Proc. Indian Acad. Sci. (Math. Sci.) (2020) 130:6
https://doi.org/10.1007/s12044-019-0530-8

A simple method to extract zeros of certain Eisenstein series of small level

ARADHITA CHATTOPADHYAYA

Department of Centre for High Energy Physics, Indian Institute of Science, Bangalore 560 012, India
Present Address: Department of Mathematics, Trinity College Dublin, Ireland
Email: aradhitac@iisc.ac.in; aradhita@maths.tcd.ie

MS received 11 March 2019; revised 15 June 2019; accepted 22 June 2019

Abstract. This paper provides a simple method to extract the zeros of some weight two Eisenstein series of level N where $N = 2, 3, 5$ and 7. The method is based on the observation that these Eisenstein series are integral over the graded algebra of modular forms on $SL(2, Z)$ and their zeros are 'controlled' by those of E_4 and E_6 in the fundamental domain of $\Gamma_0(N)$.

Keywords. Zeros of Eisenstein series; zeros of modular forms of $\Gamma_0(N)$.

2000 Mathematics Subject Classification. 11F11, 11F03.

1. Introduction

In the work of Rankin and Swinnerton-Dyer [16], the location of zeros of all Eisenstein series E_k (of weight $k \geq 4$, even) of full modular group $SL_2(\mathbb{Z})$ had been determined. In the fundamental domain this was found to be always on the arc $|\tau| = 1$, with $2\pi/3 \geq \arg(\tau) \geq \pi/2$. The method has been generalized to Fricke groups in recent works of [17] and to the subgroups of $SL_2(\mathbb{Z})$ in [13,14]. The zeros of weight two Eisenstein series $E_2(q)$ were studied by [11,19]. In this work, we shall find the zeros of Eisenstein series $\tilde{E}_N$ (in general, $\tilde{E}_N$ is defined as negative of what it is defined here in (1.1) which are holomorphic modular forms of weight 2 of $\Gamma_0(N)$ defined as

$$\tilde{E}_N(\tau) := \frac{1}{N-1}(NE_2(N\tau) - E_2(\tau)), \tag{1.1}$$

where $E_2(\tau)$ is the quasimodular Eisenstein series defined by

$$E_2(\tau) := 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n, \qquad q = \mathrm{e}^{2\pi i\tau}, \tag{1.2}$$

with $\sigma_1(n)$ being the sum over all the divisors of n.

The method which we present is quite different from that of [16], however can only be applied for $N = 2, 3, 5, 7$. Our main observation is that $\tilde{E}_N$ is integral over the graded algebra

$$M(SL_2(\mathbb{Z})) = \bigoplus_{k\geq 0} M_k(SL_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$$

Published online: 09 January 2020

Table 1. Zeros of $\tilde{E}_N(\tau)$ for $N = 2, 3, 5, 7$.

N	2	3	5	7
τ	$-\frac{1}{i+1}$	$-\frac{1}{e^{2\pi i/3}+2}$	$-\frac{1}{i+2}$, $-\frac{1}{i+3}$	$-\frac{1}{e^{2\pi i/3}+3}$, $-\frac{1}{e^{2\pi i/3}+5}$

with $E_4,\ E_6$ being the Eisenstein series of weight 4 and 6 defined as

$$E_4(\tau) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \tag{1.3}$$

$$E_6(\tau) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \tag{1.4}$$

where $q = e^{2\pi i \tau},\ \tau \in \mathbb{H}$.

Let us denote the fundamental domain for $\Gamma_0(N)$ by F_N. For $N = 2, 3, 5, 7$, the zeros of $\tilde{E}_N$ are 'controlled' (2) by those of E_4 and E_6 in F_N.

We now state the main result of our paper which would be proved in section 4.

Theorem 1. *All the zeros of $\tilde{E}_N(\tau)$ in the fundamental domain of $\Gamma_0(N)$ lie as in Table 1.*

Before we move into the proof of the theorem, we shall first provide a brief description where these modular forms occur in the context of string theory in physics.

2. Physics motivation

Partition functions defined in string theory are generally related to modular functions. One such partition function defined over a $K3$ manifold is elliptic genus.

$$\begin{aligned} F(\tau, z) &= \mathrm{Tr}_{RR}\left[(-1)^{F_{K3}+\bar{F}_{K3}} e^{2\pi i z F_{K3}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\right], \\ &= \sum_{b=0}^{1} \sum_{j \in 2\mathbb{Z}+b,\ n \in \mathbb{Z}} c_b(4n - j^2) e^{2\pi i n \tau + 2\pi i j z}. \end{aligned} \tag{2.1}$$

The trace in the above equation is taken over the Ramond–Ramond sector of the $\mathcal{N} = (4, 4)$ super conformal field theory of $K3$ with central charge (6, 6) and F refers to the Fermion number, L_0 and $\bar{L}_0$ are the scaling operators in the left moving and right moving CFTs. The result of the above trace can be given by a weak Jacobi form of index 1 and weight 0 [10].

$$F(\tau, z) = 8A(\tau, z) = 8\left(\frac{\theta_2^2(\tau, z)}{\theta_2^2(\tau, 0)} + \frac{\theta_3^2(\tau, z)}{\theta_3^2(\tau, 0)} + \frac{\theta_4^2(\tau, z)}{\theta_4^2(\tau, 0)}\right). \tag{2.2}$$

Here $\theta_i(\tau, z)$ are the Jacobi theta functions.

There are symplectic automorphisms of $K3$ related to the conjugacy classes of Mathieu group M_{24} and we can define a partition function similar to the elliptic genus for these orbifolds of $K3$ called 'twisted elliptic genus' of $K3$. This is defined as

$$F^{(r,s)}(\tau,z) = \frac{1}{N}\mathrm{Tr}_{RR\ g'^r}\left[(-1)^{F_{K3}+\bar{F}_{K3}} g'^s \mathrm{e}^{2\pi i z F_{K3}} q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right],$$
$$= \sum_{b=0}^{1}\sum_{j\in 2\mathbb{Z}+b,\ n\in\mathbb{Z}/N} c_b^{(r,s)}(4n-j^2)\mathrm{e}^{2\pi i n\tau+2\pi i j z}.$$
$$0\le r,s\le N-1. \tag{2.3}$$

In particular, if $N=2,3,5,7$, we can write the twisted elliptic genus of $K3$ as follows [6,7]:

$$F^{(0,0)}(\tau,z) = \frac{8}{N}A(\tau,z),$$
$$F^{(0,s)}(\tau,z) = \frac{8}{N(N+1)}A(\tau,z) - \frac{2}{N+1}B(\tau,z)\tilde{E}_N(\tau),$$
$$\text{for } 1\le s\le (N-1),$$
$$F^{(r,rk)} = \frac{8}{N(N+1)}A(\tau,z) + \frac{2}{N(N+1)}\tilde{E}_N\left(\frac{\tau+k}{N}\right)B(\tau,z),$$
$$\text{for } 1\le r\le (N-1),\ 1\le k\le (N-1) \tag{2.4}$$

where $B(\tau,z)=\frac{\theta_1^2(\tau,z)}{\eta^6(\tau)}$. $B(\tau,z)$ is a weak Jacobi form of weight -2 and index 1.

This quantity twisted elliptic genus is an ingredient in computing the statistical entropy of black holes in $\mathcal{N}=4$ type IIA/B string theory in 4 space-time dimensions [6,8].

Another instance where these modular functions $\tilde{E}_N$ is observed is in the context of computing new supersymmetric index in heterotic strings [2,15,18] compactified on order N orbifolds of $K3\times T^2$ and $E_8\times E_8$. This is defined by

$$\mathcal{Z}_{\text{new}}(q,\bar{q}) = \frac{1}{\eta^2(\tau)}\mathrm{Tr}_R\left(F\mathrm{e}^{i\pi F}q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right). \tag{2.5}$$

Here the trace is performed over the Ramond sector in the internal CFT with central charges $(c,\bar{c})=(22,9)$. F refers to the world sheet fermion number of the right moving $\mathcal{N}=2$ supersymmetric internal CFT. For a general embedding in case of an order 2 orbifold of $K3$, the new super-symmetric index can be given by

$$\begin{aligned}\mathcal{Z}_{\text{new}} = -\frac{1}{\eta^{24}}\Bigg\{ & 2\Gamma_{2,2}^{(0,0)}E_4E_6 \\ & +\Gamma_{2,2}^{(0,1)}\left[(E_6+2\tilde{E}_2(\tau)E_4)\left(\hat{b}\tilde{E}_2^2(\tau)+\left(\frac{2}{3}-\hat{b}\right)E_4\right)\right] \\ & +\Gamma_{2,2}^{(1,0)}\left[\left(E_6-\tilde{E}_2\left(\frac{\tau}{2}\right)E_4\right)\left(\frac{\hat{b}}{4}\tilde{E}_2^2\left(\frac{\tau}{2}\right)+\left(\frac{2}{3}-\hat{b}\right)E_4\right)\right] \\ & +\Gamma_{2,2}^{(1,1)}\left[\left(E_6-\tilde{E}_2\left(\frac{\tau+1}{2}\right)E_4\right)\left(\frac{\hat{b}}{4}\tilde{E}_2^2\left(\frac{\tau+1}{2}\right)+\left(\frac{2}{3}-\hat{b}\right)E_4\right)\right]\Bigg\},\end{aligned} \tag{2.6}$$

where $\Gamma_{2,2}^{(r,s)}$ is a lattice sum depending on the moduli of the torus T^2 in the heterotic theory and $\hat{b}$ is a rational number. Similarly for other orbifolds of $K3$, one can still write the $\mathcal{Z}_{\text{new}}$ in terms of Eisenstein series of $SL(2,\mathbb{Z})$ and $\Gamma_0(N)$ [4]. This index is used to calculate the gauge and gravitational coupling corrections in heterotic strings which predicts the existence of different Calabi Yau 3-folds using heterotic-type II string duality [5].

3. Notations and preliminaries

For an even integer $k \geq 2$, we denote the space of all modular forms on $SL(2,\mathbb{Z})$ by M_k, the space of all cusp forms on $SL(2,\mathbb{Z})$ by S_k, the space of all modular forms on $\Gamma_0(N)$ by $M_k(\Gamma_0(N))$ and the space of all cusp forms on $\Gamma_0(N)$ by $S_k(\Gamma_0(N))$.

Let $f(z)$ be a holomorphic function in the upper half plane and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q})$, then

$$f|_k\gamma = \det(\gamma)^{k/2}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

With $\omega_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ denoting the usual Fricke involution on M_k, one can readily verify the fact that $\tilde{E}_N|_2\, \omega_N = -\tilde{E}_N$.

Let F_Γ be the fundamental domain (see [1] for more details and definition) of $\Gamma := SL_2(\mathbb{Z})$ given by the region in the upper half plane such as $\tau = x + iy$, where $-1/2 \leq x < 1/2$ and $x^2 + y^2 \geq 1$. Then F_N, the fundamental domain for $\Gamma_0(N)$ for a prime N can be given as [1]

$$F_N = F_\Gamma \cup \bigcup_{k=0}^{N-1} ST^k(F_\Gamma), \tag{3.1}$$

where $\{\{ST^s\}_{s=0}^{N-1} \cup \mathrm{Id}\}$ is a set of coset representatives of $\Gamma_0(N)$ in Γ.

The valence formula: Let $f \neq 0$ be a modular form of weight $k \geq 2$, even in $SL(2,\mathbb{Z})$, $v_p(f)$ is the order of the zero of f at point $p \in F_\Gamma$ and $\rho = \mathrm{e}^{2\pi i/3}$. Then the valence formula gives

$$v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\rho(f) + \sum_{p\in\Gamma,\, p\neq i,\rho} v_p = \frac{k}{12}. \tag{3.2}$$

From the above valence formula, one obtains that in the fundamental domain of the full modular group the only zeros of E_4 and E_6 lie at $\tau = \mathrm{e}^{2\pi i/3}$ and $\tau = i$ respectively.

$$E_6(i) = 0, \qquad E_4(\mathrm{e}^{2\pi i/3}) = 0. \tag{3.3}$$

4. Proof of Theorem 1

We shall begin this section with the following lemmas regarding the zeros of Eisenstein series which are essential to prove Theorem 1.

Lemma 1. *If $\tau \in F_N$ and $E_6(\tau) = 0$, then either $\tau = i$ or $\tau = \frac{-1}{i+s}$. Similarly if $\tau \in F_N$ and $E_4(\tau) = 0$, then either $\tau = \mathrm{e}^{2\pi i/3}$ or $\tau = \frac{-1}{\mathrm{e}^{2\pi i/3}+s}$, where $0 \leq s \leq N-1$ and these are the only possible zeros of E_4 and E_6 in F_N.*

Proof. The lemma follows from the expression of fundamental domain F_N (3.1) and the zeros of E_4 and E_6 in F_Γ. □

Lemma 2. *There exists a polynomial expression for $N = 2, 3, 5, 7$,*

$$(\tilde{E}_N(\tau))^{N+1} = \sum_{i=0}^{N-1} a_i (\tilde{E}_N(\tau))^i m_{N+1-i}(\tau),$$

where a_i are constants and $m_{N+1-i} \in M_{2(N+1-i)}(SL_2(\mathbb{Z}))$. These polynomials are given by

$$\tilde{E}_2(\tau)^3 = \frac{1}{4}E_6(\tau) + \frac{3}{4}E_4\tilde{E}_2(\tau), \tag{4.1}$$

$$\tilde{E}_3(\tau)^4 = \frac{1}{27}E_4^2(\tau) + \frac{8}{27}E_6(\tau)\tilde{E}_3(\tau) + \frac{2}{3}\tilde{E}_3(\tau)^2 E_4(\tau), \tag{4.2}$$

$$\begin{aligned}\tilde{E}_5(\tau)^6 = {} & \frac{1}{3125}E_6^2(\tau) + \frac{24}{3125}E_6E_4\tilde{E}_5(\tau) + \frac{9}{125}E_4^2\tilde{E}_5(\tau)^2 + \frac{8}{25}E_6 \\ & \tilde{E}_5(\tau)^3 + \frac{3}{5}\tilde{E}_5(\tau)^4 E_4(\tau),\end{aligned} \tag{4.3}$$

$$\begin{aligned}\tilde{E}_7(\tau)^8 = {} & \frac{1}{7^7}E_4^4 + \frac{48}{7^7}E_4^2E_6\tilde{E}_7 + \left(\frac{64}{64827}E_6^2 + \frac{92}{453789}E_4^3\right)\tilde{E}_7^2 \\ & + \frac{32}{2401}E_4E_6\tilde{E}_7^3 + \frac{30}{2401}E_4^2\tilde{E}_7^4 + \frac{16}{49}E_6\tilde{E}_7^5 + \frac{4}{7}\tilde{E}_7^6E_4.\end{aligned} \tag{4.4}$$

Proof. Considering the q expansion of both sides of the above equations up to Strum's bound, the lemma follows. The program zeros.nb is given with the arxiv version of this paper [3] for reference. □

Proof of Theorem 1. From Lemma 1, we know the location of all zeros of E_4 and E_6 in F_N. Let us denote these sets of zeros in F_N as $\mathbb{L}_{4,N}$ and $\mathbb{L}_{6,N}$ respectively. Also, let us denote the set of all zeros of $\tilde{E}_N$ in F_N as $\tilde{\mathbb{L}}_N$. From Lemma 2, we observe that

$$\tilde{\mathbb{L}}_2 \subseteq \mathbb{L}_{6,2}, \qquad \tilde{\mathbb{L}}_3 \subseteq \mathbb{L}_{4,3}, \qquad \tilde{\mathbb{L}}_5 \subseteq \mathbb{L}_{6,5}, \qquad \tilde{\mathbb{L}}_7 \subseteq \mathbb{L}_{4,7}.$$

Let us chose an element $\omega_{4,N}$ from $\mathbb{L}_{4,N}$ for $N = 3, 7$ and $\omega_{6,N}$ from $\mathbb{L}_{6,N}$ for $N = 2, 5$. Now we re-write the resulting polynomial equations (4.1)–(4.4) as a product of two factors as follows:

$$\tilde{E}_2(\omega_{6,2})\left(\tilde{E}_2(\omega_{6,2})^2 - \frac{3}{4}E_4(\omega_{6,2})\right) = 0, \tag{4.5}$$

$$\tilde{E}_3(\omega_{4,3})\left(\tilde{E}_3(\omega_{4,3})^3 - \frac{8}{27}E_6(\omega_{4,3})\right) = 0, \tag{4.6}$$

$$\tilde{E}_5(\omega_{6,5})^2\left(\tilde{E}_5(\omega_{6,5})^4 - \frac{3}{5}\tilde{E}_5(\omega_{6,5})^2E_4(\omega_{6,5}) - \frac{9}{125}E_4(\omega_{6,5})^2\right) = 0, \tag{4.7}$$

$$\tilde{E}_7^2(\omega_{4,7})\left(\tilde{E}_7(\omega_{4,7})^6 - \frac{16}{49}E_6(\omega_{4,7})\tilde{E}_7(\omega_{4,7})^3 - \frac{64}{64827}E_6(\omega_{4,7})^2\right) = 0. \tag{4.8}$$

Since the right-hand side of the above set of equations are zero so at least one of these factors must be zero. Now we need to check the numerical values of these factors to determine the location of zeros of $\tilde{E}_N$ in F_N. □

So we estimate the bounds of $\tilde{E}_N(\omega)$, where $\omega = \omega_{6,N}$ for $N = 2, 5$ and $\omega = \omega_{4,N}$ for $N = 3, 7$. We write the q expansion of $\tilde{E}_N$ as follows:

$$\tilde{E}_N(\tau) = \sum_{n=0}^{m} a_n q^n + \sum_{n=m+1}^{\infty} a_n q^n = \sum_{n=0}^{m} a_n q^n + R(m, q).$$

Now we have

$$\begin{aligned} |R(m,q)| &= |\sum_{n=m+1}^{\infty} a_n (x+iy)^n| \leq \sum_{n=m+1}^{\infty} |a_n|(|x|+|y|)^n \\ &\leq \sum_{n=m+1}^{\infty} b_n(|x|+|y|)^n < \sum_{n=m+1}^{\infty} b_n(|x|+|y|+\epsilon)^n, \end{aligned} \tag{4.9}$$

where $b_n = (N+1)n(n+1)/2 > (N+1)\sigma_1(n) > |a_n|$, $q = x + iy$, $\epsilon > 0$ (ϵ is needed to approximate $|x|$ and $|y|$ in the computer program. Also, $\sum_{n=1}^{m} a_n q^n$ is done up to machine precision.)

Note that our choice of b_n is such that $\sum_{n=m+1}^{\infty} b_n(|x| + |y| + \epsilon)^n$ can be estimated exactly in terms of $|x| + |y| + \epsilon$ and m. Also, note that $\sum_{n=m+1}^{\infty} b_n(|x| + |y| + \epsilon)^n$ is convergent only if $|x| + |y| + \epsilon < 1$.

In the region $|x| + |y| + \epsilon < 1$, we estimate the numerical value of $\tilde{E}_N(\omega)$ up to first $m + 1$ terms in the q expansion and using equation (4.9) we estimate the upper-bound of $|R(m, q)|$ in F_N. From this we have estimated an upper bound of $\tilde{E}_N(\omega)$ for the values of ω as $\omega = \tau$ as in Table 1) and lower bound of $|\tilde{E}_N(\omega)|$ for the rest of the values of $\omega \in \mathbb{L}_{4,N}$ for $N = 3, 7$ and $\omega \in \mathbb{L}_{6,N}$ for $N = 2, 5$ (see Table 2 for reference).

However if $|x| + |y| \geq 1$ (or close to 1), we use the following results:

$$\begin{aligned} \tilde{E}_N\left(\frac{\tau+s}{N}\right) &= \tilde{E}_N\left(\frac{\tau+s-N}{N}\right), \\ \tilde{E}_N\left(-\frac{1}{\tau+s}\right) &= -\frac{(\tau+s)^2}{N}\tilde{E}_N\left(\frac{\tau+s}{N}\right), \end{aligned} \tag{4.10}$$

which is true for any $k \in \mathbb{Z}$. This happens at $\omega \in \mathbb{L}_{6,5}$ at $\omega = \frac{-1}{i+4}$ and $\omega \in \mathbb{L}_{4,7}$ at $\omega = \frac{-1}{e^{2\pi i/3}+5}, \frac{-1}{e^{2\pi i/3}+6}$. At these points we can evaluate the result at $\omega = \frac{-1}{i-1}, \frac{-1}{e^{2\pi i/3}-2}, \frac{-1}{e^{2\pi i/3}-1}$ respectively. These results can be related back to the fundamental domain points using equation (4.11). This implies that

$$\tilde{E}_N\left(-\frac{1}{\tau+s-N}\right) = \frac{(\tau+s-N)^2}{(\tau+s)^2}\tilde{E}_N\left(-\frac{1}{\tau+s}\right), \tag{4.11}$$

where $0 \leq s \leq N-1$. Putting $\omega = -\frac{1}{\tau+s}$ in (4.11), we have

$$\tilde{E}_N\left(\frac{1}{1/\omega+N}\right) = (1/\omega+N)^2\omega^2\tilde{E}_N(\omega). \tag{4.12}$$

Note that if $\tau \in F_\Gamma$, then for $0 \leq s \leq N-1$, we have $\frac{-1}{\tau+s} \in F_N$ and $\frac{-1}{\tau+s-N} \notin F_N$. So if $\omega = \omega_{4,N}$ for $N = 3, 7$ and $\omega = \omega_{6,N}$ for $N = 2, 5$ then equation (4.12) implies that there is a relation between the value of $\tilde{E}_N$ at a point outside $\mathbb{L}_{4,N}$ (respectively $\mathbb{L}_{6,N}$) and a point inside $\mathbb{L}_{4,N}$ (respectively $\mathbb{L}_{6,N}$). So at the point $\tau = \frac{1}{1/\omega+N}$, where $|\mathfrak{Re}(e^{2\pi i\tau})| + |\mathfrak{Im}(e^{2\pi i\tau})| < 1$ (and not very close to 1) we repeat the process as before.

Table 2. Numerical estimates of $|\tilde{E}_N(\omega)|$ up to q^{100} terms in the expansion of $\tilde{E}_N(\omega)$ using machine precision up to 1000 digits in Mathematica and $R^{>}(100, q)$ is an upper bound on the remainder term $R(100, q)$.

N	ω	$\lvert\tilde{E}_N(\omega)\rvert$	$R^{>}(100, q)$
2	i	1.0449	10^{-269}
	$\frac{-1}{i+1}$	0.0	10^{-132}
3	$e^{2\pi i/3}$	0.948674	10^{-226}
	$\frac{-1}{e^{2\pi i/3}+2}$	3.14185×10^{-77}	10^{-73}
5	i	1.01127	10^{-269}
	$\frac{-1}{i+1}$	0.77254	
	$\frac{-1}{i+2}$	2.96211×10^{-53}	10^{-33}
	$\frac{-1}{i+3}$	1.65345×10^{-25}	10^{-12}
	$\frac{-1}{i-1}$	0.77254	10^{-269}
7	$e^{2\pi i/3}$	0.98289	10^{-226}
	$\frac{-1}{e^{2\pi i/3}+2}$	0.615423	10^{-73}
	$\frac{-1}{e^{2\pi i/3}+3}$	2.63251×10^{-32}	10^{-16}
	$\frac{-1}{e^{2\pi i/3}+4}$	2.66684	10^{-7}
	$\frac{-1}{e^{2\pi i/3}-2}$	-2.63251×10^{-32}	10^{-12}
	$\frac{-1}{e^{2\pi i/3}-1}$	0.615423	10^{-73}

Thus we can estimate the bounds of $|\tilde{E}_N(\tau)|$ which in turn puts the bounds on $|\tilde{E}_N(\omega)|$ (numerical estimates can be found in table 2).

Estimates. Now we shall list the estimates of $|\tilde{E}_N(\omega)|$ up to q^m terms in the expansion of $\tilde{E}_N(\omega)$, where $\omega \in \mathbb{L}_{4,N}$ for $N = 3, 7$ and $\omega \in \mathbb{L}_{6,N}$ for $N = 2, 5$.

For $m = 100$ in all the above cases, $|R(m, q)| < 10^{-7}$. Using these bounds we get an upper bound of $|\tilde{E}_N(\omega)|$ for $\omega = \tau$ as in Table 1 which can be given by $10^{-7} < 10^{-2}$ so the second factors in equations (4.5) to (4.8) can not be zero at these points, so $\tilde{E}_N(\omega) = 0$. We also obtain the lower bound of $|\tilde{E}_N(\omega)|$ for rest of the points in $\mathbb{L}_{4,N}$ for $N = 3, 7$ and $\mathbb{L}_{6,N}$ for $N = 2, 5$ and the bound is $>10^{-2}$. This shows that $\tilde{E}_N \neq 0$ at those points. So the only points in $\tilde{\mathbb{L}}_N$ are the ones mentioned in Table 1. For detailed calculations used above, see the Mathematica file zeros.nb attached with the arxiv version of the paper [3].

Thus we obtain the points where $\tilde{E}_N$ can be zero. Now to prove that these are the only zeros of $\tilde{E}_N$ in F_N as given in Table 1, we must check that at these points the second factor will not become zero. We shall do this as follows:

From the definition of E_4 and E_6 [(1.3) and (1.4)], we see that $E_4(i) > 1$ and $E_6(e^{2\pi i/3}) > 1$. Hence, from the modular transformation properties of E_4 and E_6 $|E_4(\omega_{6,2})| > 1$, $|E_4(\omega_{6,5})| > 1$ and $|E_6(\omega_{4,2})| > 1$, $|E_6(\omega_{4,7})| > 1$. So one can easily see that the second factor in equations (4.5) to (4.8) can only become zero if $|\tilde{E}_N(\tau)| > 10^{-2}$. So when the second factor would be zero that would never give a zero of $\tilde{E}_N(\omega)$ and vice versa. Now from the positive bounds of $|\tilde{E}_N|$ (see Table 2), we find all the zeros of $\tilde{E}_N(\omega)$ which are given as

$$\begin{aligned} \tilde{E}_2\left(\frac{-1}{i+1}\right) &= 0, \\ \tilde{E}_3\left(\frac{-1}{e^{2\pi i/3}+2}\right) &= 0, \\ \tilde{E}_5\left(\frac{-1}{i+2}\right) = \quad \tilde{E}_5\left(\frac{-1}{i+3}\right) &= 0, \\ \tilde{E}_7\left(\frac{-1}{e^{2\pi i/3}+3}\right) = \quad \tilde{E}_7\left(\frac{-1}{e^{2\pi i/3}+5}\right) &= 0. \end{aligned} \tag{4.13}$$

So these are the only possible zeros of $\tilde{E}_N$ in F_N. This completes the proof of Theorem 1.

5. Remarks

(1) The cusps of $\Gamma_0(N)$ are 0 and $i\infty$ for prime N. However from the q expansions it is obvious that $\tilde{E}_N(i\infty) = 1$. Also using Fricke involutions one can see that

$$\lim_{\epsilon\to 0} \tilde{E}_N\left(\frac{-1}{i\epsilon}\right) = \lim_{\epsilon\to 0} \frac{(-(i\epsilon)^2)}{N} \tilde{E}_N\left(\frac{i\epsilon}{N}\right),$$

so $\tilde{E}_N$ is non-zero at the cusps. So the only possible zeros of $\tilde{E}_N$ are at equations (4.13) in the fundamental domain F_N.

(2) This method does not easily generalize to $\tilde{E}_{11}$ (or higher prime numbers) because of the presence of cusp forms of weight 2 in $\Gamma_0(11)$. Also it does not easily generalize to composite numbers. However using moonshine symmetry the twisted elliptic genus of $K3$ is known to exist for all 26 conjugacy classes of M_{24} where higher N Eisenstein series are present [9,12].

(3) One may try to generalize the method to different types of modular functions where there is a possibility of finding a polynomial relation in terms of modular functions whose zeros are known.

(4) Equation (4.1) was used in finding the twisted elliptic genus, new supersymmetric index and gauge threshold corrections at one loop for the non-standard embedding of heterotic string compactified on $K3 \times T^2$, where the $K3$ was orbifolded with a $\mathbb{Z}_2$ automorphism corresponding to the 2A conjugacy class of Mathieu group M_{24} and a $1/2$ shift on one of the circles of T^2 [4].

Acknowledgements

The author would like to thank Council of Scientific and Industrial Research (CSIR) for the funding. She thanks the anonymous referee for suggestions and comments. This work would not have been possible without the help and guidance of Prof. Justin R David, Department of Centre for High Energy Physics, Indian Institute of Science, Bangalore and Prof. Soumya Das, Department of Mathematics, Indian Institute of Science, Bangalore. The author also thanks Ritwik Pal and Pramath Anamby, Department of Mathematics, Indian Institute of Science, Bangalore for useful discussions.

References

[1] Apostol T M, Modular Functions and Dirichlet Series in Number Theory (1976) (New York: Springer)

[2] Cardoso G Lopes, Curio G and Lust D, Perturbative couplings and modular forms in $N = 2$ string models with a Wilson line, *Nucl. Phys.* **B491** (1997) 147–183, [arxiv: hep-th/9608154]

[3] Chattopadhyaya A, A simple method to extract the zeros of some Eisenstein series of level N, arxiv: 1707.04502

[4] Chattopadhyaya A and David J R, $N = 2$ heterotic string compactifications on orbifolds of $K3 \times T^2$, *JHEP* **1701** (2017) 037

[5] Chattopadhyaya A and David J R, Gravitational couplings in $\mathcal{N} = 2$ string compactifications and Mathieu Moonshine, *JHEP* **1805** (2018) 211, arxiv: 1712.08791

[6] David J R, Jatkar D P and Sen A, Product representation of Dyon partition function in CHL models, *JHEP* **06** (2006) 064, [arxiv: hep-th/0602254]

[7] David J R and Sen A, CHL Dyons and Statistical Entropy Function from D1–D5 System, *JHEP* **11** (2006) 072, [arxiv: hep-th/0605210]

[8] Dijkgraaf R, Verlinde E P and Verlinde H L, Counting dyons in $N = 4$ string theory, *Nucl. Phys.* **B484** (1997) 543–561 [arxiv: hep-th/9607026]

[9] Eguchi T and Hikami K, Note on twisted elliptic genus of $K3$ surface, *Phys. Lett.* **B694** (2011) 446–455, [arXiv:1008.4924]

[10] Eguchi T, Ooguri H, Taormina A and Yang S-K, Superconformal Algebras and String Compactification on Manifolds with SU(N) Holonomy, *Nucl. Phys.* **B315** (1989) 193

[11] El Basraoui A and Sebbar A, Zeros of the Eisenstein series E_2, *Proc. Am. Math. Soc.* **138(7)** (2010) 2289–2299

[12] Gaberdiel M R, Hohenegger S and Volpato R, Mathieu twining characters for K3, *JHEP* **09** (2010) 058, [arXiv:1006.0221]

[13] Garthwaite S, Long L, Swisher H and Treneer S, Zeros of classical Eisenstein series and recent developments (2011)

[14] Garthwaite S, Long L, Swisher H and Treneer S, Zeros of some level 2 Eisenstein series, *Proc. Am. Math. Soc.* **138** (2010) 467–480

[15] Harvey J A and Moore G W, Algebras, BPS states, and strings, *Nucl. Phys.* **B463** (1996) 315–368, [arxiv: hep-th/9510182]

[16] Rankin F K C and Swinnerton-Dyer H P F, On the zeros of Eisenstein series, *Bull. London Math. Soc.* **2** (1970) 169–170

[17] Shigezumi J, On the zeros of the Eisenstein series for $\Gamma_0^*(5)$ and $\Gamma_0^*(7)$, *Kyushu J. Math.* **61(2)** (2007) 527–549

[18] Stieberger S, (0, 2) heterotic gauge couplings and their M theory origin, *Nucl. Phys.* **B541** (1999) 109–144, [arxiv: hep-th/9807124]

[19] Wood R and Young M P, Zeros of the weight two Eisenstein series, *J. Number Theory* **143** (2014) 320–333

Communicating Editor: A Raghuram