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Exceptional set in Waring–Goldbach problem: Two squares, two cubes and two sixth powers

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Abstract. Let $R(n)$ denote the number of representations of an even integer n as the sum of two squares, two cubes and two sixth powers of primes, and by $\mathcal{E}(N)$ we denote the number of even integers $n \leqslant N$ such that the expected asymptotic formula for $R(n)$ fails to hold. In this paper, it is proved that $\mathcal{E}(N) \ll N^{\frac{127}{288}+\varepsilon}$ for any $\varepsilon > 0$.

Keywords. Waring–Goldbach problem; exceptional set; Hardy–Littlewood method.

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1. Introduction

Let $n, k_1, k_2, \ldots, k_s$ be natural numbers such that $2 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_s, n > s$. Waring problem of mixed type concerns the representation of a natural number n as the form

$$n = x_1^{k_1} + \cdots + x_s^{k_s}. \tag{1.1}$$

Not very much is known about results of this kind. For references in this aspect, we refer the reader to section P12 of LeVeque's Reviews in Number Theory, the bibliography in Vaughan [7] and the recent papers by Wooley [10,12].

Let $\tilde{R}(n)$ denote the number of representations of the integer n in the shape

$$n = x_1^2 + x_2^2 + x_3^3 + x_4^3 + x_5^6 + x_6^6, \tag{1.2}$$

with $x_i \in \mathbb{N}$ $(1 \leqslant i \leqslant 6)$, and let

$$\tilde{\mathfrak{S}}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{S_2^2(q,a) S_3^2(q,a) S_6^2(q,a) e_q(-an)}{q^6}, \tag{1.3}$$

in which we write

$$S_k(q,a) = \sum_{r=1}^{q} e_q(ar^k) \quad (k = 2, 3, 6), \tag{1.4}$$

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and as usual $e(\alpha)$ denotes $e^{2\pi i\alpha}$ and $e_q(\alpha)$ denotes $e(\alpha/q)$. A heuristical application of the Hardy–Littlewood method, based on a major arc analysis only, suggests that $\tilde{R}(n)$ satisfies the asymptotic relation

$$\tilde{R}(n) = \frac{\Gamma^2(\frac{3}{2})\Gamma^2(\frac{4}{3})\Gamma^2(\frac{7}{6})}{\Gamma(2)}\tilde{\mathfrak{S}}(n)n(1+o(1)). \tag{1.5}$$

It is worth noting here that for every integer n, one has $1 \ll \tilde{\mathfrak{S}}(n) \ll 1$ (see, for example, Chapter 4 of [7]).

But proving (1.5) is beyond the grasp of modern number theory techniques. Wooley [11] applied Golubeva's method to show, subject to the truth of the generalized Riemann hypothesis, that $\tilde{R}(n) > 0$ for all large integers n. However, his method fails to obtain the anticipated asymptotic formula for $\tilde{R}(n)$.

In order to assess how frequently the formula (1.5) might fail, we define an associated exceptional set. Let $\tilde{\psi}(n)$ denote a function of n, increasing monotonically to infinity, and satisfying the condition that, when n is large, one has $\tilde{\psi}(n) = O(n^{\delta})$ for some sufficiently small positive number δ, and we denote by $\tilde{E}(N)$ the number of integers n with $1 \leqslant n \leqslant N$ for which

$$\left|\tilde{R}(n) - \frac{\Gamma^2(\frac{3}{2})\Gamma^2(\frac{4}{3})\Gamma^2(\frac{7}{6})}{\Gamma(2)}\tilde{\mathfrak{S}}(n)n\right| > \frac{n}{\tilde{\psi}(n)}.$$

In 2014, Wooley [12] showed that $\tilde{E}(N) \ll \tilde{\psi}(N)^2(\log N)^3$. In 2015, Lü and Mu [4] made an important breakthrough in the study of $\tilde{E}(N)$, and proved that $\tilde{E}(N) \ll \tilde{\psi}(N)^2(\log N)^2$.

It is reasonable to propose the conjecture that every sufficiently large even integer n can be expressed as the sum of two squares, two cubes and two sixth powers of primes. That is, for sufficiently large even integer n, the equation

$$n = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^6 + p_6^6 \tag{1.6}$$

is solvable in primes p_j. However, this conjecture is perhaps out of reach at present times.

It is of interest to investigate the exceptional set $E(N)$ of (1.6), which denotes the number of even integers $n \leqslant N$, yet cannot be represented as the sum of two squares, two cubes and two sixth powers of primes. Motivated by Wooley [12], we introduce a pruning process into the Hardy–Littlewood method in this paper to establish the following results.

Theorem 1. *For an even integer n, let $R(n)$ denote the number of representations of n as the sum of two squares, two cubes and two sixth powers of primes, and $\mathcal{E}(N)$ denote the number of even integers $n \leqslant N$ such that the asymptotic formula*

$$R(n) = \frac{\Gamma^2(\frac{3}{2})\Gamma^2(\frac{4}{3})\Gamma^2(\frac{7}{6})}{\Gamma(2)}\mathfrak{S}(n)\frac{n}{\log^6 n} + O\left(\frac{n\log\log n}{\log^7 n}\right) \tag{1.7}$$

fails to hold, *where*

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{S_2^{*2}(q,a) S_3^{*2}(q,a) S_6^{*2}(q,a) e_q(-an)}{\varphi^6(q)},$$

$$S_k^*(q,a) = \sum_{\substack{r=1 \\ (r,q)=1}}^{q} e_q(ar^k).$$

Then for any $\varepsilon > 0$, *we have* $\mathcal{E}(N) \ll N^{\frac{127}{288}+\varepsilon}$.

From Theorem 1 and Lemma 2.10 ($0 < \mathfrak{S}(n) \ll 1$ for even integer n), we get as follows.

Theorem 2. *For any* $\varepsilon > 0$, *we have* $E(N) \ll N^{\frac{127}{288}+\varepsilon}$.

2. Notation and some preliminary lemmas

For the proof of Theorem 1, in this section we introduce the necessary notation and lemmas.

Throughout this paper, by n we denote a sufficiently large even integer which satisfies $N/2 \leqslant n \leqslant N$. In addition, let $A = 10^{10}$, and $\varepsilon \in (0, 10^{-10})$ be an arbitrarily small positive constant not necessarily the same in different formulae. The letter p, with or without subscripts, is reserved for a prime number. The constants in O-terms and $\ll$-symbols are absolute or depend at most on ε and A. We use $e(\alpha)$ to denote $e^{2\pi i\alpha}$ and $e_q(\alpha) = e(\alpha/q)$. By $A \sim B$ we mean that $B < A \leqslant 2B$. We denote by (m,n) the greatest common divisor of m and n. As usual, $\varphi(n)$ and $\Lambda(n)$ stand for Euler's function and Von Mangoldt function respectively. We denote by $\sum_{r(q)}$ and $\sum_{r(q)^*}$ sums with r running over a complete system and a reduced system of residues modulo q respectively. The letter χ denotes a Dirichlet character mod q, and χ_0 denotes the Dirichlet principal character mod q. By $\sum_{\chi(q)}$ we denote a sum with χ running over the Dirichlet characters mod q.

For the application of the Hardy–Littlewood method, we need to define the Farey dissection. For this purpose, we set

$$Q_0 = \log^A N, \quad Q_1 = N^{\frac{1}{6}}, \quad Q_2 = N^{\frac{5}{6}},$$

and for $(a,q) = 1$, $1 \leqslant a \leqslant q$, put

$$\mathfrak{M}(q,a) = \left(\frac{a}{q} - \frac{1}{qQ_2}, \frac{a}{q} + \frac{1}{qQ_2}\right], \quad \mathfrak{M} = \bigcup_{1 \leqslant q \leqslant Q_1} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}(q,a),$$

$$\mathfrak{I}_0 = \left(-\frac{1}{Q_2}, 1 - \frac{1}{Q_2}\right], \quad \mathfrak{m}_1 = \mathfrak{I}_0 \setminus \mathfrak{M}.$$

Then we have the Farey dissection

$$\mathfrak{J}_0 = \mathfrak{M} \bigcup \mathfrak{m}_1. \tag{2.1}$$

The application of the Hardy–Littlewood method consists of two steps. The first step is to estimate the integral over the minor arc $\mathfrak{m}_1$. For this purpose, Lemma 2.1 and Lemma 2.5 are required. The second step is to evaluate the integral over the major arc $\mathfrak{M}$. But the major arc $\mathfrak{M}$ is too large for the integral over it to be evaluated directly, and we have to prune it further. Let

$$\mathfrak{M}_0(q,a) = \left(\frac{a}{q} - \frac{Q_0^{100}}{N}, \frac{a}{q} + \frac{Q_0^{100}}{N}\right], \quad \mathfrak{M}_0 = \bigcup_{1 \leqslant q \leqslant Q_0^{100}} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \mathfrak{M}_0(q,a),$$

$$\mathfrak{m}_2 = \mathfrak{M} \setminus \mathfrak{M}_0.$$

Then we have the dissection

$$\mathfrak{M} = \mathfrak{M}_0 \bigcup \mathfrak{m}_2. \tag{2.2}$$

Now the integral over $\mathfrak{M}_0$ may be evaluated by Lemma 2.8 and Lemma 2.2, and the integral over $\mathfrak{m}_2$ can be estimated by Lemma 2.6, Lemma 2.7 and Lemma 2.1.

Now we state the lemmas required in this paper.

Lemma 2.1. *Let* $2 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_s$ *be natural numbers such that*

$$\sum_{i=j+1}^{s} \frac{1}{k_i} \leqslant \frac{1}{k_j}, \quad 1 \leqslant j \leqslant s-1$$

and

$$f_k(\alpha) = \sum_{p \leqslant N^{\frac{1}{k}}} e(\alpha p^k).$$

Then we have

$$\int_0^1 \left| \prod_{i=1}^{s} f_{k_i}(\alpha) \right|^2 d\alpha \ll N^{\frac{1}{k_1} + \cdots + \frac{1}{k_s} + \varepsilon}.$$

Proof. It follows easily from Lemma 1 of [1]. □

Lemma 2.1 will be used in the estimation of the integral over $\mathfrak{m}_j$, $j = 1, 2$.

Lemma 2.2. *Let*

$$G_k(\chi, a) = \sum_{r(q)} \chi(r) e_q(ar^k), \quad S_k^*(q,a) = G_k(\chi_0, a).$$

Then for $(q,a)=1$, *we have*

(i) $G_k(\chi,a) \ll q^{\frac{1}{2}+\varepsilon}$.

In particular, for $(p,a)=1$, *we have*

(ii) $|S_k^*(p,a)| \leqslant ((k,p-1)-1)\,p^{\frac{1}{2}}+1$;
(iii) $S_k^*(p^l,a)=0$ *for* $l \geqslant \gamma(p)$, where

$$\gamma(p)=\begin{cases}\theta+2, & \text{if } p^\theta \| k, \quad p\neq 2 \text{ or } p=2, \theta=0;\\[2ex] \theta+3, & \text{if } p^\theta \| k, \quad p=2, \theta>0.\end{cases}$$

Proof. For (i), see Chapter VI of [8]. For (ii), see Lemma 4.3 of [7]. For (iii), see Lemma 8.3 of [3]. □

Lemma 2.2 will be used in the proofs of Lemma 2.6, Lemma 2.10 and the evaluation of the integral over $\mathfrak{M}_0$.

Lemma 2.3. *For* $x \geqslant 2$, $2 \leqslant T \leqslant x$, *we have*

$$\sum_{n\leqslant x}\Lambda(n)\chi(n)=\delta_\chi x-\sum_{\substack{\rho=\beta+i\gamma\\ |\gamma|\leqslant T}}\frac{x^\rho}{\rho}+O\left(\frac{x\log^2(xq)}{T}\right), \tag{2.3}$$

where δ_χ *takes value* 1 *or* 0 *according as* $\chi=\chi_0$ *or not, and* $\rho=\beta+i\gamma$ *denotes a non-trivial zero of the Dirichlet* L*-function* $L(s,\chi)$.

Proof. See Chapter 19 of [2]. □

Lemma 2.4. *For* $q \geqslant 1$, *and real numbers* $\frac{1}{2}\leqslant\sigma\leqslant 1$, $T\geqslant 2$, *let* $N(\sigma,T,\chi)$ *denote the number of zeroes* $\rho=\beta+i\gamma$ *of the Dirichlet* L*-function* $L(s,\chi)$ *in the region*

$$\sigma\leqslant\beta\leqslant 1,\quad |\gamma|\leqslant T.$$

Define

$$N(\sigma,T,q)=\sum_{\chi(q)}N(\sigma,T,\chi).$$

Then we have

$$N(\sigma,T,q)\ll\begin{cases}(qT)^{\frac{5-4\sigma}{3}}\log^9(qT), & \frac{1}{2}\leqslant\sigma<\frac{4}{5};\\[2ex] (qT)^{\frac{5(1-\sigma)}{2}}\log^{14}(qT), & \frac{4}{5}\leqslant\sigma\leqslant 1.\end{cases}$$

Proof. The conclusion is due, in all essentials, to Theorem 12.1 of [5]. □

Lemma 2.3 and Lemma 2.4 will be used in the proof of Lemma 2.6.

Lemma 2.5. *For* $\alpha \in \mathfrak{m}_1$, *we have*

(*i*) $f_3(\alpha) \ll N^{\frac{11}{36}+\varepsilon}$;

(*ii*) $f_6(\alpha) \ll N^{\frac{95}{576}+\varepsilon}$.

Proof. For $\alpha \in \mathfrak{m}_1$, we have $N^{\frac{1}{6}} \ll q \ll N^{\frac{5}{6}}$. Then it follows from Lemma 8.5 of [13] that

$$\begin{aligned} f_3(\alpha) &\ll (\log N) \max_{P \leqslant \sqrt[3]{N}/2} \left| \sum_{p \sim P} e(\alpha p^3) \right| \\ &\ll (N^{\frac{1}{3}})^{\frac{11}{12}+\varepsilon} + (N^{\frac{1}{6}})^{-\frac{1}{6}} \cdot N^{\frac{1}{3}+\varepsilon} \ll N^{\frac{11}{36}+\varepsilon}. \end{aligned}$$

Similarly, by Lemma 2.4 of [13], we obtain

$$\begin{aligned} f_6(\alpha) &\ll (\log N) \max_{P \leqslant \sqrt[6]{N}/2} \left| \sum_{p \sim P} e(\alpha p^6) \right| \\ &\ll (N^{\frac{1}{6}})^{\frac{95}{96}+\varepsilon} + N^{\frac{1}{12}+\varepsilon} \ll N^{\frac{95}{576}+\varepsilon}. \end{aligned}$$

This completes the proof of Lemma 2.5. □

Lemma 2.5 will be used in the estimation of the integral over $\mathfrak{m}_1$.

Lemma 2.6. *For* $\alpha = \frac{a}{q} + \lambda$, $(a, q) = 1$, *let*

$$V(\alpha) = \frac{N^{\frac{1}{2}} \log^{18} N}{q^{\frac{1}{2}-\varepsilon}(1 + |\lambda| N)^{\frac{1}{2}}}. \tag{2.4}$$

Then for $\alpha \in \mathfrak{m}_2$, *we have*

$$f_2(\alpha) \ll V(\alpha) + N^{\frac{3}{8}+\varepsilon}.$$

Proof. By the orthogonality of the Dirichlet characters, we obtain

$$f_2(\alpha) = \frac{1}{\varphi(q)} \sum_{\chi(q)} G_2(\bar{\chi}, a) \sum_{N^{\frac{3}{8}} < l \leqslant N^{\frac{1}{2}}} \frac{\chi(l)\Lambda(l)e(\lambda l^2)}{\log l} + O(N^{\frac{3}{8}}). \tag{2.5}$$

Let $T = N^{\frac{1}{8}} q^{\frac{1}{2}} (1 + |\lambda| N)$. Then on taking into account (2.5), Lemma 2.3 and integration by parts, we have

$$\begin{aligned} f_2(\alpha) =& \frac{S_2^*(q,a)}{\varphi(q)} \int_2^{N^{\frac{1}{2}}} \frac{e(\lambda u^2)}{\log u} du \\ & - \frac{1}{\varphi(q)} \sum_{\chi(q)} G_2(\bar{\chi}, a) \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma| \leqslant T}} \int_{N^{\frac{3}{8}}}^{N^{\frac{1}{2}}} \frac{u^{\rho-1} e(\lambda u^2)}{\log u} du \\ & + O(N^{\frac{3}{8}} \log^2 N). \end{aligned} \tag{2.6}$$

For $N^{\frac{3}{8}} \leqslant X < Y \leqslant 2X \leqslant N^{\frac{1}{2}}$, it follows from Lemmas 4.2 and 4.3 of [6] that

$$\int_X^Y \frac{e(\lambda u^2)}{\log u} du \ll \frac{N^{\frac{1}{2}}}{1 + |\lambda| N}, \tag{2.7}$$

$$\int_X^Y \frac{u^{\rho-1} e(\lambda u^2)}{\log u} du \ll \frac{N^{\frac{\beta}{2}}}{1 + |\lambda| N}, \quad \text{if } |\gamma| \leqslant 1 + |\lambda| N \tag{2.8}$$

and

$$\int_X^Y \frac{u^{\rho-1} e(\lambda u^2)}{\log u} du \ll \frac{N^{\frac{\beta}{2}}}{|\gamma|}, \quad \text{if } 100(1 + |\lambda| N) < |\gamma| \leqslant T. \tag{2.9}$$

On the other hand, if $(1 + |\lambda| N) < |\gamma| \leqslant 100(1 + |\lambda| N)$, then by Lemma 4.5 of [6], we have

$$\begin{aligned} \int_X^Y \frac{u^{\rho-1} e(\lambda u^2)}{\log u} du &= \int_{X^2}^{Y^2} \frac{u^{\frac{\rho}{2}-1} e(\lambda u)}{\log u} du \\ &\ll \frac{N^{\frac{\beta}{2}}}{|\gamma|^{\frac{1}{2}}}. \end{aligned} \tag{2.10}$$

We thus conclude from (2.7)–(2.10) that

$$\begin{aligned} & \sum_{\chi(q)} \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma| \leqslant T}} \left| \int_X^Y \frac{u^{\rho-1} e(\lambda u^2)}{\log u} du \right| \\ & \ll \sum_{\chi(q)} \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma| \leqslant 1+|\lambda| N}} \frac{N^{\frac{\beta}{2}}}{1 + |\lambda| N} + \sum_{\chi(q)} \sum_{\substack{\rho=\beta+i\gamma \\ (1+|\lambda| N) < |\gamma| \leqslant 100(1+|\lambda| N)}} \frac{N^{\frac{\beta}{2}}}{|\gamma|^{\frac{1}{2}}} \end{aligned}$$

$$+\sum_{\chi(q)}\sum_{\substack{\rho=\beta+i\gamma\\100(1+|\lambda|N)<|\gamma|\leqslant T}}\frac{N^{\frac{\beta}{2}}}{|\gamma|}$$
$$=\sum\nolimits_1+\sum\nolimits_2+\sum\nolimits_3,\ \text{say.} \tag{2.11}$$

From now on, we concentrate on the sums in (2.11). To this end, we introduce a dissection. We split the range of $\beta = \mathrm{Re}(\rho)$ into $L = 1 + \left[\log N\right]$ equal parts $\sigma_j = \frac{j}{L} < \beta \leqslant \sigma_{j+1} (0 \leqslant j \leqslant L-1)$. Then by Lemma 2.4, we obtain

$$\begin{aligned}
\sum\nolimits_2 &\ll \sum_{j=0}^{L-1}\sum_{\chi(q)}\sum_{\substack{\rho=\beta+i\gamma\\ \sigma_j<\beta\leqslant\sigma_{j+1}\\ |\gamma|\leqslant 100(1+|\lambda|N)}}\frac{N^{\frac{\beta}{2}}}{(1+|\lambda|N)^{\frac{1}{2}}}\\
&\ll \frac{1}{(1+|\lambda|N)^{\frac{1}{2}}}\sum_{j=0}^{L-1}N^{\frac{\sigma_{j+1}}{2}}\sum_{\chi(q)}\sum_{\substack{\rho=\beta+i\gamma\\ \beta\geqslant\sigma_j\\ |\gamma|\leqslant 100(1+|\lambda|N)}}1\\
&\ll \frac{\log N}{(1+|\lambda|N)^{\frac{1}{2}}}\max_{0\leqslant\sigma\leqslant 1}N^{\frac{\sigma}{2}}\cdot N\left(\sigma, 100(1+|\lambda|N), q\right)\\
&\ll \frac{\log^{15} N}{(1+|\lambda|N)^{\frac{1}{2}}}(N^{\frac{1}{4}}(q(1+|\lambda|N))+N^{\frac{2}{5}}(q(1+|\lambda|N))^{\frac{1}{2}}+N^{\frac{1}{2}}).
\end{aligned} \tag{2.12}$$

Next, for $\alpha \in \mathfrak{m}_2$, we have $q(1+|\lambda|N) \ll N^{\frac{1}{6}}$ and

$$\sum\nolimits_2 \ll \frac{N^{\frac{1}{2}}\log^{15} N}{(1+|\lambda|N)^{\frac{1}{2}}}. \tag{2.13}$$

A similar argument gives

$$\begin{aligned}
\sum\nolimits_1 &\ll \frac{\log^{15} N}{1+|\lambda|N}(N^{\frac{1}{4}}(q(1+|\lambda|N))+N^{\frac{2}{5}}(q(1+|\lambda|N))^{\frac{1}{2}}+N^{\frac{1}{2}}).\\
&\ll \frac{N^{\frac{1}{2}}\log^{15} N}{1+|\lambda|N},
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
\sum\nolimits_3 &\ll \max_{(1+|\lambda|N)\leqslant T_1\leqslant T}\frac{\log^{16} N}{T_1}(N^{\frac{1}{4}}(qT_1)+N^{\frac{2}{5}}(qT_1)^{\frac{1}{2}}+N^{\frac{1}{2}})\\
&\ll \frac{N^{\frac{1}{2}}\log^{16} N}{(1+|\lambda|N)^{\frac{1}{2}}}+N^{\frac{3}{8}+\varepsilon}q^{\frac{1}{2}}.
\end{aligned} \tag{2.15}$$

From (2.13)–(2.15), we have

$$\begin{aligned}\frac{1}{\varphi(q)}\sum_{\chi(q)} G_2(\bar{\chi},a)\sum_{\substack{\rho=\beta+i\gamma\\|\gamma|\leqslant T}}\int_{N^{\frac{3}{8}}}^{N^{\frac{1}{2}}}\frac{u^{\rho-1}e(\lambda u^2)}{\log u}du\\ \ll \frac{\log q}{q^{\frac{1}{2}-\varepsilon}}(\log N)\max_{N^{\frac{3}{8}}\leqslant X<Y\leqslant 2X\leqslant N^{\frac{1}{2}}}\sum_{\chi(q)}\sum_{\substack{\rho=\beta+i\gamma\\|\gamma|\leqslant T}}\left|\int_X^Y\frac{u^{\rho-1}e(\lambda u^2)}{\log u}du\right|\\ \ll V(\alpha)+N^{\frac{3}{8}+\varepsilon}\end{aligned} \tag{2.16}$$

and

$$\frac{S_2^*(q,a)}{\varphi(q)}\int_2^{N^{\frac{1}{2}}}\frac{e(\lambda u^2)}{\log u}du \ll V(\alpha), \tag{2.17}$$

where the inequality $\varphi(q) \gg \frac{q}{\log q}$, Lemma 2.2(i) and (2.7) are employed. Now by (2.6), (2.16) and (2.17), Lemma 2.6 is proved. □

Lemma 2.7. *For* $1 \leqslant a \leqslant q$ *with* $(a,q)=1$, *put*

$$\mathcal{I}(q,a)=\left(\frac{a}{q}-\frac{1}{qQ_0},\frac{a}{q}+\frac{1}{qQ_0}\right],\quad \mathcal{I}=\bigcup_{1\leqslant q\leqslant Q_0}\bigcup_{\substack{a=-2q\\(a,q)=1}}^{3q}\mathcal{I}(q,a).$$

Then we have

$$(i)\qquad \int_{\mathcal{I}}|V(\alpha)|^2d\alpha=\sum_{1\leqslant q\leqslant Q_0}\sum_{\substack{a=-2q\\(a,q)=1}}^{3q}\int_{\mathcal{I}(q,a)}|V(\alpha)|^2d\alpha \ll Q_0^2, \tag{2.18}$$

$$(ii)\qquad \int_{\mathcal{I}}|V(\alpha)|^{\frac{1}{4}}d\alpha=\sum_{1\leqslant q\leqslant Q_0}\sum_{\substack{a=-2q\\(a,q)=1}}^{3q}\int_{\mathcal{I}(q,a)}|V(\alpha)|^{\frac{1}{4}}d\alpha \ll N^{-\frac{7}{8}}Q_0^2, \tag{2.19}$$

where $V(\alpha)$ is defined by (2.4).

Proof. By (2.4), we have

$$\begin{aligned}\sum_{1\leqslant q\leqslant Q_0}\sum_{\substack{a=-2q\\(a,q)=1}}^{3q}\int_{\mathcal{I}(q,a)}|V(\alpha)|^2d\alpha\\ \ll \sum_{1\leqslant q\leqslant Q_0}q^{-1+\varepsilon}\sum_{\substack{a=-2q\\(a,q)=1}}^{3q}\int_{|\lambda|\leqslant\frac{1}{Q_0}}\frac{N\log^{36}N}{1+|\lambda|N}d\lambda\end{aligned}$$

$$\ll \sum_{1\leqslant q\leqslant Q_0} q^{-1+\varepsilon} \sum_{\substack{a=-2q\\(a,q)=1}}^{3q} \left(\int_{|\lambda|\leqslant \frac{1}{N}} N\log^{36} N d\lambda + \int_{\frac{1}{N}\leqslant|\lambda|\leqslant\frac{1}{Q_0}} \frac{\log^{36} N}{|\lambda|} \mathrm{d}\lambda \right)$$
$$\ll Q_0^2.$$

Hence the proof of Lemma 2.7(i) is completed, and (ii) can be proved in a similar way. □

Lemma 2.6 and Lemma 2.7 will be used in the estimation of the integral over $\mathfrak{m}_2$.

Lemma 2.8. *Let*

$$u_k(\lambda) = \sum_{2<n\leqslant N} \frac{e(n\lambda)}{n^{1-\frac{1}{k}}\log n}.$$

Then for $\alpha = \frac{a}{q} + \lambda \in \mathfrak{M}_0$, *we have*

$$f_k(\alpha) = \frac{S_k^*(q,a)}{\varphi(q)} u_k(\lambda) + O(N^{\frac{1}{k}} \exp(-\log^{\frac{1}{3}} N)),$$

where $S_k^*(q,a)$ *is defined in Lemma* 2.2.

Proof. See Lemma 7.15 of [3]. □

Lemma 2.8 will be used in the evaluation of the integral over $\mathfrak{M}_0$.

Lemma 2.9. *Let* $\mathcal{L}^*(p,n)$ *denote the number of solutions to the congruence*

$$x_1^2 + x_2^2 + x_3^3 + x_4^3 + x_5^6 + x_6^6 \equiv n \pmod{p}, \quad 1 \leqslant x_i \leqslant p-1.$$

Then for even integer n, *we have*

$$\mathcal{L}^*(p,n) > 0. \tag{2.20}$$

Proof. By the orthogonality of additive characters, we have

$$\begin{aligned} p\mathcal{L}^*(p,n) &= \sum_{a=1}^{p} S_2^{*2}(p,a) S_3^{*2}(p,a) S_6^{*2}(p,a) e_p(-an) \\ &= (p-1)^6 + E_p, \end{aligned} \tag{2.21}$$

where

$$E_p = \sum_{a=1}^{p-1} S_2^{*2}(p,a) S_3^{*2}(p,a) S_6^{*2}(p,a) e_p(-an). \tag{2.22}$$

By Lemma 2.2(ii), we obtain

$$|E_p| \leqslant (p-1)(\sqrt{p}+1)^2(2\sqrt{p}+1)^2(5\sqrt{p}+1)^2. \tag{2.23}$$

It's easy to verify that $|E_p| < (p-1)^6$ for $p \geqslant 17$, hence we have $\mathcal{L}^*(p,n) > 0$ for $p \geqslant 17$. On the other hand, when $p = 2, 3, 5, 7, 11, 13$, we can check by hand that $\mathcal{L}^*(p,n) > 0$. Now Lemma 2.9 is proved. □

Lemma 2.9 will be required in the proof of Lemma 2.10.

Lemma 2.10. *For even integer n, the series*

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \sum_{a(q)^*} \frac{S_2^{*2}(q,a)S_3^{*2}(q,a)S_6^{*2}(q,a)e_q(-an)}{\varphi^6(q)}$$

is convergent and $\mathfrak{S}(n) > 0$, where $S_k^*(q,a)$ is defined in Lemma 2.2.

Proof. The convergence of $\mathfrak{S}(n)$ follows from Lemma 2.2(i) easily. Let

$$A(q,n) = \sum_{a(q)^*} \frac{S_2^{*2}(q,a)S_3^{*2}(q,a)S_6^{*2}(q,a)e_q(-an)}{\varphi^6(q)}.$$

Noting the fact that $A(q,n)$ is multiplicative in q, by Lemma 2.2(iii), we know that

$$\begin{aligned}\mathfrak{S}(n) &= \prod_p (1 + A(p,n)) \\ &= \left(\prod_{p<23} (1 + A(p,n))\right)\left(\prod_{p\geqslant 23} (1 + A(p,n))\right).\end{aligned} \tag{2.24}$$

For $p \geqslant 23$, by Lemma 2.2(ii), we have

$$|A(p,n)| \leqslant \frac{(p-1)(p^{\frac{1}{2}}+1)^2(2p^{\frac{1}{2}}+1)^2(5p^{\frac{1}{2}}+1)^2}{(p-1)^6} \leqslant \frac{250}{p^2}.$$

So we get

$$\prod_{p\geqslant 23} (1 + A(p,n)) \geqslant \prod_{p\geqslant 23} \left(1 - \frac{250}{p^2}\right) \geqslant c_1 > 0. \tag{2.25}$$

It is easy to see that

$$1 + A(p,n) = \frac{p\mathcal{L}^*(p,n)}{(p-1)^6}. \tag{2.26}$$

From (2.26) and (2.20), we have $1 + A(p, n) > 0$. Therefore, we get

$$\prod_{p<23} (1 + A(p, n)) \geqslant c_2 > 0. \tag{2.27}$$

Thus we conclude from (2.24)–(2.25) and (2.27) that $\mathfrak{S}(n) > 0$. This completes the proof of Lemma 2.10. □

Lemma 2.10 was used in the proof of Theorem 2.

3. Auxiliary estimates

We are now equipped to establish the auxiliary estimates in this paper, and we initiate our proof by recalling the Farey dissections (2.1) and (2.2) that

$$\begin{aligned} R(n) = &\int_{\mathfrak{M}_0} f_2^2(\alpha) f_3^2(\alpha) f_6^2(\alpha) e(-\alpha n) d\alpha \\ &+ \int_{\mathfrak{m}_1} f_2^2(\alpha) f_3^2(\alpha) f_6^2(\alpha) e(-\alpha n) d\alpha \\ &+ \int_{\mathfrak{m}_2} f_2^2(\alpha) f_3^2(\alpha) f_6^2(\alpha) e(-\alpha n) d\alpha, \end{aligned}$$

where $f_k(\alpha)$ is defined in Lemma 2.1.

3.1 *The evaluation of the integral over $\mathfrak{M}_0$*

PROPOSITION.

For $\frac{N}{2} < n \leqslant N$, we have

$$\begin{aligned} &\int_{\mathfrak{M}_0} f_2^2(\alpha) f_3^2(\alpha) f_6^2(\alpha) e(-\alpha n) d\alpha \\ &\quad = \frac{\Gamma^2(\frac{3}{2})\Gamma^2(\frac{4}{3})\Gamma^2(\frac{7}{6})}{\Gamma(2)} \mathfrak{S}(n) \frac{n}{\log^6 n} + O\left(\frac{n \log\log n}{\log^7 n}\right). \end{aligned}$$

Proof. For $\alpha = \frac{a}{q} + \lambda$, let $g_k(\alpha) = \frac{S_k^*(q,a)}{\varphi(q)} u_k(\lambda)$. Then it follows from Lemma 2.8 that

$$\begin{aligned} &\int_{\mathfrak{M}_0} f_2^2(\alpha) f_3^2(\alpha) f_6^2(\alpha) e(-\alpha n) d\alpha \\ &\quad = \int_{\mathfrak{M}_0} g_2^2(\alpha) g_3^2(\alpha) g_6^2(\alpha) e(-\alpha n) d\alpha + O\left(\int_{\mathfrak{M}_0} N^2 \exp(-6 \log^{\frac{1}{3}} N) d\alpha\right) \\ &\quad = \int_{\mathfrak{M}_0} g_2^2(\alpha) g_3^2(\alpha) g_6^2(\alpha) e(-\alpha n) d\alpha + O\left(n \exp(-\log^{\frac{1}{4}} n)\right). \end{aligned} \tag{3.1}$$

It is easy to see that

$$\begin{aligned}\int_{\mathfrak{M}_0} g_2^2(\alpha)g_3^2(\alpha)g_6^2(\alpha)e(-\alpha n)d\alpha \\ = \sum_{q\leqslant Q_0^{100}} A(q,n)\int_{|\lambda|\leqslant \frac{Q_0^{100}}{N}} u_2^2(\lambda)u_3^2(\lambda)u_6^2(\lambda)e(-\lambda n)d\lambda.\end{aligned}\tag{3.2}$$

Moreover, it follows from [3, Lemma 7.16] that

$$\begin{aligned}&\int_{|\lambda|\leqslant \frac{Q_0^{100}}{N}} u_2^2(\lambda)u_3^2(\lambda)u_6^2(\lambda)e(-\lambda n)d\lambda \\ &= \int_0^1 u_2^2(\lambda)u_3^2(\lambda)u_6^2(\lambda)e(-\lambda n)d\lambda + O\left(\int_{\frac{Q_0^{100}}{N}}^1 \frac{1}{\lambda^2\log^6 N}d\lambda\right) \\ &= \int_0^1 u_2^2(\lambda)u_3^2(\lambda)u_6^2(\lambda)e(-\lambda n)d\lambda + O(NQ_0^{-100}).\end{aligned}\tag{3.3}$$

Similar to [3, Lemma 7.19], we have

$$\begin{aligned}&\int_0^1 u_2^2(\lambda)u_3^2(\lambda)u_6^2(\lambda)e(-\lambda n)d\lambda \\ &= \frac{\Gamma^2(\frac{3}{2})\Gamma^2(\frac{4}{3})\Gamma^2(\frac{7}{6})}{\Gamma(2)}\frac{n}{\log^6 n} + O\left(\frac{n\log\log n}{\log^7 n}\right).\end{aligned}\tag{3.4}$$

By (3.3) and (3.4), we obtain

$$\begin{aligned}&\int_{|\lambda|\leqslant \frac{Q_0^{100}}{N}} u_2^2(\lambda)u_3^2(\lambda)u_6^2(\lambda)e(-\lambda n)d\lambda \\ &= \frac{\Gamma^2(\frac{3}{2})\Gamma^2(\frac{4}{3})\Gamma^2(\frac{7}{6})}{\Gamma(2)}\frac{n}{\log^6 n} + O\left(\frac{n\log\log n}{\log^7 n}\right).\end{aligned}\tag{3.5}$$

From Lemma 2.2(i) and the inequality $\varphi(q) \gg \frac{q}{\log q}$, we get

$$\begin{aligned}\sum_{q\leqslant Q_0^{100}} A(q,n) &= \mathfrak{S}(n) + O\left(\sum_{q>Q_0^{100}} q^{-2+\varepsilon}\right) \\ &= \mathfrak{S}(n) + O(Q_0^{-200+\varepsilon}).\end{aligned}\tag{3.6}$$

On combining (3.1), (3.2), (3.5) and (3.6), we get

$$\begin{aligned}&\int_{\mathfrak{M}_0} f_2^2(\alpha)f_3^2(\alpha)f_6^2(\alpha)e(-\alpha n)d\alpha \\ &= \frac{\Gamma^2(\frac{3}{2})\Gamma^2(\frac{4}{3})\Gamma^2(\frac{7}{6})}{\Gamma(2)}\mathfrak{S}(n)\frac{n}{\log^6 n} + O\left(\frac{n\log\log n}{\log^7 n}\right).\end{aligned}$$

This completes the proof of the proposition. □

3.2 *The estimation of the integrals over* $\mathfrak{m}_j$ $(j = 1, 2)$

We denote by $Z_j(N)$ the set of even integers n, $\frac{N}{2} < n \leqslant N$ for which the inequality

$$\left|\int_{\mathfrak{m}_j} f_2^2(\alpha) f_3^2(\alpha) f_6^2(\alpha) e(-\alpha n) d\alpha\right| \geqslant \frac{n \log\log n}{\log^7 n} \geqslant \frac{n}{\log^7 n} \tag{3.7}$$

holds. For simplicity, we abbreviate the cardinality of $Z_j(N)$ to Z_j. Next, define the complex number $\xi_j(n)$ by taking $\xi_j(n) = 0$ for $n \notin Z_j(N)$, and for $n \in Z_j(N)$ by means of the equation

$$\begin{aligned}&\left|\int_{\mathfrak{m}_j} f_2^2(\alpha) f_3^2(\alpha) f_6^2(\alpha) e(-\alpha n) d\alpha\right| \\ &\quad = \xi_j(n) \int_{\mathfrak{m}_j} f_2^2(\alpha) f_3^2(\alpha) f_6^2(\alpha) e(-\alpha n) d\alpha.\end{aligned} \tag{3.8}$$

Plainly, one has $|\xi_j(n)| = 1$ whenever $\xi_j(n)$ is non-zero. Thus, we have

$$\begin{aligned}&\sum_{n \in Z_j(N)} \xi_j(n) \int_{\mathfrak{m}_j} f_2^2(\alpha) f_3^2(\alpha) f_6^2(\alpha) e(-\alpha n) d\alpha \\ &\quad = \int_{\mathfrak{m}_j} f_2^2(\alpha) f_3^2(\alpha) f_6^2(\alpha) K_j(\alpha) d\alpha,\end{aligned} \tag{3.9}$$

where the exponential sum $K_j(\alpha)$ is defined by

$$K_j(\alpha) = \sum_{n \in Z_j(N)} \xi_j(n) e(-\alpha n).$$

Let

$$I_j = \int_{\mathfrak{m}_j} f_2^2(\alpha) f_3^2(\alpha) f_6^2(\alpha) K_j(\alpha) d\alpha \quad (j = 1, 2).$$

By (3.7)–(3.9), we get

$$I_j \geqslant \sum_{n \in Z_j(N)} \frac{n}{\log^7 n} \gg \frac{Z_j N}{\log^7 N}. \tag{3.10}$$

3.2.1 *The estimation of* Z_1

We now establish our estimate for Z_1. An application of the Cauchy–Schwarz inequality yields the inequality

$$\begin{aligned}I_1 \ll &\left(\max_{\alpha \in \mathfrak{m}_1} |f_3(\alpha)||f_6(\alpha)|\right) \left(\int_0^1 |f_2(\alpha) K_1(\alpha)|^2 d\alpha\right)^{\frac{1}{2}} \\ &\left(\int_0^1 |f_2(\alpha) f_3(\alpha) f_6(\alpha)|^2 d\alpha\right)^{\frac{1}{2}}.\end{aligned} \tag{3.11}$$

By [9, equation (2.4)] and the upper bound $|\xi_j(n)| \leqslant 1$, we obtain

$$
\begin{aligned}
\int_0^1 |f_2(\alpha)K_j(\alpha)|^2 d\alpha &= \sum_{p_1,p_2\leqslant N^{\frac{1}{2}}} \sum_{\substack{m,n\in Z_j(N)\\ p_1^2-p_2^2=n-m}} \xi_j(m)\overline{\xi_j(n)} \\
&\ll \sum_{p_1,p_2\leqslant N^{\frac{1}{2}}} \sum_{\substack{m,n\in Z_j(N)\\ p_1^2-p_2^2=n-m}} 1 \\
&\ll N^{\varepsilon}(Z_j N^{\frac{1}{2}} + Z_j^2). \qquad (3.12)
\end{aligned}
$$

On combining Lemma 2.1, Lemma 2.5 with (3.11)–(3.12), we find that

$$
\begin{aligned}
I_1 &\ll N^{\frac{559}{576}+\varepsilon}(Z_1 N^{\varepsilon} + Z_1^{\frac{1}{2}} N^{\frac{1}{4}+\varepsilon}) \\
&\ll Z_1 N^{\frac{559}{576}+\varepsilon} + Z_1^{\frac{1}{2}} N^{\frac{703}{576}+\varepsilon}. \qquad (3.13)
\end{aligned}
$$

Hence, (3.10) and (3.13) reveal that

$$
Z_1 \ll N^{\frac{127}{288}+\varepsilon}. \qquad (3.14)
$$

3.2.2 *The estimation of Z_2*

In order to estimate Z_2, we note that from Lemma 2.6, we have

$$
\begin{aligned}
I_2 &\ll \int_{\mathfrak{m}_2} |V^{\frac{1}{4}}(\alpha) f_2^{\frac{7}{4}}(\alpha) f_3^2(\alpha) f_6^2(\alpha) K_2(\alpha)| d\alpha \\
&\quad + N^{\frac{3}{32}+\varepsilon} \int_{\mathfrak{m}_2} |f_2^{\frac{7}{4}}(\alpha) f_3^2(\alpha) f_6^2(\alpha) K_2(\alpha)| d\alpha \\
&\ll \int_{\mathfrak{m}_2} |V^2(\alpha) f_3^2(\alpha) f_6^2(\alpha) K_2(\alpha)| d\alpha \\
&\quad + N^{\frac{21}{32}+\varepsilon} \int_{\mathfrak{m}_2} |V^{\frac{1}{4}}(\alpha) f_3^2(\alpha) f_6^2(\alpha) K_2(\alpha)| d\alpha \\
&\quad + N^{\frac{3}{32}+\varepsilon} \int_{\mathfrak{m}_2} |f_2^{\frac{7}{4}}(\alpha) f_3^2(\alpha) f_6^2(\alpha) K_2(\alpha)| d\alpha \\
&= I_{21} + I_{22} + I_{23}\text{, say.} \qquad (3.15)
\end{aligned}
$$

In view of the fact that $\mathfrak{m}_2 \subset \mathcal{I}$, where $\mathcal{I}$ is defined in Lemma 2.7, and the trivial bound $|K_2(\alpha)| \leqslant Z_2$, we get

$$
\begin{aligned}
I_{21} &\ll \max_{\alpha\in\mathfrak{m}_2} (|f_3(\alpha)|^2 |f_6(\alpha)|^2 |K_2(\alpha)|) \times \int_{\mathfrak{m}_2} |V(\alpha)|^2 d\alpha \\
&\ll \frac{N Z_2}{\log^{10A} N} \times \int_{\mathcal{I}} |V(\alpha)|^2 d\alpha \\
&\ll \frac{N Z_2}{\log^{A} N}, \qquad (3.16)
\end{aligned}
$$

here we employed the stronger form of Hua's inequality, Lemma 2.7 and the upper bound

$$\max_{\alpha\in\mathfrak{m}_2}|f_j(\alpha)| \ll \frac{N^{\frac{1}{j}}}{\log^{10A} N}, \quad (j=3,6) \tag{3.17}$$

which follows from arguments similar to the proof of Lemma 2.6. Moreover, we deduce from Cauchy–Schwarz inequality, (3.12) and (3.17) that

$$\begin{aligned} I_{22} &\ll N^{\frac{21}{32}+\varepsilon} \max_{\alpha\in\mathfrak{m}_2}(|f_3(\alpha)|^2|f_6(\alpha)|^2|K_2(\alpha)|) \times \int_{\mathfrak{m}_2}|V(\alpha)|^{\frac{1}{4}}d\alpha \\ &\ll \frac{N^{\frac{25}{32}+\varepsilon}Z_2}{\log^A N}. \end{aligned} \tag{3.18}$$

Similarly, it follows from Hölder inequality, Hua's inequality, Lemma 2.1, (3.12) and (3.17)that

$$\begin{aligned} I_{23} &\ll N^{\frac{3}{32}+\varepsilon} \max_{\alpha\in\mathfrak{m}_2}\left(|f_6(\alpha)||K_2(\alpha)|^{\frac{1}{4}}\right) \times \int_{\mathfrak{m}_2}|f_2^{\frac{7}{4}}(\alpha)f_3^2(\alpha)f_6(\alpha)K_2^{\frac{3}{4}}(\alpha)|d\alpha \\ &\ll N^{\frac{3}{32}+\varepsilon} \max_{\alpha\in\mathfrak{m}_2}\left(|f_6(\alpha)||K_2(\alpha)|^{\frac{1}{4}}\right) \times \left(\int_0^1|f_2(\alpha)f_3(\alpha)f_6(\alpha)|^2d\alpha\right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^1|f_3(\alpha)|^8d\alpha\right)^{\frac{1}{8}}\left(\int_0^1|f_2(\alpha)K_2(\alpha)|^2d\alpha\right)^{\frac{3}{8}} \\ &\ll N^{\frac{31}{32}+\varepsilon}Z_2 + N^{\frac{37}{32}+\varepsilon}Z_2^{\frac{5}{8}}. \end{aligned} \tag{3.19}$$

A combination of (3.15), (3.16) together with (3.18),(3.19) then yields

$$I_2 \ll \frac{Z_2 N}{\log^A N} + Z_2^{\frac{5}{8}}N^{\frac{37}{32}+\varepsilon.} \tag{3.20}$$

It follows from (3.10) and (3.20) that

$$Z_2 \ll N^{\frac{5}{12}+\varepsilon}. \tag{3.21}$$

4. Proof of Theorem 1

Let $Z(N)$ denote the number of even integers n in the interval $[N/2, N]$ such that the asymptotic formula (1.8) fails to hold. On recalling (3.14) and (3.21), we arrive at the conclusion that

$$Z(N) \leqslant Z_1 + Z_2 \ll N^{\frac{127}{288}+\varepsilon}. \tag{4.1}$$

From (4.1), we get

$$\begin{aligned} \mathcal{E}(N) &\leqslant N^{\frac{127}{288}} + \sum_{0\leqslant j\leqslant J} Z\left(\frac{N}{2^j}\right) \\ &\ll N^{\frac{127}{288}+\varepsilon}, \end{aligned} \tag{4.2}$$

where J is chosen in such a way that $2^{J-1} < N^{\frac{161}{288}} \leqslant 2^J$. Now by (4.2), the proof of Theorem 1 is completed.

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