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On weighted signed color partitions

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Abstract. In this paper, we provide combinatorial interpretations of certain proved Rogers–Ramanujan type identities using signed color partitions with attached weights. The approach of using the signed color partitions is interesting since negative exponents do not make an explicit appearance in these identities.

Keywords. $(n+t)$-color partitions; signed partitions; combinatorial interpretation; Rogers–Ramanujan type identities.

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1. Introduction

Euler included a result in the chapter titled "De Partitio Numerorum", given in [5], that every positive integer is uniquely represented as the sum or difference of distinct power of 3. He wrote this in terms of the generating function

$$\sum_{\nu=-\infty}^{\infty} x^{\nu} = \prod_{\nu=0}^{\infty} (x^{-3^n} + x^{3^n} + 1), \tag{1.1}$$

which converges for no values of x. But in [2], Andrews treated (1.1) as an identity in formal Laurent series. He showed great interest in this section and asked the question: Why have we thought so little about partition generating functions in which some of the partitions might have some negative parts? He also explored Euler's eye-catching identity (1.1) and found some new and appealing results. He called such partitions as 'signed partitions' in which parts may appear with + or − sign.

DEFINITION 1.1

In [8], a signed partition Θ of an integer ν, denoted by $\Theta \dashv \nu$, is a partition pair (θ_1, θ_2), where

$$\nu = \theta_1 + \theta_2,$$

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θ_1 (resp. θ_2) is the positive (resp. negative) subpartition of Θ and $\theta_1^1, \theta_1^2, \ldots, \theta_1^{l(\theta_1)}$ (resp. $\theta_2^1, \theta_2^2, \ldots, \theta_2^{l(\theta_2)}$) are the positive (resp. negative) parts of Θ.

Remark 1. The unrestricted signed partitions of an integer are infinitely many. We are particularly interested in finite sets of signed partitions and this can be done by imposing some suitable restrictions on parts.

Example 1.1. Let $\theta_1 = 6+3+3$ and $\theta_2 = -3-2-1$. Then $\Theta = (6+3+3, -3-2-1)$ is a signed partition of 6.

Signed partitions fit naturally while interpreting many classical q-series identities. For instance, refer [2] for interpretations of Göllnitz–Gordon identities using signed partition. Further in [9], Sills provided a bijection between the ordinary and signed partitions for the Göllnitz–Gordon identity.

In [7], Keith explored four combinatorial theorems by presenting bijections between restricted signed partitions and ordinary partitions. He also studied the behavior of signed partitions of zero in arithmetic progression. In [8], McLaughlin and Sills provided interpretations of Rogers–Ramanujan type identities, which belonged to the family of mod 36 with the missing member of the family in the same flavor.

Recently, Gupta and Rana [6] introduced signed color partitions and provided combinatorial interpretations of 100 Rogers–Ramanujan type identities which are listed in [4,10]. This rich collection of combinatorial interpretations of Rogers–Ramanujan type identities allows to explore signed color partitions. But still a particular type of Rogers–Ramanujan type identities lack in their interpretations using signed partitions or signed color partitions. In this paper, we address this issue by introducing weight attached to signed color partitions. We will study the following Rogers–Ramanujan type identities:

$$\sum_{\nu=0}^{\infty} \frac{(q;q^3)_\nu (q^2;q^3)_{\nu+1} q^{3\binom{\nu+1}{2}}}{(q^3;q^3)_\nu (q^3;q^6)_{\nu+1}} = \frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}(q^2;q^2)_\infty, \text{ [4; } I(50)] \tag{1.2}$$

$$\sum_{\nu=0}^{\infty}(-1)^\nu \frac{(q;q^2)_{2\nu} q^{2\nu(\nu+2)}}{(-q^2;q^2)_{2\nu+1}(q^4;q^4)_\nu} = \frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}[q^8,-q^7,-q^9;q^8]_\infty, \text{ [4; } I(78)] \tag{1.3}$$

$$\sum_{\nu=0}^{\infty} \frac{(-q;q)_\nu (-1;q^3)_\nu q^{\binom{\nu+1}{2}}}{(q;q)_{2\nu}(-1;q)_\nu} = \frac{(-q;q)_\infty}{(q;q)_\infty}[q^6,q,q^5;q^6]_\infty [q^8,q^4;q^{12}]_\infty, \text{ [4; } I(158)] \tag{1.4}$$

$$\sum_{\nu=0}^{\infty}(-1)^\nu \frac{(q;q)_{4\nu} q^{2\nu^2}}{(q^4;q^4)_\nu (q^4;q^4)_{2\nu}} = \frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}[q^8,-q^3,-q^5;q^8]_\infty, \text{ [4; } I(76)]. \tag{1.5}$$

Here we have employed the standard q-series notation

$$(a;q)_\nu = \prod_{k=0}^{\nu-1}(1-aq^k)$$

and

$$[a_1,a_2,\ldots,a_k;q]_\nu = (a_1;q)_\nu (a_2;q)_\nu \cdots (a_k;q)_\nu,$$

and also

$$[a_1, a_2, \ldots, a_k; q]_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_k; q)_\infty.$$

We are going to handle these identities by using signed color partitions which have their roots in the colored partitions introduced by Agarwal and Andrews in [1] and has applications in the hard hexagon model of Baxter [3].

DEFINITION 1.2

An n-color partition is a partition where a part n can appear in n colors denoted by subscripts: $n_1, n_2, \ldots, n_n$. The parts are ordered first by size and then by color. For any integer $t \geq 0$, an $(n+t)$-color partition is a partition in which a part n can appear in $(n+t)$-color partitions as $n_1, n_2, \ldots, n_{n+t}$. Note that if $t > 0$, the partition can contain a part of size 0 but only one copy of zero '0_t' is allowed. The weighted difference of two parts $(m_i)_{x_i}, (m_j)_{x_j}, m_i \geq m_j$ in an $(n+t)$-color partition $(m_r)_{x_r} + (m_{r-1})_{x_{r-1}} + \cdots + (m_1)_{x_1}$ such that $(m_r)_{x_r} \geq (m_{r-1})_{x_{r-1}} \geq \cdots \geq (m_1)_{x_1}$, is $m_i - m_j - x_i - x_j$ and is denoted by $(((m_i)_{x_i} - (m_j)_{x_j}))$.

DEFINITION 1.3

A signed color partition Θ is a signed partition pair (θ_1, θ_2), where θ_1 and θ_2 are the $(n+t)$-color partitions.

Example 1.2. $\Theta = (9_3 + 4_2 + 1_1, -6_2 - 3_1 - 1_1)$ is an example of signed color partitions of 4, where $\theta_1 = 9_3 + 4_2 + 1_1$ and $\theta_2 = -6_2 - 3_1 - 1_1$.

2. Main results

In this section, we provide the signed color partition-theoretic interpretation of (1.2). We first rewrite the summation side of (1.2) as follows:

$$\begin{aligned}\sum_{\nu=0}^{\infty} h_1(\nu) q^\nu &= \sum_{\nu=0}^{\infty} \frac{(q;q^3)_\nu (q^2;q^3)_{\nu+1} q^{3\binom{\nu+1}{2}}}{(q^3;q^3)_\nu (q^3;q^6)_{\nu+1}} \\ &= \sum_{\nu=0}^{\infty} \frac{q^{3\binom{\nu+1}{2}}}{(q^3;q^3)_\nu (q^3;q^6)_{\nu+1}} \\ &\qquad (1-q^{3\nu+2}) \prod_{j=1}^{\nu} (1-q^{3j-1})(1-q^{3j-2}) \\ &= \sum_{\nu=0}^{\infty} \frac{(-1)^{2\nu+1} q^{(9\nu^2+9\nu+4)/2}}{(q^3;q^3)_\nu (q^3;q^6)_{\nu+1}} \\ &\qquad (1-q^{-3\nu-2}) \prod_{j=1}^{\nu} (1-q^{-3j+1})(1-q^{-3j+2})\end{aligned}$$

$$= \sum_{\nu=0}^{\infty}\Big(\frac{q^{(9\nu^2+9\nu+4)/2}}{(q^3;q^3)_\nu(q^3;q^6)_{\nu+1}}\Big)\Big((-1)^{2\nu+1}(1-q^{-3\nu-2})\prod_{j=1}^{\nu}(1-q^{-3j+1})(1-q^{-3j+2})\Big).$$

The factor $\frac{q^{(9\nu^2+9\nu+4)/2}}{(q^3;q^3)_\nu(q^3;q^6)_{\nu+1}}$ generates the positive parts and $(-1)^{2\nu+1}(1-q^{-3\nu-2})\prod_{j=1}^{\nu}(1-q^{-(3j-2)})(1-q^{-(3j-1)})$ generates the negative parts of the signed color partitions. Let $g_1(\nu)$ be the generating function for the product side of (1.2). Then

$$\sum_{\nu=0}^{\infty} g_1(\nu)q^\nu = \frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}(q^2;q^2)_\infty$$
$$= \frac{(q^2;q^4)_\infty(q^4;q^6)_\infty}{(q^3;q^6)^2_\infty}.$$

Theorem 2.1. *Let Θ_1 denote the signed color partition pairs (θ_1,θ_2) and θ_1 be an $(n+5)$-color partition such that*

(*a*) $m_k \equiv \begin{cases} 2 \pmod 3 & \text{if } k=1,\\ 0 \pmod 3 & \text{if } k>1,\end{cases}$
(*b*) $x_1 \geq 7$ *and* $\equiv 1 \pmod 3$, $x_1 = m_1+5$,
(*c*) $x_k \geq 9$ *and* $\equiv 0 \pmod 3$ *for* $2 \leq k \leq r$,
(*d*) $(((m_k)_{x_k} - (m_{k-1})_{x_{k-1}})) \geq -9$ *and* $\equiv 0 \pmod 3$ *for* $2 \leq k \leq r$,
(*e*) *Each partition is to be counted with weight* $w_{\theta_1} = 1$.

Let θ_2 be an n-color partition such that

(*f*) $(m_l)_{x_l} \neq (m_{l-1})_{x_{l-1}}$, *for all l*,
(*g*) $m_l \equiv 1$ *or* $2 \pmod 3$. *If* $m_l \equiv 1 \pmod 3$, *then* $-3r-2 \leq m_l \leq -2$ *and if* $m_l \equiv 2 \pmod 3$, *then* $-3r+2 \leq m_l \leq -1$,
(*h*) $x_l = 1$, $1 \leq l \leq r$,
(*i*) *The weight count for each partition is* $w_{\theta_2} = (-1)^{(\text{number of parts in } \theta_2+1)}$.

Then

$$\sum_{\nu=0}^{\infty}\left(\sum_{\Theta_1\dashv\nu} w_{\theta_1}w_{\theta_2}\right)q^\nu = \sum_{\nu=0}^{\infty} h_1(\nu)q^\nu. \tag{2.1}$$

Let $g_1(\nu)$ be the weighted number of 2-colored partitions of ν with parts $\equiv \pm 2, 3 \pmod 6$, where parts $\equiv \pm 2 \pmod 6$ are distinct and occur with color 1. Each partition is counted with weight $(-1)^{(\text{number of parts } \equiv \pm 2 \pmod 6)}$. Then

$$h_1(\nu) = g_1(\nu) \quad \forall\ \nu.$$

Example 2.1. For $\nu = 9$, the relevant partition of $h_1(9) = 8 = g_1(9)$ (see Table 1).

Hence $h_1(9) = \sum_{\Theta_1\dashv\nu} w_{\theta_1}w_{\theta_2} = 8$. Table 2 presents the partitions corresponding to $g_1(\nu) = 8$ for $\nu = 9$ and their weight respectively.

Table 1. Weighted calculations for signed color partitions of $\nu = 9$.

Relevant partitions $\Theta_1 : (\theta_1, \theta_2)$	w_{θ_1}	w_{θ_2}	$w_{\theta_1} w_{\theta_2}$
$(9_9 + 2_7, -2_1)$	1	$(-1)^2$	1
$(11_{16}, -2_1)$	1	$(-1)^2$	1
$(12_{12} + 2_7, -5_1)$	1	$(-1)^2$	1
$(12_9 + 2_7, -5_1)$	1	$(-1)^2$	1
$(15_{15} + 2_7, -1_1 - 2_1 - 5_1)$	1	$(-1)^4$	1
$(15_{12} + 2_7, -1_1 - 2_1 - 5_1)$	1	$(-1)^4$	1
$(15_9 + 2_7, -1_1 - 2_1 - 5_1)$	1	$(-1)^4$	1
$(18_9 + 9_9 + 2_7, -1_1 - 2_1 - 4_1 - 5_1 - 8_1)$	1	$(-1)^6$	1

Table 2. Relevant partitions corresponding to $g_1(\nu)$.

Partition enumerated by $g_1(\nu)$	Corresponding weight
9_1	1
9_2	1
$4_1 + 3_1 + 2_1$	$(-1)^2$
$4_1 + 3_2 + 2_1$	$(-1)^2$
$3_2 + 3_2 + 3_2$	1
$3_2 + 3_2 + 3_1$	1
$3_2 + 3_1 + 3_1$	1
$3_1 + 3_1 + 3_1$	1

Proof. To generate $(n + 5)$-color partitions corresponding to

$$\frac{q^{(9r^2+9r+4)/2}}{(q^3; q^3)_r (q^3; q^6)_{r+1}},$$

first consider the factor $q^{(9r^2+9r+4)/2}$ which generates $(n + 5)$-color partitions of the type $(9r)_9 + (9r - 9)_9 + \cdots + 18_9 + 9_9 + 2_7$. It corresponds to the following two-line array where the first array depicts the parts and the second array gives the color of the corresponding part:

$$\begin{pmatrix} 9r & 9r - 9 & \cdots & 18 & 9 & 2 \\ 9 & 9 & \cdots & 9 & 9 & 7 \end{pmatrix}.$$

The factor $\frac{1}{(q^3;q^3)_r}$ generates the r non negative numbers $\equiv 0 \pmod 3$, say $u_1 \times 3, u_2 \times 6, \ldots, u_r \times 3r$, where u_k's are non negative. By introducing this factor, the k-th part is increased by $3(u_r + u_{r-1} + \cdots + u_{r-k+1})$. It modifies the above two-line array to

$$\begin{pmatrix} 9r + 3u_r + \cdots + 3u_1 & 9r - 9 + 3u_r + \cdots + 3u_2 & \cdots & 18 + 3u_r + 3u_{r-1} & 9 + 3u_r & 2 \\ 9 & 9 & \cdots & 9 & 9 & 7 \end{pmatrix}.$$

The factor $\frac{1}{(q^3;q^6)_{r+1}}$ generates the $(r+1)$ non negative multiples of $6k-3$, $1 \leq k \leq r+1$, say $v_1 \times 3$, $v_2 \times 9, \ldots, v_{r+1} \times (6r+3)$, where v_k's are non negative. This factor increases the k-th part by $6v_{r+1} + 6v_r + \cdots + 6v_{r-k+3} + 3v_{r-k+2}$ and the corresponding subscript by $3v_r - k + 2$. The modified two-line array becomes

$$\begin{pmatrix} \begin{array}{c} 9r + 3u_r + \cdots + 3u_1 \\ +6v_{r+1} + \cdots + 6v_2 + 3v_1 \end{array} & \cdots & 9 + 3u_r + 6v_{r+1} + 3v_r & 2 + 3v_{r+1} \\ 9 + 3v_1 & \cdots & 9 + 3v_r & 7 + 3v_{r+1} \end{pmatrix}.$$

For $2 \leq k \leq r+1$, the k-th and $(k-1)$-th parts of the $(n+5)$-color partition are

$$m_k = 9k + 3u_r + \cdots + 3u_k + 6v_{r+1} + \cdots + 6v_{k-1} + 3v_k, \tag{2.2}$$
$$x_k = 9 + 3v_k, \tag{2.3}$$
$$m_{k-1} = 9(k-1) + 3u_r + \cdots + 3u_{k-1} + 6v_{r+1} + \cdots + 6v_{k-2} + 3v_{k-1}, \tag{2.4}$$
$$x_{k-1} = 9 + 3v_{k-1} \tag{2.5}$$

and

$$m_1 = 2 + 3v_{r+1}, \tag{2.6}$$
$$x_1 = 7 + 3v_{r+1}. \tag{2.7}$$

Thus for $1 \leq k \leq r+1$,

$$m_k - m_{k-1} - x_k - x_{k-1} = -9 + 3u_k.$$

So Theorem 2.1(d) holds.

Clearly (2.7) implies Theorem 2.1(b) and (2.3) implies Theorem 2.1(c) and from (2.2) and (2.6), we get Theorem 2.1(a).

The term $(1 + q^{-3\nu-2}) \prod_{j=1}^{\nu} (1 - q^{-3j-2})(1 - q^{-3j+1})$ clearly generates n-color partitions satisfying Theorems 2.1(f)–(i). □

3. Combinatorial interpretations of (1.3)–(1.5)

In this section, we interpret three identities (1.3)–(1.5) using similar combinatorial arguments as done in Section 2. To study the identity (1.3), first consider its summation side

$$\sum_{\nu=0}^{\infty} h_2(\nu) q^{\nu} = \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{(q;q^2)_{2\nu} q^{2\nu(\nu+2)}}{(-q^2;q^2)_{2\nu+1}(q^4;q^4)_{\nu}}$$
$$= \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{q^{6\nu^2+4\nu}}{(-q^2;q^4)_{\nu+1}(q^8;q^8)_{\nu}} \prod_{j=1}^{2\nu} (1 - q^{-(2j-1)}).$$

As a matter of fact we consider that the factor $(-1)^{\nu} \frac{q^{6\nu^2+4\nu}}{(-q^2;q^4)_{\nu+1}(q^8;q^8)_{\nu}}$ generates the positive parts and $\prod_{j=1}^{2\nu} (1 - q^{-(2j-1)})$ generate the negative parts of the signed color partitions. Let $g_2(\nu)$ be the generating function for the product side of (1.3). Then

Table 3. Weight calculation for signed color partitions of $\nu = 16$.

Relevant partitions $\Theta_2 : (\theta_1, \theta_2)$	w_{θ_1}	w_{θ_2}	$w_{\theta_1} w_{\theta_2}$
$(14_{18}, 0)$	-1	1	-1
$(14_{10} + 0_4, 0)$	-1	1	-1
$(18_{14} + 0_4, -3_1 - 1_1)$	-1	1	-1
$(16_8 + 2_6, -3_1 - 1_1)$	-1	1	-1
$(18_6 + 0_4, -3_1 - 1_1)$	-1	1	-1

$$\sum_{\nu=0}^{\infty} g_2(\nu) q^{\nu} = \frac{(q^2; q^4)_\infty [q^8, -q^7, -q^9; q^8]_\infty}{(q^4; q^4)_\infty}$$
$$= \frac{(-q^7; q^8)_\infty (-q^9; q^8)_\infty}{(-q^2; q^8)_\infty (-q^6; q^8)_\infty}.$$

Theorem 3.1. *Let Θ_2 denote the signed color partition pairs (θ_1, θ_2) and θ_1 be an $(n+4)$-color partition such that*

(*a*) $m_k, x_k \equiv 0 \pmod 2$, $\forall\, k$,
(*b*) $m_1 = x_1 - 4$,
(*c*) $(((m_k)_{x_k} - (m_{k-1})_{x_{k-1}})) \equiv 0 \pmod 8$, $\forall\, k$,
(*d*) *The weight count for each partition is* $w_{\theta_1} = \sum_{i=1}^{r+1} x_i - 2r - 2$.

Let θ_2 be an n-color partition such that

(*e*) $(m_l)_{x_l} \neq (m_{l-1})_{x_{l-1}}$, $\forall\, l$,
(*f*) $m_l \equiv 1 \pmod 2$ *and* $x_l = 1$, $\forall\, l$,
(*g*) $m_l < 2r$,
(*h*) *The weight count for each partition is* $w_{\theta_2} = (-1)^{(\textit{number of parts in } \theta_2)}$.

Then

$$\sum_{\nu=0}^{\infty} \left(\sum_{\Theta_2 \dashv \nu} w_{\theta_1} w_{\theta_2} \right) q^{\nu} = \sum_{\nu=0}^{\infty} h_2(\nu). \tag{3.1}$$

Let $g_2(\nu)$ be the weighted number of partitions of ν into parts ≥ 2 and $\equiv \pm 2 \pmod 8$ and distinct parts $\equiv \pm 1 \pmod 8$. Each partition is counted with the weight $(-1)^{(\textit{number of parts} \equiv \pm 2 \pmod 8)}$. Then

$$h_2(\nu) = g_2(\nu) \quad \forall\, \nu.$$

Example 3.1. For $\nu = 16$, the weight calculation for signed color partitions is shown in Table 3. Hence $h_2(16) = \sum_{\Theta_2 \dashv \nu} w_{\theta_1} w_{\theta_2} = -5$.

To study the identity (1.4), consider

$$\sum_{\nu=0}^{\infty} h_3(\nu)q^{\nu} = \sum_{\nu=0}^{\infty} \frac{(-q;q)_{\nu}(-1;q^3)_{\nu} q^{\binom{\nu+1}{2}}}{(q;q)_{2\nu}(-1;q)_{\nu}}$$
$$= \sum_{\nu=0}^{\infty} \frac{(-q;q)_{\nu} q^{\nu(3\nu-1)/2}}{(q;q)_{2\nu}} \prod_{j=1}^{\nu-1}(1-q^{-j}+q^{-2j}).$$

In the last expression, the factor $\frac{(-q;q)_{\nu} q^{\nu(3\nu-1)/2}}{(q;q)_{2\nu}}$ generates the positive parts and $\prod_{j=1}^{\nu-1}(1-q^{-j}+q^{-2j})$ generates the negative parts of the signed color partitions. Let $g_3(\nu)$ be the generating function for the product side of (1.4). Then

$$\sum_{\nu=0}^{\infty} g_3(\nu)q^{\nu} = \frac{(-q;q)_{\infty}}{(q;q)_{\infty}}[q^6,q,q^5;q^6]_{\infty}[q^8,q^4;q^{12}]_{\infty}$$
$$= \frac{(-q;q)_{\infty}}{(q^2;q^{12})_{\infty}(q^3;q^{12})_{\infty}(q^9;q^{12})_{\infty}(q^{10};q^{12})_{\infty}}$$
$$= \sum_{\nu=0}^{\infty} a_3(\nu)q^{\nu} \sum_{\nu=0}^{\infty} b_3(\nu)q^{\nu}$$
$$= \sum_{\nu=0}^{\infty}\sum_{k=0}^{\nu} a_3(\nu-k)b_3(k)q^{\nu},$$

where

$$\sum_{\nu=0}^{\infty} a_3(\nu)q^{\nu} = (-q;q)_{\infty}$$

and

$$\sum_{\nu=0}^{\infty} b_3(\nu)q^{\nu} = \frac{1}{(q^2;q^{12})_{\infty}(q^3;q^{12})_{\infty}(q^9;q^{12})_{\infty}(q^{10};q^{12})_{\infty}}.$$

Theorem 3.2. *Let Θ_3 denote the signed color partition pairs (θ_1,θ_2) and θ_1 be an n-color partition such that*

(*a*) $(((m_k)_{x_k}-(m_{k-1})_{x_{k-1}})) \geq 1$,
(*b*) *Each partition is to be counted with weight $w_{\theta_1}=1$.*

Let θ_2 be the n-color partition such that

(*c*) $(m_l)_{x_l} \neq (m_{l-1})_{x_{l-1}}$, *for all l*,
(*d*) $x_l = 1, 2$, *for all l*,
(*e*) $m_l < r$, *for all l*,
(*f*) *the weight count for each partition is $w_{\theta_2} = (-1)^{(\text{number of distinct parts of } \theta_2)}$.*

Then

$$\sum_{\nu=0}^{\infty}\left(\sum_{\Theta_3 \dashv \nu} w_{\theta_1}w_{\theta_2}\right)q^{\nu} = \sum_{\nu=0}^{\infty} h_3(\nu). \tag{3.2}$$

Let $a_3(\nu)$ denote the number of distinct partitions of ν and $b_3(\nu)$ denote the number of partitions of ν such that the parts are $\equiv \pm 2, \pm 3 \pmod{12}$. Then

$$g_3(\nu) = \sum_{k=0}^{\nu} a_3(\nu - k) b_3(k) = h_3(\nu) \quad \forall\, \nu.$$

Lastly, we consider the identity (1.4), where

$$\begin{aligned}\sum_{\nu=0}^{\infty} h_4(\nu) q^{\nu} &= \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{(q;q)_{4\nu} q^{2\nu^2}}{(q^4;q^4)_{\nu}(q^4;q^4)_{2\nu}} \\ &= \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{q^{6\nu^2}}{(-q^2;q^4)_{\nu}(q^8;q^8)_{\nu}} \\ &\qquad \prod_{j=1}^{\nu} (1 - q^{-(4j-1)})(1 - q^{-(4j-3)}).\end{aligned}$$

In the above expression the factor $(-1)^{\nu} \frac{q^{6\nu^2}}{(-q^2;q^4)_{\nu}(q^8;q^8)_{\nu}}$ generates the positive parts and $\prod_{j=1}^{2\nu}(1-q^{-(4j-1)})(1-q^{-(4j-3)})$ generate the negative parts of the signed color partition. If the corresponding product side of (1.4) is generated by $g_4(\nu)$, then

$$\begin{aligned}\sum_{\nu=0}^{\infty} g_4(\nu) q^{\nu} &= \frac{(q^2;q^4)_{\infty}}{(q^4;q^4)_{\infty}} [q^8, -q^7, -q^9; q^8]_{\infty} \\ &= \frac{(-q^3;q^8)_{\infty}(-q^5;q^8)_{\infty}}{(-q^2;q^8)_{\infty}(-q^6;q^8)_{\infty}}.\end{aligned}$$

Theorem 3.3. *Let Θ_4 denote the signed color partition pairs (θ_1, θ_2) and θ_1 be an n-color partition such that*

(*a*) $m_k \equiv x_k \pmod 2$ *for all* k,
(*b*) $x_k \geq 6$ *for all* k,
(*c*) $(((m_k)_{x_k} - (m_{k-1})_{x_{k-1}})) \equiv 0 \pmod 8$, *for all* k,
(*d*) *the weight count for each partition is* $w_{\theta_1} = \sum_{i=1}^{r} x_i - 2r$.

Let θ_2 be an n-color partition such that

(*e*) $m_{l_{x_l}} \neq m_{l-1_{x_{l-1}}}$ *for all* l,
(*f*) $m_l \equiv \pm 1 \pmod 4$ *and* $x_l = 1$ *for all* l,
(*g*) $m_l < 2r$ *for all* l,
(*h*) *the weight count for each partition is* $w_{\theta_2} = (-1)^{(\text{number of parts of } \theta_2)}$.

Then

$$\sum_{\nu=0}^{\infty} \left(\sum_{\Theta_4 \dashv \nu} w_{\theta_1} w_{\theta_2} \right) q^{\nu} = \sum_{\nu=0}^{\infty} h_4(\nu). \tag{3.3}$$

Let $g_4(\nu)$ be the weighted number of partitions of ν into parts $\equiv \pm 2 \pmod 8$ and distinct parts $\equiv \pm 3 \pmod 8$. Each partition is counted with the weight $(-1)^{(\text{number of parts } \pm 2 \pmod 8)}$.

Then

$$h_4(\nu) = g_4(\nu) \quad \forall\, \nu.$$

Proof. The proofs of Theorem 3.1–Theorem 3.3 can be proved along the lines of Theorem 2.1. □

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References

[1] Agarwal A K and Andrews G E, Rogers–Ramanujan identities for partitions with "*N* copies of *N*", *J. Combin. Theory, Series A* **45(1)** (1987) 40–49
[2] Andrews G E, Euler's "De partitio numerorum", *Bull. Amer. Math. Soc.* **44(4)** (2007) 561–574
[3] Baxter R J, Rogers–Ramanujan identities in the hard hexagon model, *J. Stat. Phys.* **26(3)** (1981) 427–452
[4] Chu W and Zhang W, Bilateral bailey lemma and Rogers–Ramanujan identities. *Adv. Appl. Math.* **42(3)** (2009) 358–391
[5] Euler L, Introductio in analysin infinitorum, volume 2, (1748) (Marcum Michaelem Bousquet)
[6] Gupta V and Rana M, On some combinatorics of Rogers–Ramanujan type identities using signed color partitions, Current Trends in Mathematical Analysis and Its Interdisciplinary Applications (2019) (Springer) pp. 99–114
[7] Keith W J, A bijective toolkit for signed partitions, *Ann. Combin.* **15(1)** (2011) 95
[8] McLaughlin J and Sills A V, Combinatorics of Ramanujan–Slater type identities, *Integers* **9(14)** (2009)
[9] Sills A V, On the ordinary and signed Göllnitz–Gordon partitions, arXiv preprint: arXiv:1812.07550 (2018)
[10] Slater L J, Further identities of the Rogers–Ramanujan type, *Proc. London Math. Soc.* **2(1)** (1952) 147–167

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