 2$.

Example 2.16. Assume that $n \geq 2$. Consider the ring in Example 2.12, i.e., $R = (\mathbb{Z}_2 + A)/I$ with the endomorphism α, where A, I and α are as defined in Example 2.12. Then R is right α-skew central reversible. For simplicity, we identify the elements of $\mathbb{Z}_2 + A$ with their images in R. For $A = (a_0, 0, \ldots, 0, a_1)$, $B = (b_0c, 0, \ldots, 0, b_1c) \in V_n(R)$, we have $AB = 0$ but $B\bar{\alpha}(A) \notin Z(V_n(R))$ as $b_0cb_1 + b_1cb_0 \notin Z(R)$, entailing $V_n(R)$ is not right $\bar{\alpha}$-skew central reversible.

PROPOSITION 2.17

For a reduced ring R and an endomorphism α of R, R is right α-skew central reversible if and only if $V_n(R)$ is right $\bar{\alpha}$-skew central reversible for $n \geq 2$.

Proof. Assume that $n \geq 2$. Suppose R is right α-skew central reversible. Let $A = (a_1, a_2, \ldots, a_n)$, $B = (b_1, b_2, \ldots, b_n) \in V_n(R)$ such that $AB = 0$. Then we have the following equations:

$$a_1b_1 = 0, \tag{3}$$
$$a_1b_2 + a_2b_1 = 0, \tag{4}$$
$$a_1b_3 + a_2b_2 + a_3b_1 = 0, \tag{5}$$
$$\vdots$$
$$a_1b_{n-1} + a_2b_{n-2} + \cdots + a_{n-1}b_1 = 0, \tag{6}$$
$$a_1b_n + a_2b_{n-1} + \cdots + a_{n-1}b_2 + a_nb_1 = 0. \tag{7}$$

Since R is reduced, therefore from (3), we have

$$b_1a_1 = 0. \tag{8}$$

Multiplying (4) on the left-hand side by b_1 and using (8), we get $b_1a_2b_1 = 0$ and so $(a_2b_1)^2 = 0$. Since R is reduced, therefore, $a_2b_1 = 0$. Then from (4), $a_1b_2 = 0$. Thus we get

$$a_1b_2 = a_2b_1 = 0 \quad \text{and so } b_1a_2 = b_2a_1 = 0. \tag{9}$$

Multiplying (5) on the left-hand side by b_1, and using (8) and (9), we get $a_3b_1 = 0$ as R is reduced. Then (5) becomes

$$a_1b_3 + a_2b_2 = 0. \tag{10}$$

Multiplying (10) on the left-hand side by b_2, we obtain $a_2b_2 = 0$ and $a_1b_3 = 0$ by similar arguments. Thus we have

$$a_ib_j = 0 \quad \text{for all } 2 \leq i + j \leq 4. \tag{11}$$

Assume that

$$a_ib_j = 0 \quad \text{for all } 2 \leq i + j \leq n. \tag{12}$$

Since R is reduced, therefore,

$$b_j a_i = 0 \quad \text{for all } 2 \le i + j \le n. \tag{13}$$

Multiplying (7) on the left-hand side by $b_1, b_2, \ldots, b_{n-1}$ in turn, and using (13), we obtain $a_i b_j = 0$ for all $i + j = n + 1$. By induction, we have

$$a_i b_j = 0 \quad \text{for all } 2 \le i + j \le n + 1. \tag{14}$$

Since R is right α-skew central reversible, therefore, $b_j \alpha(a_i) \in Z(R)$ for all $2 \le i + j \le n + 1$ and consequently, $B\bar{\alpha}(A) \in Z(V_n(R))$. Hence $V_n(R)$ is right $\bar{\alpha}$-skew central reversible. The converse is straightforward by Remark 2.9(1). □

Recall that for a ring R and an (R, R)-bimodule M, the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$. This ring is isomorphic to the ring of all matrices $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

COROLLARY 2.18

For a reduced ring R and an endomorphism α of R, the following are equivalent:

(1) *R is right α-skew central reversible.*
(2) *$T(R, R)$ is right $\bar{\alpha}$-skew central reversible.*
(3) *$R[x]/(x^n)$ is right $\bar{\alpha}$-skew central reversible for $n \ge 2$.*

For a ring R and for $p \in Z(R)$, define

$$R_p = \left\{ \begin{pmatrix} pa + b & a \\ 0 & b \end{pmatrix} : a, b \in R \right\}.$$

It is easy to see that R_p is a subring of $M_2(R)$. Also, for an endomorphism α of R with $\alpha(p) = p$, the restriction of $\bar{\alpha}$ on R_p induces an endomorphism of R_p.

PROPOSITION 2.19

For a central unit element p of a ring R with an endomorphism α such that $\alpha(p) = p$, R is right α-skew central reversible if and only if R_p is right $\bar{\alpha}$-skew central reversible.

Proof. We first establish the following Claim.

Claim 1. $Z(R_p) = \left\{ \begin{pmatrix} pa + b & a \\ 0 & b \end{pmatrix} : a, b \in Z(R) \right\}.$

Proof of Claim 1. Let $a, b \in Z(R)$. Then for any $x, y \in R$, we have

$$\begin{aligned} \begin{pmatrix} pa + b & a \\ 0 & b \end{pmatrix} \begin{pmatrix} px + y & x \\ 0 & y \end{pmatrix} &= \begin{pmatrix} (pa + b)(px + y) & (pa + b)x + ay \\ 0 & by \end{pmatrix} \\ &= \begin{pmatrix} (px + y)(pa + b) & (px + y)a + xb \\ 0 & yb \end{pmatrix} \\ &= \begin{pmatrix} px + y & x \\ 0 & y \end{pmatrix} \begin{pmatrix} pa + b & a \\ 0 & b \end{pmatrix} \end{aligned}$$

as $p \in Z(R)$. This gives

$$\begin{pmatrix} pa+b & a \\ 0 & b \end{pmatrix} \in Z(R_p).$$

Let $a, b \in R$ be such that

$$\begin{pmatrix} pa+b & a \\ 0 & b \end{pmatrix} \in Z(R_p).$$

Then for any $r \in R$, we have

$$\begin{pmatrix} pa+b & a \\ 0 & b \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} pa+b & a \\ 0 & b \end{pmatrix}.$$

This gives $ar = ra$ and $br = rb$, entailing $a, b \in Z(R)$. Hence

$$Z(R_p) = \left\{ \begin{pmatrix} pa+b & a \\ 0 & b \end{pmatrix} : a, b \in Z(R) \right\}.$$

Claim 2. If for $a, b \in R$, $pa + b, b \in Z(R)$, then $a \in Z(R)$.

Proof of Claim 2. Let $a, b \in R$ such that $pa + b, b \in Z(R)$. Since $Z(R)$ is a subring of R, therefore, $pa = (pa + b) - b \in Z(R)$. Thus for any $r \in R$, we have

$$p(ar) = (pa)r = r(pa) = p(ra)$$

as $p \in Z(R)$. Also, since p is a unit, we have $ar = ra$, yielding $a \in Z(R)$.

Let R be right α-skew central reversible and let $A, B \in R_p$ such that $AB = 0$, where

$$A = \begin{pmatrix} pa+b & a \\ 0 & b \end{pmatrix} \quad \text{and}$$
$$B = \begin{pmatrix} px+y & x \\ 0 & y \end{pmatrix}.$$

This gives $(pa + b)(px + y) = 0$ and $by = 0$. Since R is right α-skew central reversible, therefore, $(px + y)(p\alpha(a) + \alpha(b)) = (px + y)\alpha(pa + b) \in Z(R)$ and $y\alpha(b) \in Z(R)$. Therefore, by Claim 1 and Claim 2, we have

$$\begin{aligned} B\bar{\alpha}(A) &= \begin{pmatrix} px+y & x \\ 0 & y \end{pmatrix}\begin{pmatrix} p\alpha(a)+\alpha(b) & \alpha(a) \\ 0 & \alpha(b) \end{pmatrix} \\ &= \begin{pmatrix} (px+y)\,(p\alpha(a)+\alpha(b)) & (px+y)\,\alpha(a)+x\alpha(b) \\ 0 & y\alpha(b) \end{pmatrix} \in Z(R_p). \end{aligned}$$

Hence R_p is right $\bar{\alpha}$-skew central reversible. Conversely, assume that R_p is right $\bar{\alpha}$-skew central reversible. Let $a, b \in R$ be such that $ab = 0$. Then

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \in R_p$$

such that $AB = 0$. By assumption, $B\bar{\alpha}(A) \in Z(R_p)$ and consequently, $b\alpha(a) \in Z(R)$. Hence R is right α-skew central reversible. □

COROLLARY 2.20

For a central unit element p of a ring R, R is central reversible if and only if R_p is central reversible.

For a ring R and for $p, q \in Z(R)$, define

$$R_{p,q} = \left\{ \begin{pmatrix} pa+qb+c & 0 & 0 \\ a & qb+c & b \\ 0 & 0 & c \end{pmatrix} : a, b, c \in R \right\}.$$

Then $R_{p,q}$ is a subring of $M_3(R)$. Also, for an endomorphism α of R with $\alpha(p) = p$ and $\alpha(q) = q$, the restriction of $\bar{\alpha}$ on $R_{p,q}$ induces an endomorphism of $R_{p,q}$.

PROPOSITION 2.21

For central unit elements p, q of a ring R with an endomorphism α such that $\alpha(p) = p$ and $\alpha(q) = q$, R is right α-skew central reversible if and only if $R_{p,q}$ is right $\bar{\alpha}$-skew central reversible.

Proof. By applying arguments similar to those given in Proposition 2.19, we see that

$$Z(R_{p,q}) = \left\{ \begin{pmatrix} pa+qb+c & 0 & 0 \\ a & qb+c & b \\ 0 & 0 & c \end{pmatrix} : a, b, c \in Z(R) \right\}.$$

Claim 3. If $a, b, c \in R$ such that $pa + qb + c, qb + c, c \in Z(R)$, then $a, b \in Z(R)$.

Proof of Claim 3. Let $a, b, c \in R$ be such that $pa + qb + c, qb + c, c \in Z(R)$. Since $Z(R)$ is a subring of R, therefore, $pa = (pa + qb + c) - (qb + c) \in Z(R)$ and $qb = (qb + c) - c \in Z(R)$. Since p and q are central units in R, therefore, by the arguments given in the proof of Claim 2 of Proposition 2.19, we have $a, b \in Z(R)$.

Let R be right α-skew central reversible and let $A, B \in R_{p,q}$ be such that $AB = 0$, where

$$A = \begin{pmatrix} pa+qb+c & 0 & 0 \\ a & qb+c & b \\ 0 & 0 & c \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} px+qy+z & 0 & 0 \\ x & qy+z & y \\ 0 & 0 & z \end{pmatrix}.$$

Then we have

$$\begin{aligned} (pa+qb+c)(px+qy+z) &= 0, \\ (qb+c)(qy+z) &= 0, \\ cz &= 0. \end{aligned}$$

Since R is right α-skew central reversible, therefore,

$$\begin{aligned} (px+qy+z)(p\alpha(a)+q\alpha(b)+\alpha(c)) &\in Z(R), \\ (qy+z)(q\alpha(b)+\alpha(c)) &\in Z(R), \\ z\alpha(c) &\in Z(R), \end{aligned}$$

and so by Claim 3, we can conclude that $B\bar{\alpha}(A) \in Z(R_{p,q})$. The converse is straightforward. □

COROLLARY 2.22

For central unit elements p, q of a ring R, R is central reversible if and only if $R_{p,q}$ is central reversible.

For a ring R and for $p, q, r \in Z(R)$, define

$$R_{p,q,r} = \left\{ \begin{pmatrix} pa+qb+rc+d & a & 0 & 0 \\ 0 & qb+rc+d & 0 & 0 \\ 0 & b & rc+d & c \\ 0 & 0 & 0 & d \end{pmatrix} : a, b, c, d \in R \right\}.$$

Of course, $R_{p,q,r}$ is a subring of $M_4(R)$ and for an endomorphism α of R which fix the elements p, q, r; the restriction of $\bar{\alpha}$ on $R_{p,q,r}$ induces an endomorphism of $R_{p,q,r}$. By applying arguments similar to those given in Propositions 2.19 and 2.21, we have the following.

PROPOSITION 2.23

For central unit elements p, q, r of a ring R with an endomorphism α such that $\alpha(p) = p$, $\alpha(q) = q$ and $\alpha(r) = r$, R is right α-skew central reversible if and only if $R_{p,q,r}$ is right $\bar{\alpha}$-skew central reversible.

COROLLARY 2.24

For central unit elements p, q, r of a ring R, R is central reversible if and only if $R_{p,q,r}$ is central reversible.

Remark 2.25. In this way, we can construct subrings of $M_n(R)$ which preserve the 'right α-skew central reversible' as well as 'central reversible' property of R.

For an algebra R over a commutative ring S, the *Dorroh extension* [10] of R by S is the abelian group $D = R \oplus S$ with multiplication given by

$$(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2).$$

For an S-endomorphism α of R and the Dorroh extension D of R by S, the mapping $\bar{\alpha} : D \to D$ defined by $\bar{\alpha}(r, s) = (\alpha(r), s)$ is an S-algebra endomorphism.

PROPOSITION 2.26

Let R be an algebra over a commutative ring S and let α be an S-endomorphism of R with $\alpha(1) = 1$. Then R is right α-skew central reversible if and only if the Dorroh extension D of R by S is right $\bar{\alpha}$-skew central reversible.

Proof. Note that any $s \in S$ can be written as $s = s1 \in R$ and so $R = \{r + s : (r, s) \in D\}$. Assume that R is right α-skew central reversible and let $(r_1, s_1), (r_2, s_2) \in D$ such that $(r_1, s_1)(r_2, s_2) = 0$. This gives $r_1r_2 + s_1r_2 + s_2r_1 = 0$ and $s_1s_2 = 0$. Therefore, $(r_1, s_1)(r_2, s_2) = 0$ is equivalent to $(r_1 + s_1)(r_2 + s_2) = 0$ with $s_1s_2 = 0$. Since R is right α-skew central reversible and S is commutative, therefore, $(r_2 + s_2)\alpha(r_1 + s_1) \in Z(R)$ and $s_2s_1 = 0$, entailing $(r_2 + s_2)(\alpha(r_1) + s_1) \in Z(R)$ with $s_2s_1 = 0$ as $\alpha(1) = 1$. Thus $(r_2, s_2)\bar{\alpha}(r_1, s_1) = (r_2, s_2)(\alpha(r_1), s_1) \in Z(D)$ and so D is right $\bar{\alpha}$-skew central reversible. Conversely, assume that D is right $\bar{\alpha}$-skew central reversible. Clearly, $e = (1, 0) \in D$ such that $e^2 = e$ and $eDe \cong R$. The rest follows directly from Proposition 2.10(1). □

The condition '$\alpha(1) = 1$' in Proposition 2.26 is not superfluous by the following example.

Example 2.27. The argument here is due to [5, Example 2.16]. Consider $R = \mathbb{H} \oplus \mathbb{H}$ with the usual addition and multiplication and let $\alpha : R \to R$ be an endomorphism defined by $\alpha((a, b)) = (0, b)$. It is easy to show that R is right α-skew central reversible. Since any ring can be regarded as a $\mathbb{Z}$-algebra, we consider the Dorroh extension D of R by $\mathbb{Z}$. For $a = ((1, 0), -1), b = ((i, 0), 0) \in D$, we have $ab = 0$ but $b\bar{\alpha}(a) = ((-i, 0), 0) \notin Z(D)$. Therefore, D is not right $\bar{\alpha}$-skew central reversible.

A nonzero element of a ring is called *regular* if it is neither a left nor a right zero divisor. For a multiplicatively closed subset S of a ring R consisting of regular elements, we denote by $S^{-1}R$, the localization of R at S. For an endomorphism α of R with $\alpha(S) \subseteq S$, the mapping $\bar{\alpha} : S^{-1}R \to S^{-1}R$ defined by $\bar{\alpha}(s^{-1}r) = \alpha(s)^{-1}\alpha(r)$ for $s \in S$ and $r \in R$, induces an endomorphism of $S^{-1}R$.

PROPOSITION 2.28

For a multiplicatively closed subset S of a ring R consisting of central regular elements and an endomorphism α of R with $\alpha(S) \subseteq S$ and $\alpha(1) = 1$, R is right α-skew central reversible if and only if $S^{-1}R$ is right $\bar{\alpha}$-skew central reversible.

Proof. Let R be right α-skew central reversible and let $A = s_1^{-1}r_1$, $B = s_2^{-1}r_2 \in S^{-1}R$ be such that $AB = 0$, where $s_1, s_2 \in S$ and $r_1, r_2 \in R$. Then we have

$$0 = AB = \left(s_1^{-1}r_1\right)\left(s_2^{-1}r_2\right) = (s_1s_2)^{-1}\, r_1r_2$$

as $s_1, s_2 \in S \subseteq Z(R)$. This gives $r_1r_2 = 0$. By assumption, $r_2\alpha(r_1) \in Z(R)$. Thus

$$B\bar{\alpha}(A) = \left(s_2^{-1}r_2\right)\left(\alpha(s_1)^{-1}\alpha(r_1)\right) = (s_2\alpha(s_1))^{-1}\, r_2\alpha(r_1) \in Z(S^{-1}R).$$

Therefore $S^{-1}R$ is right $\bar{\alpha}$-skew central reversible. The converse is straightforward by Remark 2.9(1) as $\alpha(1) = 1$. □

The ring of *Laurent polynomials* in x over a ring R, consisting of all formal sums $\sum_{i=k}^{n} r_i x^i$ with the usual addition and multiplication, where $r_i \in R$ and k, n are integers, is denoted by $R[x; x^{-1}]$. For a ring R with an endomorphism α, $\bar{\alpha} : R[x; x^{-1}] \to R[x; x^{-1}]$ defined by

$$\bar{\alpha}\left(\sum_{i=k}^{n} r_i x^i\right) = \sum_{i=k}^{n} \alpha(r_i) x^i$$

induces an endomorphism of $R[x; x^{-1}]$.

COROLLARY 2.29

For a ring R with an endomorphism α such that $\alpha(1) = 1$, $R[x]$ is right $\bar{\alpha}$-skew central reversible if and only if $R[x; x^{-1}]$ is right $\bar{\alpha}$-skew central reversible.

Proof. Since $S = \{1, x, x^2, \ldots\}$ is a multiplicatively closed subset of $R[x]$ such that $R[x; x^{-1}] = S^{-1}R[x]$, it follows directly from Proposition 2.28. □

Recall that for a ring R with an endomorphism α, a *skew polynomial* ring (also called an *Ore extension of endomorphism type*) $R[x; \alpha]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication $xr = \alpha(r)x$ for all $r \in R$.

For a ring R with an endomorphism α, the mapping $\bar{\alpha} : R[x; \alpha] \to R[x; \alpha]$ defined by

$$\bar{\alpha}\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} \alpha(a_i) x^i$$

induces an endomorphism of $R[x; \alpha]$.

Next we show that for a right α-skew central reversible ring R, $R[x; \alpha]$ need not be right $\bar{\alpha}$-skew central reversible.

Example 2.30. Consider the ring in Example 2.12, i.e., $R = (\mathbb{Z}_2 + A)/I$ with the endomorphism α, where A, I and α are as defined in Example 2.12. Then R is right α-skew central reversible. Note that $\alpha^2 = 1_R$. For simplicity, we identify the elements of $\mathbb{Z}_2 + A$ with their images in R.

For $f(x) = a_0 + a_1x^2 + a_2x^4$, $g(x) = b_0c + b_1cx^2 + b_2cx^4 \in R[x; \alpha]$, we have $f(x)g(x) = 0$ but $g(x)\bar{\alpha}(f(x)) \notin Z(R[x; \alpha])$ as $b_0cb_1 + b_1cb_0$ is not central in R. Therefore, $R[x; \alpha]$ is not right $\bar{\alpha}$-skew central reversible.

A multiplicatively closed subset S of a ring R is said to satisfy the left *Ore* condition (or, is called a left *Ore* set) if for each $r \in R$ and $s \in S$, $Sr \cap Rs \neq \emptyset$. Following [11, Theorem 6.2], a multiplicatively closed subset S of a ring R consisting of regular elements satisfies the left Ore condition if and only if the localization of R at S exists.

It is easy to see that $S = \{1, x, x^2, \ldots\}$ is a left Ore subset of $R[x; \alpha]$. Therefore, by localizing $R[x; \alpha]$ with respect to S, we can obtain the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$, whose elements are finite sums of elements of the form $x^{-j}rx^i$, where $r \in R$ and i, j are non-negative integers, and the multiplication is subject to $xr = \alpha(r)x$ and $rx^{-1} = x^{-1}\alpha(r)$ for all $r \in R$.

For a monomorphism α of a ring R, we refer to the Jordan's construction of an overring of R by α (see [15] for details). Let $A(R, \alpha) = \{x^{-i}rx^i : r \in R, i \geq 0\}$ be a subset of the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$. Then for each $j \geq 0$, we can write $x^{-i}rx^i = x^{-(i+j)}\alpha^j(r)x^{(i+j)}$. It follows that $A(R, \alpha)$ forms a subring of $R[x, x^{-1}; \alpha]$ with the following operations:

$$\left(x^{-i}rx^i\right) + \left(x^{-j}sx^j\right) = x^{-(i+j)}\left(\alpha^j(r) + \alpha^i(s)\right)x^{(i+j)},$$

$$\left(x^{-i}rx^i\right)\left(x^{-j}sx^j\right) = x^{-(i+j)}\left(\alpha^j(r)\alpha^i(s)\right)x^{(i+j)}.$$

The mapping $\bar{\alpha} : A(R,\alpha) \to A(R,\alpha)$ defined by $\bar{\alpha}\left(x^{-i}rx^{i}\right) = x^{-i}\alpha(r)x^{i}$, induces an automorphism of $A(R,\alpha)$. The ring $A(R,\alpha)$ is called the *Jordan extension* of R by α.

PROPOSITION 2.31

For a ring R with an automorphism α, R is right α-skew central reversible if and only if the Jordan extension $A(R,\alpha)$ of R by α is right $\bar{\alpha}$-skew central reversible.

Proof. To prove the result, we first establish the following Claim.

Claim 1. If $r \in Z(R)$, then $\alpha^{k}(r) \in Z(R)$ for all $k \geq 1$.

Proof of Claim 1. It is enough to show that $\alpha(r) \in Z(R)$ when $r \in Z(R)$. Let $r \in Z(R)$ and let $s \in R$. Since α is surjective, there exists $s_1 \in R$ such that $\alpha(s_1) = s$. Thus $\alpha(r)s = \alpha(r)\alpha(s_1) = \alpha(rs_1) = \alpha(s_1 r) = \alpha(s_1)\alpha(r) = s\alpha(r)$, yielding $\alpha(r) \in Z(R)$.

Claim 2. If $r \in Z(R)$, then $x^{-i}rx^{i} \in Z(A(R,\alpha))$.

Proof of Claim 2. Let $r \in Z(R)$ and let $x^{-j}sx^{j} \in A(R,\alpha)$. Then by Claim 1, we have

$$\begin{aligned}(x^{-i}rx^{i})(x^{-j}sx^{j}) &= x^{-(i+j)}(\alpha^{j}(r)\alpha^{i}(s))x^{(i+j)}\\ &= x^{-(i+j)}(\alpha^{i}(s)\alpha^{j}(r))x^{(i+j)}\\ &= (x^{-j}sx^{j})(x^{-i}rx^{i}),\end{aligned}$$

entailing $x^{-i}rx^{i} \in Z(A(R,\alpha))$.

Let R be right α-skew central reversible and let $A = x^{-i}rx^{i}$, $B = x^{-j}sx^{j} \in A(R,\alpha)$ such that $AB = 0$. This gives $0 = (x^{-i}rx^{i})(x^{-j}sx^{j}) = x^{-(i+j)}(\alpha^{j}(r)\alpha^{i}(s))x^{(i+j)}$, entailing $\alpha^{j}(r)\alpha^{i}(s) = 0$. Since R is right α-skew central reversible, therefore, $\alpha^{i}(s)\alpha^{j+1}(r) = \alpha^{i}(s)\alpha(\alpha^{j}(r)) \in Z(R)$. By Claim 2, we have

$$B\bar{\alpha}(A) = (x^{-j}sx^{j})(x^{-i}\alpha(r)x^{i}) = x^{-(i+j)}(\alpha^{i}(s)\alpha^{j+1}(r))x^{(i+j)} \in Z(A(R,\alpha)).$$

Hence $A(R,\alpha)$ is right $\bar{\alpha}$-skew central reversible. The converse is straightforward by Remark 2.9(1). □

COROLLARY 2.32

For an automorphism α of a ring R, R is central reversible if and only if the Jordan extension $A(R,\alpha)$ of R by α is central reversible.

For a subring S of a ring R and for $n \geq 1$, define

$$[R, S]_n = \{(r_1, \ldots, r_n, s, s, \ldots) : s \in S, r_i \in R, 1 \leq i \leq n\}.$$

Clearly, $[R, S]_n$ is a ring under componentwise addition and multiplication. Note that

$$Z([R, S]_n) = \{(r_1, \ldots, r_n, s, s, \ldots) : s \in Z(S), r_i \in Z(R), 1 \leq i \leq n\}.$$

For an endomorphism α of R with $\alpha(S) \subseteq S$, the mapping $\bar{\alpha} : [R, S]_n \to [R, S]_n$ defined by

$$\bar{\alpha}\left((r_1, \ldots, r_n, s, s, \ldots)\right) = (\alpha(r_1), \ldots, \alpha(r_n), \alpha(s), \alpha(s), \ldots)$$

induces an endomorphism of $[R, S]_n$.

PROPOSITION 2.33

Let S be a subring of a ring R and α an endomorphism of R with $\alpha(S) \subseteq S$. Then R is right α-skew central reversible if and only if $[R, S]_n$ is right $\bar{\alpha}$-skew central reversible for $n \geq 1$.

Proof. Assume that $n \geq 1$. Let R be right α-skew central reversible and let $A = (a_1, \ldots, a_n, s, s, \ldots)$, $B = (b_1, \ldots, b_n, t, t, \ldots) \in [R, S]_n$ such that $AB = 0$, where $a_i, b_i \in R$ for $1 \leq i \leq n$ and $s, t \in S$. This gives $a_i b_i = 0$ and so $b_i \alpha(a_i) \in Z(R)$ as R is right α-skew central reversible. Also, $\alpha(S) \subseteq S$ implies that S is right α-skew central reversible by Remark 2.9(1) and so $t\alpha(s) \in Z(S)$, entailing $B\bar{\alpha}(A) \in Z([R, S]_n)$. Hence $[R, S]_n$ is right $\bar{\alpha}$-skew central reversible. The converse is straightforward. □

COROLLARY 2.34

For a subring S of a ring R, R is central reversible if and only if $[R, S]_n$ is central reversible for $n \geq 1$.

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