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Rainbow 2-connectivity of edge-comb product of a cycle and a Hamiltonian graph

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Abstract. An edge-colored graph G is rainbow k-connected, if for every two vertices of G, there are k internally disjoint rainbow paths, i.e., if no two edges of each path are colored the same. The minimum number of colors needed for which there exists a rainbow k-connected coloring of G, $rc_k(G)$, is the rainbow k-connection number of G. Let G and H be two connected graphs, where O is an orientation of G. Let $\vec{e}$ be an oriented edge of H. The edge-comb product of G (under the orientation O) and H on $\vec{e}$, $G^O \rhd_{\vec{e}} H$, is a graph obtained by taking one copy of G and $|E(G)|$ copies of H and identifying the i-th copy of H at the edge $\vec{e}$ to the i-th edge of G, where the two edges have the same orientation. In this paper, we provide sharp lower and upper bounds for rainbow 2-connection numbers of edge-comb product of a cycle and a Hamiltonian graph. We also determine the rainbow 2-connection numbers of edge-comb product of a cycle with some graphs, i.e. complete graph, fan graph, cycle graph, and wheel graph.

Keywords. Cycle; edge-comb product; Hamiltonian graph; rainbow 2-connectivity; rainbow path.

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1. Introduction

All graphs considered in this paper are simple, finite and undirected. Let G be a nontrivial connected graph, $l \in \mathbb{N}$, and $c : E(G) \to \{1, 2, \ldots, l\}$ be an edge-coloring of G, where the adjacent edges may be colored the same. An edge-colored path in G is *rainbow* if the colors of its edges are different. We say that an edge-coloring c on G is *rainbow-connected* if for any two distinct vertices in G, there exists a rainbow path (with respect to c) which connect them. In this case, the coloring c is called a *rainbow edge-coloring* or a *rainbow coloring* of G. As introduced in [3], the *rainbow connection number* of G, denoted by $rc(G)$, is the minimum number of colors needed for a rainbow coloring on G that results in a rainbow-connected graph.

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Chakraborty *et al.* [2] showed that determining the rainbow connectivity of a graph is NP-hard. However, many authors investigated bounds, algorithms and computational complexity of the rainbow connections of some graphs (see [13,14]). Some of them investigated that rainbow connection numbers of graphs resulted from graph operations, such as Cartesian product [1,12], strong product [1,8], lexicographic product [1,8,12], direct product [8], corona product [5], power graph [16] and amalgamation [6].

In 2009, Chartrand *et al.* [4] introduced the concept of *rainbow k-connectivity* as follows: An edge-coloring of a graph G is called *rainbow k-connected* if, for any two different vertices u and v of G, there are k internally disjoint u–v rainbow paths which connect them (i.e., the edges of each path have distinct colors). The minimum number of colors for which there exists a rainbow k-connected coloring of G is the *rainbow k-connection number* of G, denoted by $rc_k(G)$.

By Whitney's theorem [20], a graph G is k-connected if and only if any two distinct vertices u and v of a graph G have k internally disjoint u–v paths which connect them. Hence, the function $rc_k(G)$ will only be defined for k-connected graph.

The following definition of k-distance and k-diameter of a graph G are referred from Hsu and Łuczak [10]. Let G be a k-connected graph and u, v be any two distinct vertices of G. The *k-distance*, $d_k(u, v)$ between vertices u and v is defined as the minimum integer ℓ for which there are at least k internally disjoint paths of length at most ℓ between u and v in G. The *k-diameter* of G, denoted by $\text{diam}_k(G)$, is defined as the maximum k-distance, $d_k(u, v)$ over all pairs u, v of vertices of G.

The concept of rainbow k-connectivity has many applications in transferring classified information in communication networks security [4]. In [11], Li and Sun stated that it is difficult to determine the exact value or a nice bound of the rainbow k-connectivity for a general graph. While the value of rainbow k-connection numbers of some graphs are known, such as those complete graphs, regular complete bipartite graphs [4], complete multipartite graphs [11] and non-commuting graph [19]. Some other researchers gave upper bounds for the rainbow k-connection numbers for some graph classes, i.e., dense graphs [7], random graphs [7,9] and Cayley graphs [15]. In [18], Susanti *et al.* gave a lower bound for the rainbow k-connection number of a k-connected graph G, namely, $rc_k(G) \geq \text{diam}_k(G)$. Meanwhile, in 2015, Susanti *et al.* [17] investigated upper bounds for rainbow 2-connectivity of Cartesian product of a path and a cycle. They continued their investigation for rainbow 2-connectivity of Cartesian product of a 2-connected graph and a path [18].

In this paper, we define the edge-comb product of two graphs on an arc and find sharp lower and upper bounds for rainbow 2-connection number of *edge-comb product* of a cycle C_m and a Hamiltonian graph. We also determine the rainbow 2-connection number of *edge-comb product* of a cycle and some graphs, i.e. complete graph, fan graph, cycle graph and wheel graph.

2. Main results

The edge-comb product of two graphs is defined as follows.

DEFINITION 1

An *orientation* of an undirected graph G is an assignment of exactly one direction to each of the edges of G. Let G and H be two connected graphs, where O is an orientation of G. Let $\vec{e}$ be an oriented edge of H. The *edge-comb product* of G (under the orientation O)

and H on $\vec{e}$, denoted by $G^o \rhd_{\vec{e}} H$, is a graph obtained by taking one copy of G and $|E(G)|$ copies of H and identifying the i-th copy of H at the edge $\vec{e}$ to the i-th edge of G, where the two edges have the same orientation.

The following theorem provides lower and upper bounds for rainbow 2-connection number of edge-comb product of a cycle and a Hamiltonian graph. For simplifying, we define $[a, b] = \{x \in \mathbb{Z} | a \leq x \leq b\}$.

Theorem 2. *Let m and n be two positive integer at least 3. Let H be a Hamiltonian graph of order n. Let C be a Hamiltonian cycle in H and $\vec{e}$ be a directed edge in C. Then the rainbow 2-connection number of edge-comb product of a cycle C_m with a cyclic orientation O and H is*

$$\mathrm{diam}_2(C_m^o \rhd_{\vec{e}} H) \leq rc_2(C_m^o \rhd_{\vec{e}} H) \leq m(n-3) + 2.$$

Proof. It is obvious that $\mathrm{diam}_2(C_m^o \rhd_{\vec{e}} H) \leq rc_2(C_m^o \rhd_{\vec{e}} H)$. Let $C_m = (v_1, v_2, \ldots, v_m, v_1)$. Let H be a Hamiltonian graph with the notations of the vertices according to $C = (w_1, w_2, \ldots, w_n, w_1)$, where $e = (w_1, w_2)$ and the edge set $E(H)$. Let $H^i = (V(H^i), E(H^i))$ be a copy of graph H for $i \in [1, m]$ and C^i be a copy of the Hamiltonian cycle C in H^i. Let $V({C_m}^o \rhd_{\vec{e}} H) = \{v_i | i \in [1, m]\} \bigcup \{v_k^i | i \in [1, m], k \in [3, n]\}$ and $E({C_m}^o \rhd_{\vec{e}} H) = \{v_i v_{i+1} | i \in [1, m]$, where $v_m v_{m+1} = v_m v_1\} \bigcup \bigcup_{i=1}^m E(H^i)$.

Next, we shall consider a proof of the upper bound. We first color the cycle C_m by using m different colors. Then, for $i \in [1, m]$, we color edges $v_i v_3^i$ with one additional color and another additional color for edges $v_{i+1} v_n^i$. After that, we use the same color used by $v_i v_{i+1}$ for edges $v_3^i v_4^i$ and the remaining $m(n-4)$ edges, i.e., $v_k^i v_{k+1}^i$ with $i \in [1, m]$ and $k \in [4, n-1]$ contained in all C^i are colored with $m(n-4)$ additional colors. Finally, we color the remaining edges in all H^i with one of the colors that we already used before.

We shall show that this coloring is a rainbow 2-connected $(m(n-3)+2)$-coloring of ${C_m}^o \rhd_{\vec{e}} H$. Consider any two vertices $x, y \in V({C_m}^o \rhd_{\vec{e}} H)$. We divided the proof into four cases:

Case 1. For $x = v_i$ and $y = v_j$, $x, y \in V(C_m)$ with $i \neq j$ and $i, j \in [1, m]$, there exist two internally disjoint $x - y$ rainbow paths, i.e. $v_i, v_{i+1}, \ldots, v_j$ and $v_i, v_{i-1}, \ldots, v_j$.
Case 2. For $x, y \in V(H^i)$, $i \in [1, m]$ where $x = v_k^i$ and $y = v_l^i$, $k \neq l$, $k, l \in [3, n]$, there exist two internally disjoint $x - y$ rainbow paths which connect them, i.e. $v_k^i, v_{k+1}^i, \ldots, v_l^i$ and $v_k^i, v_{k-1}^i, \ldots, v_3^i, v_i, v_{i-1}, \ldots, v_{i+1}, v_n^i, \ldots, v_l^i$.
Case 3. For $x \in V(C_m)$ and $y \in V(H^i)$, we divide into three subcases:

(i) For $x = v_i \in V(C_m)$, $i \in [1, m]$ and $y = v_k^i \in V(H^i)$, $i \in [1, m]$, $k \in [3, n]$, there exist two internally disjoint $x - y$ rainbow paths, i.e. $v_i, v_3^i, \ldots, v_k^i$ and $v_i, v_{i-1}, \ldots, v_{i+1}, v_n^i, \ldots, v_k^i$.
(ii) For $x = v_{i+1} \in V(C_m)$, $i \in [1, m]$ and $y = v_k^i \in V(H^i)$, $i \in [1, m]$, $k \in [1, n-2]$, there exist two internally disjoint $x - y$ rainbow paths, i.e. $v_{i+1}, v_n^i, \ldots, v_k^i$ and $v_{i+1}, v_{i+2}, \ldots, v_i, v_3^i, \ldots, v_k^i$.
(iii) For $x = v_s \in V(C_m)$, $i \neq s$, $s \in [1, m]$ and $y = v_k^i \in V(H^i)$, $i \in [1, m]$, $k \in [3, n]$, there exist two internally disjoint $x - y$ rainbow paths, i.e. $v_s, v_{s+1}, \ldots, v_i, v_3^i, \ldots, v_k^i$ and $v_s, v_{s-1}, \ldots, v_{i+1}, v_n^i, \ldots, v_k^i$.

Case 4. For $x = v_k^i \in V(H^i)$ and $y = v_l^s \in V(H^s), i \neq s, i, s \in [1, m], k, l \in [3, n]$, there exist two internally disjoint $x - y$ rainbow paths which connect them, i.e. $v_k^i, v_{k+1}^i, \ldots, v_n^i, v_{i+1}, v_{i+2}, \ldots, v_s, v_1^s, \ldots, v_l^s$ and $v_k^i, v_{k-1}^i, \ldots, v_3^i, v_i, v_{i-1}, \ldots, v_{s+1}, v_{n-2}^s, \ldots, v_l^s$.

Hence $rc_2(C_m^o \rhd_{\vec{e}} H) \leq m(n-3)+2$. Thus, we complete the proof. □

The following theorem gives an example of an edge-comb product of a cycle and a Hamiltonian graph whose rainbow 2-connection number achieves the lower bound of Theorem 2.

Before we proceed to the next theorem, we shall use the following definition. A *fan* graph, F_n, of order $n+1$ is the graph obtained by joining all the vertices of a path, P_n to a further vertex called the *centre*. We call the edges of P_n as *rims* and the edges connecting the centre to the vertices of P_n as *spokes*. The edges connecting the centre to the endpoints of P_n are called *outer spokes*.

Theorem 3. *Let m and n be two positive integers at least* 3. *Let e be an outer spoke of F_n. The rainbow* 2*-connection number of the edge-comb product of a cycle C_m with a cyclic orientation O and a fan F_n is*

$$rc_2(C_m^o \rhd_{\vec{e}} F_n) = m + n - 2 = \mathrm{diam}_2(C_m^o \rhd_{\vec{e}} F_n).$$

Proof. Since $\mathrm{diam}_2(C_m^o \rhd_{\vec{e}} F_n) = m + n - 2$, it is clear that $\mathrm{diam}_2(C_m^o \rhd_{\vec{e}} F_n) \leq rc_2(C_m^o \rhd_{\vec{e}} F_n)$. Let $V(C_m^o \rhd_{\vec{e}} F_n) = \{v_i | 1 \leq i \leq m\} \bigcup \{v_k^i | i \in [1, m], k \in [1, n-1]\}$ and $E(C_m^o \rhd_{\vec{e}} F_n) = \{v_i v_{i+1} | i \in [1, m]\} \bigcup \{v_i v_k^i | i \in [1, m] \text{ and } k \in [1, n-1]\} \bigcup \{v_k^i v_{k+1}^i | i \in [1, m], k \in [1, n-1]\}$, where $v_m v_{m+1} = v_m v_1$ and $v_{n-1}^i v_n^i = v_{n-1}^i v_{i+1}$.

Next, we shall consider the proof of the upper bound. First, we color the cycle C_m by using a rainbow $rc(C_m)$-coloring of C_m for $m \in [3, 4]$ or by using m different colors otherwise. After that, for $i \in [1, m]$ and $m \in [3, 4]$, we color the edges $v_i v_k^i, k \in [1, n-1]$ with one additional color and another additional color for edges $v_{i+1} v_{n-1}^i$. For $m \geq 5$, we use the same color used by $v_i v_{i+1}$ for edges $v_i v_k^i, k \in [1, n-1]$ and one additional color for all $v_{i+1} v_{n-1}^i$. Next, for $m \in [3, 4]$, we color the copies of F_n with $n-2$ additional colors for all $v_k^i v_{k+1}^i$ with $k \in [1, n-2]$. For $m \geq 5$, we use $n-3$ additional colors to color the copies of F_n, i.e., the same color used by $v_i v_{i+1}$ for $v_1^i v_2^i$ and $n-3$ additional colors for all $v_k^i v_{k+1}^i$ with $k \in [2, n-2]$.

We shall show that the above coloring is a rainbow 2-connected $(m+n-2)$-coloring of $C_m^o \rhd_{\vec{e}} F_n$. Consider any two distinct vertices $x, y \in V(C_m^o \rhd_{\vec{e}} F_n)$. We divided the proof into four cases:

Case 1. For $x = v_i$ and $y = v_j, x, y \in V(C_m)$ with $i \neq j$ and $i, j \in [1, m]$, if $v_i \sim v_j$, then there exist two internally disjoint $x - y$ rainbow paths, i.e. v_i, v_j and v_i, v_{n-1}^i, v_j. For $v_i \nsim v_j$, there exist two internally disjoint $x - y$ rainbow paths, i.e. $v_i, v_{i+1}, \ldots, v_j$ and $v_i, v_{i-1}, \ldots, v_j$.

Case 2. For $x = v_k^i$ and $y = v_l^i$, $x, y \in V(F_n^i), k \neq l, k, l \in [1, n-1]$, there exist two internally disjoint $x - y$ rainbow paths which connect them, i.e. $v_k^i, v_{k+1}^i, \ldots, v_l^i$ and $v_k^i, v_i, v_{i+1}, v_{n-1}^i, v_{n-2}^i, \ldots, v_l^i$ for $m \in [3, 4]$ or $v_k^i, v_i, v_{i-1}, \ldots, v_{i+1}, v_{n-1}^i, v_{n-2}^i, \ldots, v_l^i$ for $m \geq 5$.

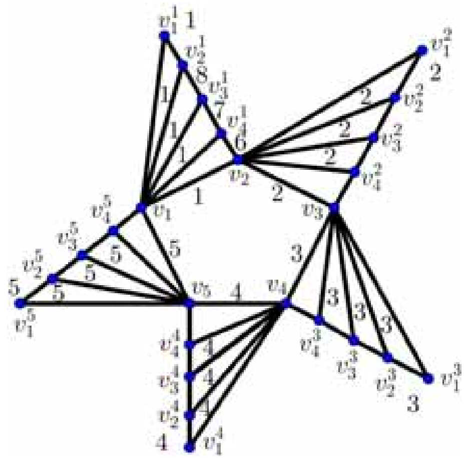

Figure 1. A rainbow 2-connected coloring on $C_m^o \triangleright_{\vec{e}} F_n$ for $m = n = 5$.

Case 3. For $x \in V(C_m)$ and $y \in V(F_n^i)$, we divide into three subcases:

(i) For $x = v_i \in V(C_m), i \in [1, m]$ and $y = v_k^i \in V(F_n^i), i \in [1, m], k \in [1, n-1]$, there exist two internally disjoint $x - y$ rainbow paths which connect x and y, i.e. v_i, v_k^i and $v_i, v_{i+1}, v_{n-1}^i, \ldots, v_k^i$ for $m \in [3, 4]$ or $v_i, v_{i-1}, \ldots, v_{i+1}, v_{n-1}^i, \ldots, v_k^i$ for $m \geq 5$.

(ii) For $x = v_{i+1} \in V(C_m), i \in [1, m]$ and $y = v_k^i \in V(F_n^i), i \in [1, m], k \in [1, n-1]$, there exist two internally disjoint $x - y$ rainbow paths which connect them, i.e. $v_{i+1}, v_{n-1}^i, \ldots, v_k^i$ and v_{i+1}, v_i, v_k^i for $m \in [3, 4]$ or $v_{i+1}, v_{i+2}, \ldots, v_i, v_k^i$ for $m \geq 5$.

(iii) For $x = v_s \in V(C_m), i \neq s, s \in [1, m]$ and $y = v_k^i \in V(F_n^i), i \in [1, m], k \in [1, n-1]$, there exist two internally disjoint $x - y$ rainbow paths which connect x and y, i.e. $v_s, v_{s+1}, \ldots, v_i, v_k^i$ and $v_s, v_{s-1}, \ldots, v_{i+1}, v_{n-1}^i, \ldots, v_k^i$.

Case 4. For $x = v_k^i, x \in V(F_n^i)$ and $y = v_l^s, y \in V(F_n^s), i, s \in [1, m], k, l \in [1, n-1]$, there exist two internally disjoint $x - y$ rainbow paths which connect them, i.e., $v_k^i, v_{k+1}^i, \ldots, v_{n-1}^i, v_{i+1}, v_{i+2}, \ldots, v_s, v_l^s$ and $v_k^i, v_i, v_{i-1}, \ldots, v_{s+1}, v_{n-1}^s, \ldots, v_l^s$.

So, this completes the proof. □

As shown in Figure 1, we present a rainbow 2-connected coloring on $C_m^o \triangleright_{\vec{e}} F_n$ for $m = n = 5$.

Besides the fan graph, there is another graph which satisfies the lower bound of rainbow 2-connectivity of edge-comb product of Theorem 2, i.e., the complete graph, K_n, on n vertices.

Theorem 4. *Let m and n be two positive integers at least* 3. *The rainbow* 2-*connection number of edge-comb product of* C_m *with a cyclic orientation O and* K_n *satisfies*

$$rc_2({C_m}^o \triangleright_{\vec{e}} K_n) = \begin{cases} m = \mathrm{diam}_2({C_m}^o \triangleright_{\vec{e}} K_n), & \text{for } m \in [3,4] \text{ and } n \geq 3, \\ & \text{or } m \geq 5 \text{ and } n = 3, \\ & \text{or } m, n \geq 5; \\ m + 1 = \mathrm{diam}_2({C_m}^o \triangleright_{\vec{e}} K_n) + 1, & \text{for } m \geq 5 \text{ and } n = 4. \end{cases}$$

Proof. Let $V(C_m^o \rhd_{\vec{e}} K_n) = \{v_i | i \in [1, m]\} \bigcup \{v_k^i | i \in [1, m] \text{ and } k \in [1, n-2]\}$ and $E(C_m^o \rhd_{\vec{e}} K_n) = \{v_i v_{i+1} | i \in [1, m]\} \bigcup \{v_i v_k^i | i \in [1, m] \text{ and } k \in [1, n-2]\} \bigcup \{v_{i+1} v_{i,i+1}^k | i \in [1, m] \text{ and } k \in [1, n-2]\} \bigcup \{v_k^i v_l^i | i \in [1, m], k \neq l, k, l \in [1, n-2]\}$, where $v_m v_{m+1} = v_m v_1$.

We divide the proof of the lower bound into two cases:

Case 1. $m \in [3, 4]$ and $n \geq 3$ or $m \geq 5$ and $n = 3$ or $m, n \geq 5$. It is obvious that $\text{diam}_2(C_m^o \rhd_{\vec{e}} K_n) \leq rc_2(C_m^o \rhd_{\vec{e}} K_n)$ for $m \in [3, 4]$ and $n \geq 3$ or $m \geq 5$ and $n = 3$ or $m, n \geq 5$.

Case 2. $m \geq 5$ and $n = 4$. Here, we want to show that $rc_2(C_m^o \rhd_{\vec{e}} K_n) \geq \text{diam}_2(C_m^o \rhd_{\vec{e}} K_n) + 1$ for $m \geq 5$ and $n = 4$. Without loss of generality, assume to the contrary that $rc_2(C_m^o \rhd_{\vec{e}} K_n) \leq \text{diam}_2(C_m^o \rhd_{\vec{e}} K_n)$. Let c' be a rainbow 2-connected coloring of $C_m^o \rhd_{\vec{e}} K_n$ using $\text{diam}_2(C_m^o \rhd_{\vec{e}} K_n)$ colors.

Consider two vertices v_2^i and v_1^{i+1}. Then two internally disjoint paths which connect them are $v_2^i, v_{i+1}, v_1^{i+1}$ and $v_2^i, v_i, v_{i-1}, \ldots, v_m, \ldots, v_{i+3}, v_{i+2}, v_1^{i+1}$ of length $\text{diam}_2(C_m^o \rhd_{\vec{e}} K_n)$. Clearly, one of the paths $v_2^i - v_1^{i+1}$ must contain $m - 2$ edges of C_m. By considering all such vertices v_2^i and v_1^{i+1}, we obtain that all the edges of C_m must be colored differently. Without loss of generality, we color $c'(v_i v_{i+1}) = i, i \in [1, m]$, where $v_m v_{m+1} = v_m v_1$.

Consider two vertices v_2^i and v_1^{i+1}. One of the paths, i.e. $v_2^i, v_i, v_{i-1}, \ldots, v_m, \ldots, v_{i+3}, v_{i+2}, v_1^{i+1}$ has length $\text{diam}_2(C_m^o \rhd_{\vec{e}} K_n)$. Because we have already colored a cycle C_m, without loss of generality, let $c'(v_i v_2^i) = \alpha$ and $c'(v_{i+2} v_1^{i+1}) = \beta$, where $\alpha \neq \beta$ and $\alpha, \beta \in [i, i+1]$.

Consider two vertices v_2^i and v_2^{i+1}. Then two internally disjoint paths which connect them are $v_2^i, v_{i+1}, v_2^{i+1}$ and $v_2^i, v_i, v_{i-1}, \ldots, v_m, \ldots, v_{i+3}, v_{i+2}, v_2^{i+1}$ of length $\text{diam}_2(C_m^o \rhd_{\vec{e}} K_n)$, and it forces us to color $c'(v_{i+2} v_2^{i+1}) = \beta$. Consider two vertices v_1^i and v_1^{i+1}. It forces us to color $c'(v_i v_1^{i+1}) = \alpha$. Consider two vertices v_2^{i+1} and v_1^{i+2}. Then two internally disjoint paths which connect them are $v_2^{i+1}, v_{i+2}, v_1^{i+2}$ and $v_2^{i+1}, v_{i+1}, v_i, v_{i-1}, \ldots, v_m, \ldots, v_{i+3}, v_1^{i+2}$ of length $\text{diam}_2(C_m^o \rhd_{\vec{e}} K_n)$. Then it forces us to color $c'(v_{i+1} v_2^{i+1}) = i+2$ and $c'(v_{i+3} v_1^{i+2}) = \beta$. We can not color $c'(v_{i+1} v_2^{i+1}) = \beta$ and $c'(v_{i+3} v_1^{i+2}) = i+2$, because if we consider two vertices v_1^{i+1} and v_1^{i+2}, it forces us to color $c'(v_{i+1} v_1^{i+1}) = \beta$ and which leads to the non-existence of two rainbow $v_1^{i+1} - v_2^{i+1}$ paths. So, consider two vertices v_2^{i+1} and v_2^{i+2}. It forces us to color $c'(v_{i+3} v_2^{i+2}) = \beta$. Consider two vertices v_i^{i+1} and v_2^{i+2}. It forces us to color $c'(v_{i+1} v_1^{i+1}) = i+2$. But, there are no two internally disjoint rainbow paths which connect v_1^{i+1} and v_2^{i+1}.

Hence, we get a contradiction. Thus, $rc_2(C_m^o \rhd_{\vec{e}} K_n) \geq \text{diam}_2(C_m^o \rhd_{\vec{e}} K_n) + 1$ for $m \geq 5$ and $n = 4$.

Next, we shall consider the proof of the upper bound. First, we color a cycle C_m by using a rainbow $rc(C_m)$-coloring of C_m for $m \in [3, 4]$ or by using m different colors otherwise. Then, for $i \in [1, m], m \in [3, 4]$ and $n \geq 3$, we color edges $v_i v_k^i$ with one additional color and another additional color for edges $v_{i+1} v_k^i$ with $k \in [1, n-2]$. For $m \geq 5$ and $n = 3$ or $m, n \geq 5$, we use the same color used by $v_i v_{i+1}$ for edges $v_i v_k^i$ and $v_{i+1} v_k^i$ with $k \in [1, n-2]$. For $m \geq 5$ and $n = 4$, we use the same color used by $v_i v_{i+1}$ for edges $v_i v_k^i$ and one additional color for all edges $v_{i+1} v_k^i$ with $k \in [1, n-2]$. Next, for $m \in [3, 4]$, we use the same color used by $v_{i+1} v_k^i$ for edges $v_k^i v_{k+1}^i$, $i \in [1, m]$ and $k \in [1, n-3]$. For $m \geq 5$, we use the same color used by $v_{i+1} v_{i+2}$ for edges $v_k^i v_{k+1}^i$, odd i, and $k \in [1, n-3]$

or the same color used by $v_{i+2}v_{i+3}$ for edges $v_k^i v_{k+1}^i$, even i and $k \in [1, n-3]$. Finally, we also use the same color used by edges $v_{i+1}v_k^i$ for edges $v_k^i v_l^i$, $k \neq l$, $l \neq k+1$ and $k, l \in [1, n-2]$.

We shall show that this coloring is a rainbow 2-connected m-coloring of $C_m{}^o \rhd_{\vec{e}} K_n$ for $m \in [3, 4]$ and $n \geq 3$ or $m \geq 5$ and $n = 3$ or $m, n \geq 5$ or a rainbow 2-connected $(m+1)$-coloring of $C_m{}^o \rhd_{\vec{e}} K_n$ for $m \geq 5$ and $n = 4$. Consider any two vertices $x, y \in V(C_m{}^o \rhd_{\vec{e}} K_n)$. We divided the proof into four cases:

Case 1. For $x = v_i$ and $y = v_j$, $x, y \in V(C_m)$ with $i \neq j$ and $i, j \in [1, m]$, there exist two internally disjoint $x - y$ rainbow paths, i.e. v_i, v_j and v_i, v_k^i, v_{i+1} for $v_i \sim v_j$ and $m \in [3, 4]$ or $v_i v_{i+1} \ldots v_j$ and $v_i, v_{i-1}, \ldots, v_j$ for $v_i \nsim v_j$ and $m \in [3, 4]$ or $m \geq 5$.
Case 2. For $x = v_k^i$ and $y = v_l^i$, $x, y \in V(K_n^i)$, $i \in [1, m]$, $k \neq l$, $k, l \in [1, n-2]$, we divide into two subcases:

- (i) For $m \in [3, 4]$ and $n \geq 3$, or $m \geq 5$ and $n = 3$, or $m, n \geq 5$, there exist two internally disjoint $x - y$ rainbow paths which connect them, i.e. v_k^i, v_l^i and $v_k^i, v_i, v_{i+1}, v_l^i$ for $m \in [3, 4]$ or v_k^i, v_{l+1}^i, v_l^i for $m, n \geq 5$.
- (ii) For $m \geq 5$ and $n = 4$, there exist two internally disjoint $x - y$ rainbow paths which connect x and y, i.e. v_k^i, v_l^i and $v_k^i, v_i, v_{i-1}, \ldots, v_1, v_m, \ldots, v_{i+1}, v_l^i$.

Case 3. For $x \in V(C_m)$ and $y \in V(K_n^i)$, we divide into three subcases:

- (i) For $x = v_i \in V(C_m)$, $i \in [1, m]$ and $y = v_k^i \in V(K_n^i)$, $i \in [1, m]$, $k \in [1, n-2]$, there exist two internally disjoint $x - y$ rainbow paths which connect x and y, i.e. v_i, v_k^i and v_i, v_l^i, v_k^i for $m \in [3, 4]$ or $v_i, v_{i-1}, \ldots, v_{i+1}, v_k^i$ for $m \geq 5$.
- (ii) For $x = v_{i+1} \in V(C_m)$, $i \in [1, m]$ and $y = v_k^i \in V(K_n^i)$, $i \in [1, m]$, $k \in [1, n-2]$, there exist two internally disjoint $x - y$ rainbow paths which connect them, i.e. v_{i+1}, v_k^i and v_{i+1}, v_i, v_k^i for $m \in [3, 4]$ or $v_{i+1}, v_{i+2}, \ldots, v_m, v_1, \ldots, v_i, v_k^i$ for $m \geq 5$.
- (iii) For $x = v_s \in V(C_m)$, $i \neq s$, $s \in [1, m]$ and $y = v_k^i \in V(K_n^i)$, $i \in [1, m]$, $k \in [1, n-2]$, there exist two internally disjoint $x - y$ rainbow paths which connect x and y, i.e. $v_s, v_{s+1}, \ldots, v_i, v_k^i$ and $v_s, v_{s-1}, \ldots, v_{i+1}, v_k^i$.

Case 4. For $x = v_k^i$, $x \in V(K_n^i)$ and $y = v_l^s$, $y \in V(K_n^s)$, $i, s \in [1, m]$, $k, l \in [1, n-2]$, $i \neq s$, there exist two internally disjoint $x - y$ rainbow paths which connect them, i.e, $v_k^i, v_{i+1}, v_{i+2}, \ldots, v_s, v_l^s$ and $v_k^i, v_i, v_{i-1}, \ldots, v_{s+1}, v_l^s$. □

We present a rainbow 2-connected coloring on $C_m^o \rhd_{\vec{e}} K_n$ for $m = n = 5$, as shown in Figure 2.

In the next theorem, we give an example of an edge-comb product of a cycle and a Hamiltonian graph whose rainbow 2-connection number achieves the upper bound of Theorem 2. We use the following lemma to prove the lower bound of Theorem 5.

Lemma 5. *Let c be a rainbow 2-connected coloring on a 2-connected graph G. Let e be an edge of G whose two endpoints have degree 2 and $c(e) = \alpha$. Then, there is no other edge f having one of the endpoints of degree 2 such that $c(f) = \alpha$.*

Proof. Suppose $f = (f_1, f_2)$ is an edge with $\deg(f_2) = 2$ and $e = (e_1, e_2)$ with $\deg(e_1) = \deg(e_2) = 2$. Since the two endpoints of e and one of the endpoint of f have degree 2, there exists exactly one cycle containing both edges. Consequently, there are no two rainbow paths which connect vertices e_1 and f_2. Thus, we have a contradiction. □

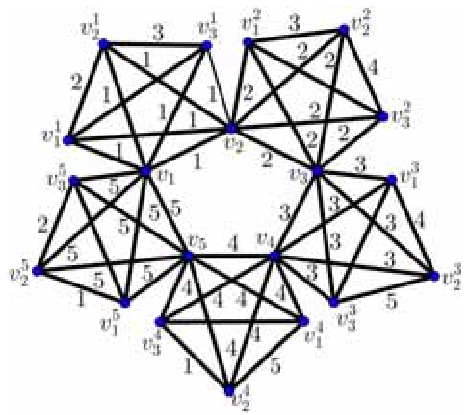

Figure 2. A rainbow 2-connected coloring on $C_m^o \rhd_{\vec{e}} K_n$ for $m = n = 5$.

Theorem 6. *Let m and n be two positive integers with $m \geq 3$ and $n \geq 4$. The rainbow 2-connection number of the edge-comb product of a cycle C_m with a cyclic orientation O and C_n is*

$$rc_2(C_m^o \rhd_{\vec{e}} C_n) = m(n-3) + 2.$$

Proof. Let $V(C_m^o \rhd_{\vec{e}} C_n) = \{v_i | i \in [1, m]\} \bigcup \{v_k^i | i \in [1, m], k \in [1, n-2]\}$ and $E(C_m^o \rhd_{\vec{e}} C_n) = \{v_i v_{i+1} | i \in [1, m]\} \bigcup \{v_i v_1^i | i \in [1, m]\} \bigcup \{v_k^i v_{k+1}^i | i \in [1, m], k \in [1, n-2]\}$, where $v_m v_{m+1} = v_m v_1$ and $v_{n-2}^i v_{n-1}^i = v_{n-2}^i v_{i+1}$. We divide the proof of the lower bound into two cases:

Case 1. $m \geq 3$ and $n = 4$. Since $\mathrm{diam}_2(C_m^o \rhd_{\vec{e}} C_n) = m(n-3) + 2$, it is obvious that $\mathrm{diam}_2(C_m^o \rhd_{\vec{e}} C_n) \leq rc_2(C_m^o \rhd_{\vec{e}} C_n)$ for $m \geq 3$ and $n = 4$.
Case 2. $m \geq 3$ and $n \geq 5$. We want to show that $rc_2(C_m^o \rhd_{\vec{e}} C_n) \geq m(n-3) + 2$. Without loss of generality, assume to the contrary that $rc_2(C_m^o \rhd_{\vec{e}} C_n) \leq m(n-3) + 1$. Let c' be a rainbow 2-connected coloring of $C_m^o \rhd_{\vec{e}} C_n$ using $m(n-3)+1$ colors. Consider two vertices v_{n-2}^i and v_1^{i+1}. Then two internally disjoint paths which connect them are $v_{n-2}^i, v_{i+1}, v_1^{i+1}$ and $v_{n-2}^i, \ldots, v_1^i, v_i, v_{i-1}, \ldots, v_{i+2}, v_{i+1}^{n-2}, \ldots, v_1^{i+1}$. Then the first path should be given with two different additional colors in order to be a rainbow path. Thus, by considering the first path of such vertices v_{n-2}^i and v_1^{i+1}, we need at least two additional colors, i.e., one color for edge $v_{i+1} v_{n-2}^i$ and another color for edge $v_{i+1} v_1^{i+1}$. By this coloring, we can never obtain a path where two edges $v_i v_1^i$ and $v_j v_1^j$ or $v_{i+1} v_{n-2}^i$ and $v_{j+1} v_{n-2}^j$ overlap each other. Now, consider all paths $v_1^i, v_2^i, \ldots, v_{n-2}^i$ with $i \in [1, m]$, by Lemma 4. We must color all the paths with $m(n-3)$ additional colors. But, we only have $m(n-3) - 1$ additional colors. So, we have a contradiction.

The proof of the upper bound of $rc_2({C_m}^o \rhd_{\vec{e}} C_n)$ is a direct consequence of Theorem 1. Hence, we complete the proof. □

As shown in Figure 3, we present a rainbow 2-connected coloring on $C_m^o \rhd_{\vec{e}} C_n$ for $m = n = 5$.

In the following theorem, we give an example of graph which edge-comb product of a cycle and a Hamiltonian graph have a rainbow 2-connection number achieving the bound

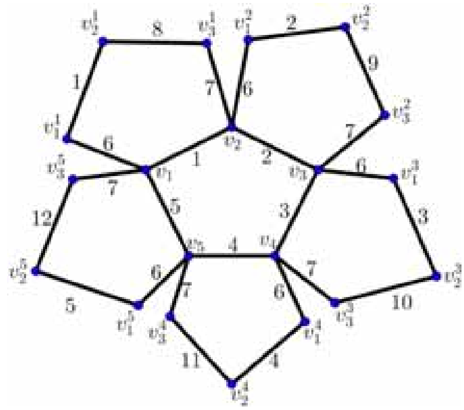

Figure 3. A rainbow 2-connected coloring on $C_m^o \triangleright_{\vec{e}} C_n$ for $m = n = 5$.

between the lower and upper bounds of Theorem 2. Before we proceed, we shall use the following definition. A *wheel graph*, W_n of order $n+1$ is a graph obtained from a cycle, C_n, by adding a new vertex called the hub and joining it to every vertex of C_n. We call the edges of C_n as *rims* and the edges connecting the hub to the vertices of C_n as *spokes*.

Theorem 7. *Let m and n be two positive integers with $m \geq 3$ and $n \geq 4$. The rainbow 2-connection number of the edge-comb product of a cycle C_m with a cyclic orientation O and a wheel W_n with e a rim of W_n is*

$$rc_2(C_m^o \triangleright_{\vec{e}} W_n) \leq m + n - 2$$

and equality holds for $n = 4$.

Proof. Let $V(C_m^o \triangleright_{\vec{e}} W_n) = \{v_i | i \in [1, m]\} \bigcup \{v_{i,i+1}^k | i \in [1, m], k \in [0, n-2]\}$ and $E(C_m^o \triangleright_{\vec{e}} W_n) = \{v_i v_{i+1} | i \in [1, m]\} \bigcup \{v_i v_k^i | i \in [1, m]$ and $k \in [0, 1]\} \bigcup \{v_{i+1} v_k^i | i \in [1, m]$ and $k \in \{0, n-2\}\} \bigcup \{v_k^i v_{k+1}^i | i \in [1, m], k \in [1, n-2]\} \bigcup \{v_0^i v_k^i | i \in [1, m], k \in [1, n-2]\}$, where $v_m v_{m+1} = v_m v_1$ and $v_{n-2}^i v_{n-1}^i = v_{n-2}^i v_{i+1}$.

Since $\text{diam}(C_m^o \triangleright_{\vec{e}} W_4) = m+n-2$, it is obvious that $\text{diam}_2(C_m^o \triangleright_{\vec{e}} W_4) \leq rc_2(C_m^o \triangleright_{\vec{e}} W_4)$. Next, we shall consider the proof of the upper bound. First, we color a cycle C_m by using a rainbow $rc(C_m)$-coloring of C_m for $m \in [3, 4]$ or by using m different colors otherwise. After that, for $i \in [1, m]$ and $m \in [3, 4]$, we color edges $v_i v_k^i$, $k \in [0, 1]$ with one additional color and another additional color for edges $v_{i+1} v_k^i$, $k\{0, n-2\}$. For $m \geq 5$, we use the same color used by $v_i v_{i+1}$ for edges $v_i v_k^i$, $k \in [0, 1]$ and one additional color for all $v_{i+1} v_k^i$ and $v_0^i v_1^i$, $k \in \{0, n-2\}$. Next, for $m \in [3, 4]$, we color the copies of W_n with $n-3$ additional colors for all $v_k^i v_{k+1}^i$ with $k \in [1, n-3]$ and another additional color for all $v_0^i v_{n-2}^i$. For $m \geq 5$, we color the copies of W_n with $n-3$ additional colors for all $v_k^i v_{k+1}^i$ with $k \in [1, n-3]$. Finally, for $m \in [3, 4]$, we use the same color used by $v_k^i v_{k+1}^i$ for edges $v_0^i v_k^i$ with $k \in [1, n-3]$. For $m \geq 5$, we use the same color used by $v_k^i v_{k+1}^i$ for edges $v_0^i v_{k+1}^i$.

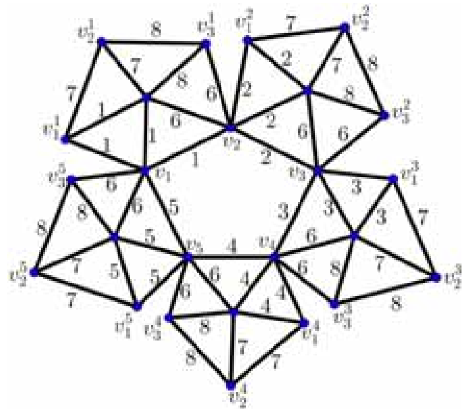

Figure 4. A rainbow 2-connected coloring on $C_m^o \rhd_{\vec{e}} W_n$ for $m = n = 5$.

We show that the above coloring is a rainbow 2-connected $(m + n - 2)$-coloring of $C_m^o \rhd_{\vec{e}} W_n$. Next, we consider any two distinct vertices $x, y \in V(C_m^o \rhd_{\vec{e}} W_n)$. We divided the proof into four cases:

Case 1. For $x = v_i$ and $y = v_j, x, y \in V(C_m)$ with $i \neq j$ and $i, j \in [1, m]$, there exist two internally disjoint $x - y$ rainbow paths, i.e. v_i, v_j and v_i, v_0^i, v_j for $v_i \sim v_j$ or $v_i, v_{i+1}, \ldots, v_j$ and $v_i, v_{i-1}, \ldots, v_j$ for $v_i \nsim v_j$.
Case 2. For $x = v_k^i$ and $y = v_l^i, x, y \in V(W_n^i), i \in [1, m], k \neq l, k, l \in [0, n - 2]$, we divide into three subcases:

(i) For $k = 0$ and $l \in [1, n - 3]$, there exist two internally disjoint $x - y$ rainbow paths which connect them, i.e. v_0^i, v_l^i and v_0^i, v_{l+1}^i, v_l^i for $m \in [3, 4]$ or $v_0^i, v_i, v_{i-1}, \ldots, v_l, v_l^{n-2}, \ldots, v_l^i$ for $m \geq 5$.

(ii) For $k = 0$ and $l = n - 2$, there exist two internally disjoint $x - y$ rainbow paths which connect x and y, i.e. v_0^i, v_{n-2}^i and $v_0^i, v_i, v_{i+1}, v_{n-2}^i$ for $m \in [3, 4]$ or $v_0^i, v_i, v_{i-1}, \ldots, v_{i+1}, v_{n-2}^i$ for $m \geq 5$.

(iii) For $k \neq l$ and $k, l \in [1, n-2]$, there exist two internally disjoint $x - y$ rainbow paths which connect x and y, i.e. $v_k^i v_{k+1}^i, \ldots, v_l^i$ and v_k^i, v_0^i, v_l^i.

Case 3. For $x \in V(C_m)$ and $y \in V(W_n^i)$, we divide into three subcases:

(i) For $x = v_i \in V(C_m), i \in [1, m]$ and $y = v_k^i \in V(W_n^i), i \in [1, m], k \in [0, n - 2]$, there exist two internally disjoint $x - y$ rainbow paths which connect x and y, i.e. $v_i, v_1^i, \ldots, v_k^i$ and v_i, v_0^i, v_k^i.

(ii) For $x = v_{i+1} \in V(C_m), i \in [1, m]$ and $y = v_k^i \in V(W_n^i), i \in [1, m], k \in [0, n - 2]$, there exist two internally disjoint $x - y$ rainbow paths which connect them, i.e. $v_{i+1}, v_{n-2}^i, \ldots, v_k^i$ and v_{i+1}, v_0^i, v_k^i.

(iii) For $x = v_s \in V(C_m), s \in [1, m]$ and $y = v_k^i \in V(W_n^i), i \in [1, m], k \in [0, n - 2]$, there exist two internally disjoint $x - y$ rainbow paths which connect x and y, i.e. $v_s, v_{s+1}, \ldots, v_i, v_1^i, \ldots, v_k^i$ and $v_s, v_{s-1}, \ldots, v_{i+1}, v_{n-2}^i, \ldots, v_k^i$.

Case 4. For $x = v_k^i, x \in V(W_n^i)$ and $y = v_l^s, y \in V(W_n^s), i, s \in [1, m], k, l \in [0, n - 2]$, we divide into four subcases:

(i) For $k, l = 0$, there exist two internally disjoint $x - y$ rainbow paths which connect them, i.e. $v_0^i, v_{i+1}, v_{i+2}, \ldots, v_s, v_0^s$ and $v_0^i, v_i, v_{i-1}, \ldots, v_{s+1}, v_0^s$.

(ii) For $k = 0$ and $l \in [1, n-2]$, there exist two internally disjoint $x - y$ rainbow paths which connect them, i.e. $v_0^i, v_{i+1}, v_{i+2}, \ldots, v_s, v_1^s, \ldots, v_l^s$ and $v_0^i, v_i, v_{i-1}, \ldots, v_{s+1}, v_{n-2}^s, \ldots, v_l^s$.

(iii) For $k = l, k, l \in [1, n-2]$, there exist two internally disjoint $x - y$ rainbow paths which connect them, i.e. $v_k^i, v_{k+1}^i, \ldots, v_{n-2}^i, v_{i+1}, v_{i+2}, \ldots, v_s, v_1^s, \ldots, v_k^s$ and $v_k^i, v_{k-1}^i, \ldots, v_1^i, v_i, v_{i-1}, \ldots, v_{s+1}, v_{n-2}^s, \ldots, v_k^s$.

(iv) For $k \neq l$ and $k, l \in [1, n-2]$, there exist two internally disjoint $x - y$ rainbow paths which connect them, i.e. $v_k^i, v_0^i, v_{i+1}, v_{i+2}, \ldots, v_s, v_0^s, v_l^s$ and $v_k^i, v_{k-1}^i, \ldots, v_1^i, v_i, v_{i-1}, \ldots, v_{s+1}, v_{n-2}^s, \ldots, v_l^s$ for $k < l$ or $v_k^i, v_0^i, v_i, v_{i-1}, \ldots, v_{s+1}, v_0^s, v_l^s$ and $v_k^i, v_{k+1}^i, \ldots, v_{n-2}^i, v_{i+1}, v_{i+2}, \ldots, v_s, v_1^s, \ldots, v_l^s$ for $k > l$.

Hence, we complete the proof. □

As shown in Figure 4, we present a rainbow 2-connected coloring on $C_m^o \rhd_{\vec{e}} W_n$ for $m = n = 5$.

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