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Finite groups with exactly one composite conjugacy class size

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Abstract. A composite number is a positive integer that has at least one divisor integer other than 1 and itself. In this paper, we give a detailed structural description of a group if it has a unique composite conjugacy class size.

Keywords. Finite groups; conjugacy class sizes; quasi-Frobenius groups; composite number.

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1. Introduction

Throughout this paper, all groups are finite and G denotes a group. Let x be an element of G. It is denoted by $|x^G|$ the size of the conjugacy class containing x, and by $cs(G)$ the set of conjugacy class sizes of G. An element $x \in G$ is called a primary element if its order is a prime power. A number is a composite number if it is a positive integer having at least one divisor besides 1 and itself.

A group G is a quasi-Frobenius group if $G/\mathbf{Z}(G)$ is a Frobenius group. The inverse images of the kernel and a complement of $G/\mathbf{Z}(G)$ are then called the kernel and a complement of G. Let $\Gamma(G)$ be the simple undirected graph whose vertices are the distinct sizes of noncentral conjugacy classes of G, two of them being adjacent if and only if they are not coprime numbers. Let $n(\Gamma(G))$ denote the number of connected components of $\Gamma(G)$. All further unexplained notations are standard, and the readers may refer to [10].

It is a classic topic in finite group theory, to study the influence of conjugacy class sizes on the structure of groups. Also, it is well known that there are numerous analogies between results about irreducible character degrees and conjugacy class sizes of G. Motivated by the recent joint work of Liu and Liu [11], who investigated groups with exactly one composite character degree, we study groups with exactly one composite conjugacy class size.

Camina and Camina [6] proved that if G has the property that given any three distinct conjugacy class sizes greater than 1, there is a pair which is coprime, then $|cs(G)| \leq 4$ and G is solvable. On the other hand, if $|cs(G)| = 2$, then G is nilpotent, and if $|cs(G)| = 3$, then G is solvable, by applying [7,8]. Consequently, G is solvable if there exists a unique composite number in $cs(G)$. Further, we obtain as follows.

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Theorem A. *Let G be a group having exactly one composite conjugacy class size m. Then, up to abelian direct factors, one of the following statements holds*:

(1) *G is a p-group with m a p-power*;
(2) *$G = P_1 \times P_2$, where P_1 and P_2 are Sylow p_1 and p_2 subgroups of G, with distinct primes p_1 and p_2. In this case, $cs(P_1) = \{1, p_1\}$, $cs(P_2) = \{1, p_2\}$ and $m = p_1 p_2$*;
(3) *$cs(G) = \{1, p, m\}$, where m is not a p-power. In particular, if $p \nmid m$, then G is a quasi-Frobenius group with abelian kernel and complements*; *while if $p \mid m$, then $|G/\boldsymbol{F}(G)| = p$, where $\boldsymbol{F}(G)$ is the abelian fitting subgroup of G.*

2. Preliminaries

In this section, we list some lemmas which will be used in the sequel.

Lemma 2.1. *Let G be a group. Suppose $x, y \in G$ are two elements of coprime orders. If x commutes with y and $(|x^G|, |y^G|) = 1$, then $|(xy)^G| = |x^G||y^G|$.*

Proof. Since $xy = yx$ and x, y have coprime orders, we see that $\mathbf{C}_G(xy) = \mathbf{C}_G(x) \cap \mathbf{C}_G(y)$, which implies that $|x^G| \mid |(xy)^G|$ and $|y^G| \mid |(xy)|^G$. Recall that $(|x^G|, |y^G|) = 1$. Then $|x^G||y^G| \leq |(xy)^G|$. On the other hand, $|(xy)^G| = |G : \mathbf{C}_G(xy)| = |G : \mathbf{C}_G(x)||\mathbf{C}_G(x) : \mathbf{C}_G(x) \cap \mathbf{C}_G(y)| \leq |x^G||y^G|$, yielding to $|(xy)^G| = |x^G||y^G|$. □

Lemma 2.2 ([9, *Lemma* 2.4]). *Let G be a group. If each primary p'-element of G has conjugacy class size 1 or m, then $m = p^a q^b$, where a, b are two integers and q is a prime distinct from p. Moreover, $G = PQ \times A$, where P is a Sylow p-subgroup of G, Q is a Sylow q-subgroup of G and $A \leq \mathbf{Z}(G)$. In particular, if $b = 0$, then G has abelian p-complements*; *if $a = 0$, then $G = P \times Q \times A$.*

3. Proof of Theorem A

Proof. If G can be written as $G = A \times B$ with $B \leq \mathbf{Z}(G)$, then $cs(G) = cs(A)$. As a consequence, G can be assumed to be a group without central direct factors. Write $\pi := \pi(G)$.

Suppose first that G is nilpotent. Then $G = P_1 \times \cdots \times P_t$, where P_i are non-central Sylow p_i-subgroups of G with $p_i \in \pi$. If $t \geq 3$, then there exists elements $x_1 \in P_1, x_2 \in P_2, x_3 \in P_3$ such that $|x_1^G| = p_1$, $|x_2^G| = p_2$ and $|x_3^G| = p_3$. By Lemma 2.1, we get $|(x_1x_2)^G| = p_1p_2$ and $|(x_2x_3)^G| = p_2p_3$, which are two distinct composite numbers in $cs(G)$, against our assumption. Hence $t \leq 2$. Further, if $t = 1$, statement (1) is trivial. Assume then $t = 2$. The same argument above deduces that $m = p_1p_2$ and $cs(P_i) = \{1, p_i\}$ with $i = 1, 2$. Thus the statement (2) of the theorem holds.

Suppose now that G is non-nilpotent. Write $cs(G) = \{1, p_1, \ldots, p_t, m\}$, where $p_1, \ldots, p_t$ are distinct primes and m is the unique composite number in $cs(G)$. Then it follows by [4, Corollary 1] that $\pi = \pi(m) \cup \{p_1, \ldots, p_t\}$. If there exists a prime $p \in cs(G)$ such that $p \nmid m$, by applying [3, Theorem 1], we have $n(\Gamma(G)) = 2$. Further, [3, Theorem 2] implies that G is a quasi-Frobenius group with abelian kernel and complement, forcing $cs(G) = \{1, p, m\}$, as required in statement (3).

Now we consider the case that each prime in $cs(G)$ divides m. Notice that [8, Theorem 1] and [6, Main Theorem] indicate $|cs(G)| = 3$ or 4. Assume first $cs(G) = \{1, p, m\}$ with

$p \mid m$. Then $G/\mathbf{Z}(G)$ is a p-group or $\mathbf{F}(G)$ is abelian satisfying $|G/\mathbf{F}(G)| = p$ according to [7, Theorem 3.2]. If the former holds, then G is a p-group since G has no central direct factors, against our assumption. Hence the latter holds, as described in statement (3).

Now suppose $|cs(G)| = 4$. It follows by [4, Theorem 2] that, our main task is to work on a non-nilpotent group G with $cs(G) = \{1, p, q, m\}$, where $m \neq pq$ is a composite number. We will complete the proof in several steps.

Step 1. Let v be an r-element satisfying $|v^G| = m$ with prime $r \in \pi$. Then $\mathbf{C}_G(v) = \mathbf{C}_G(v)_r \times \mathbf{C}_G(v)_{r'}$, where $\mathbf{C}_G(v)_r$ is a Sylow r-subgroup of $\mathbf{C}_G(v)$, and $\mathbf{C}_G(v)_{r'}$ is an abelian Hall r'-subgroup of $\mathbf{C}_G(v)$.

For any r'-element $w \in \mathbf{C}_G(v)$, we see that $\mathbf{C}_G(wv) = \mathbf{C}_G(w) \cap \mathbf{C}_G(v) \leq \mathbf{C}_G(v)$, implying $m = |v^G| \mid |(wv)^G| \in cs(G)$. Thus $\mathbf{C}_G(wv) = \mathbf{C}_G(v)$, indicating that $w \in \mathbf{Z}(\mathbf{C}_G(v))$. As a result, $\mathbf{C}_G(v) = \mathbf{C}_G(v)_r \times \mathbf{C}_G(v)_{r'}$, where $\mathbf{C}_G(v)_{r'}$ is an abelian Hall r'-subgroup of $\mathbf{C}_G(v)$, as required.

Step 2. If there exists two primary elements u and v satisfying $|u^G| = p$ and $|v^G| = q$ such that $\pi(u) \neq \pi(v)$, then u must be a q-element, while v must be a p-element.

If v is not a p-element, then, up to conjugation, we may consider $u \in \mathbf{C}_G(v)$. In this case, $\mathbf{C}_G(uv) = \mathbf{C}_G(u) \cap \mathbf{C}_G(v)$ and $(|u^G|, |v^G|) = 1$, which follows that $|(uv)^G| = pq = m$ by Lemma 2.1, a contradiction. Analogously, u must be a q-element.

Step 3. If there exist two r-elements $u, v \in G$ such that $|u^G| = p$ and $|v^G| = q$, respectively, then each primary element whose conjugacy class size is a prime must be an r-element.

By the symmetry of p and q, suppose on the contrary, there exists an s-element z satisfying $|z^G| = p$, where $s \neq r$ is a prime. If $r \neq p$, then, up to conjugation, we may consider $v \in \mathbf{C}_G(z)$. In this case, $\mathbf{C}_G(vz) = \mathbf{C}_G(v) \cap \mathbf{C}_G(z)$ and $(|v^G|, |z^G|) = 1$, which follows that $|(vz)^G| = pq = m$, by Lemma 2.1. This contradiction leads to $r = p$. Analogously, $s = q$, since otherwise, we may consider $z \in \mathbf{C}_G(v)$, and the same argument as above will deduce a contradiction.

As a consequence, u, v are two p-elements satisfying $|u^G| = p$ and $|v^G| = q$, and z is a q-element with $|z^G| = p$. Up to conjugation, we may assume that $z \in \mathbf{C}_G(u) =: U$. Moreover, $\mathbf{C}_G(uz) = \mathbf{C}_G(u) \cap \mathbf{C}_G(z) \leq \mathbf{C}_G(u)$ implies that $|u^G| \mid |(uz)^G|$, and thus $|(uz)^G| = m$ or p. Assume first that $|(uz)^G| = p$. Then $\mathbf{C}_G(u) = \mathbf{C}_G(z)$. We prove that each primary element has conjugacy class size 1 or m/p in U. Take any primary element $h \in U$. If h is a p'-element, then $\mathbf{C}_G(uh) = \mathbf{C}_G(u) \cap \mathbf{C}_G(h) \leq \mathbf{C}_G(u)$, which implies that $|(uh)^G| = p$ or m. Under this situation, $|h^U| = |U : \mathbf{C}_U(h)| = |\mathbf{C}_G(u) : \mathbf{C}_G(uh)| = 1$ or m/p; if h is a p-element, then $|h^U| = |\mathbf{C}_G(u) : \mathbf{C}_G(uh)| = |\mathbf{C}_G(z) : \mathbf{C}_G(zh)|$ as $\mathbf{C}_G(u) = \mathbf{C}_G(z)$. The same argument as above also deduces that $|h^U| = 1$ or m/p, as required. By [2, Corollary B], U is nilpotent. Write $U = U_q \times U_{q'}$, where U_q is the Sylow q-subgroup and $U_{q'}$ is the Hall q'-subgroup of U, respectively. Easily, $z \in U_q$. Note that v is a p-element with $|v^G| = q$. By Sylow's theorem, there is some $g \in G$ such that $v^g \in U_{q'}$, forcing $[v^g, z] = 1$. By Lemma 2.1, we have that $|(v^g z)^G| = pq$, a contradiction.

If $|(uz)^G| = m$, then every p'-element of U has conjugacy class sizes 1 or m/p in U. Further, by Lemma 2.2, we see that $\pi(m/p) = \{p, q\}$, leading to $\pi(G) = \{p, q\}$ as $\pi(G) = \pi(m)$. Suppose that each p-element has prime conjugacy class size. Then [5, Theorem 2] deduces a contradiction. Hence, there exists a p-element c such that $|c^G| = m$. By Step 1, we see that $\mathbf{C}_G(c) = \mathbf{C}_G(c)_p \times \mathbf{C}_G(c)_q$, where $\mathbf{C}_G(c)_p$ is the Sylow p-subgroup and $\mathbf{C}_G(c)_q$ is the abelian Sylow q-subgroup of $\mathbf{C}_G(c)$, respectively. Moreover, $\mathbf{C}_G(c)_q \nleq \mathbf{Z}(G)$, since $z \in \mathbf{C}_G(uz)_q$ and $|\mathbf{C}_G(c)|_q = |\mathbf{C}_G(uz)|_q$.

Take any arbitrary element $e \in \mathbf{C}_G(c)_q \setminus \mathbf{Z}(G)$. Then $\mathbf{C}_G(c) \leq \mathbf{C}_G(e)$, forcing $|e^G| = p, q$ or m. Recall that v is a p-element with $|v^G| = q$. By conjugation, we may consider $v^{g_1} \in \mathbf{C}_G(c)$ for some $g_1 \in G$, and thus $v^{g_1} \in \mathbf{C}_G(c)_p$. Lemma 2.1 implies that $|e^G| \neq p$, since otherwise $|(v^{g_1}e)^G| = pq = m$, against our assumption. In particular, any conjugate of z could not be contained in $\mathbf{C}_G(c)_q$, since z is a q-element with $|z^G| = p$. On the other hand, up to conjugation, we may consider $z^{g_2} \in \mathbf{C}_G(e)$. Hence, $\mathbf{C}_G(e) \geq \langle \mathbf{C}_G(c)_q, z^{g_2}\rangle > \mathbf{C}_G(c)_q$, forcing $|e^G| = q$. In this case, we may consider $u^{g_3} \in \mathbf{C}_G(e)$ for some $g_3 \in G$, since $|u^G| = p$. By Lemma 2.1, we have $|(eu^{g_3})^G| = pq$, a contradiction.

In the following, we fix $x, y \in G$ two primary elements satisfying $|x^G| = p$ and $|y^G| = q$.

Step 4. G is a $\{p, q\}$-group. Assume on the contrary that $|\pi| \geq 3$. Then there exists a prime $r \in \pi$ distinct from p and q. If $\pi(x) = \pi(y) = r$, then by Step 3, each element having conjugacy class size a prime must be an r-element. That is to say, each primary r'-element has conjugacy class size 1 or m in G. By Lemma 2.2, $|\pi(G)| = 2$, against our assumption.

Hence $\pi(x) \neq \pi(y)$. Step 2 deduces that x is a q-element and y is a p-element since $|x^G| = p$ and $|y^G| = q$. Hence, there exists a Sylow r-subgroup R of G satisfying $R \leq \mathbf{C}_G(x) \cap \mathbf{C}_G(y)^g$ for some $g \in G$. Let $v \in R$ be a non-central r-element of G. If $|v^G| = p$, then v must be a q-element by Step 2, against the choice of v. Similarly, $|v^G| \neq q$, forcing $|v^G| = m$. By Step 1, $\mathbf{C}_G(v) = \mathbf{C}_G(v)_r \times \mathbf{C}_G(v)_{r'}$, where $\mathbf{C}_G(v)_{r'}$ is an abelian Hall r'-subgroup of $\mathbf{C}_G(v)$. As $x, y^g \in \mathbf{C}_G(v)_{r'}$, we get $[x, y^g] = 1$, yielding $m = pq$. This contradiction concludes that G is a $\{p, q\}$-group.

Step 5. We will work on the contradiction. Recall that $x, y \in G$ are two primary elements satisfying $|x^G| = p$ and $|y^G| = q$. We will work on the contradiction on the case that $\pi(x) = \pi(y)$ or not.

Case 1: $\pi(x) = \pi(y)$. Without loss of generality, we assume that x is a p-element according to Step 3. Moreover, each primary element with a prime conjugacy class size must be a p-element. On the other hand, we prove that the conjugacy class size of each p-element must be a prime. Assume that it is false. Then there exists a p-element $v \in G$ satisfying $|v^G| = m$. Then $\mathbf{C}_G(v) = \mathbf{C}_G(v)_p \times \mathbf{C}_G(v)_q$ such that $\mathbf{C}_G(v)_q \leq \mathbf{Z}(\mathbf{C}_G(v))$, by Step 1.

If $\mathbf{C}_G(v)_q \leq \mathbf{Z}(G)$, then for any noncentral q-element $w \in G$, we see that $\mathbf{C}_G(w) \geq \langle w, \mathbf{C}_G(v)_q\rangle \gneq \mathbf{C}_G(v)_q$, leading to $|w^G|$ which is a proper divisor of m. Hence $|w^G|$ is a prime. By Step 3, w must be a p-element, a contradiction.

Hence $\mathbf{C}_G(v)_q \nleq \mathbf{Z}(G)$. Let $u \in \mathbf{C}_G(v)_q \setminus \mathbf{Z}(G)$. Then $\mathbf{C}_G(v) \leq \mathbf{C}_G(u)$. Note that $|u^G| = m$, by Step 3. Then $\mathbf{C}_G(v) = \mathbf{C}_G(u)$ is abelian. Moreover, $\mathbf{C}_{\mathbf{O}_p(G)}(v) = \mathbf{C}_{\mathbf{O}_p(G)}(u)$. By Thompson's $P \times Q$ lemma (see for instance [10, Theorem 8.2.8]), we obtain that $\mathbf{O}_p(G) \leq \mathbf{C}_G(u)$. Analogously, $\mathbf{O}_q(G) \leq \mathbf{C}_G(v)$. Hence $\mathbf{F}(G) \leq \mathbf{C}_G(v) = \mathbf{C}_G(u)$. On the other hand, for every $b \in \mathbf{C}_G(v)$, $\mathbf{C}_G(v) \leq \mathbf{C}_G(b)$ since $\mathbf{C}_G(v)$ is abelian. Thus $b \in \mathbf{C}_G(\mathbf{F}(G)) \leq \mathbf{F}(G)$ as G is solvable. Hence $\mathbf{C}_G(v) \leq \mathbf{F}(G)$, forcing $\mathbf{C}_G(v) = \mathbf{F}(G)$. Consequently, we conclude that each p-element with conjugacy class size m is contained in $\mathbf{F}(G)$.

On the other hand, take an arbitrary element $x_0 \in P$. If $|x_0^G| = q$, we have $x_0^g \in \mathbf{C}_G(v)$ for some $g \in G$ and thus $x_0^g \in \mathbf{F}(G)$, that is, $x_0 \in \mathbf{F}(G)$. If $|x_0^G| = p$, by [1, Lemma 6], we have $x_0 \in \mathbf{O}_p(G)$ and thus $x_0 \in \mathbf{F}(G)$. This shows $P \leq \mathbf{F}(G)$. As $|G : \mathbf{F}(G)| = m$, we have $p \mid |G : \mathbf{F}(G)|$, which contradicts $P \leq \mathbf{F}(G)$. Consequently, each p-element must have prime conjugacy class size. In this case, by [5, Theorem 2], there is a unique prime such that each p-element has conjugacy class size p, which is a contradiction.

Case 2: $\pi(x) \neq \pi(y)$. By Step 2, x is a q-element and y is a p-element. Let $\mathbf{C}_G(x)_p \in \mathrm{Syl}_p(\mathbf{C}_G(x))$ and $P \in \mathrm{Syl}_p(G)$ such that $\mathbf{C}_G(x)_p$ is contained in P. If $\mathbf{C}_G(x)_p \leq \mathbf{Z}(G)$, then $\mathbf{C}_G(x)_p = \mathbf{Z}(G)_p$, leading to $|P/\mathbf{Z}(G)_p| = p$. As a result, P is abelian and $m_p = p$. In this case, $\mathbf{C}_G(y)_q \nleq \mathbf{Z}(G)$, since otherwise, $m_q = q$ by a similar argument as above, yielding to $m = pq$, which is a contradiction. Take $z \in \mathbf{C}_G(y)_q \setminus \mathbf{Z}(G)$. This forces $|z^G| = m$, according to Step 3. Further, Step 1 indicates that $\mathbf{C}_G(z) = \mathbf{C}_G(z)_q \times \mathbf{C}_G(z)_p$ with $\mathbf{C}_G(z)_p$ abelian. Notice that $|P/\mathbf{Z}(G)_p| = p$ and $p \mid m$. We obtain that $y \in \mathbf{C}_G(z)_p = \mathbf{Z}(G)_p$, which is a contradiction.

Consequently, $\mathbf{C}_G(x)_p \nleq \mathbf{Z}(G)$. Take $u \in \mathbf{C}_G(x)_p \setminus \mathbf{Z}(G)$ such that $|u^G| = q$ or m, by Step 1. If $|u^G| = q$, then $|(xu)^G| = pq = m$, by Lemma 2.1, a contradiction. Hence $|u^G| = m$. By a similar argument as above, we see that $\mathbf{C}_G(u) = \mathbf{C}_G(u)_p \times \mathbf{C}_G(u)_q$, where $\mathbf{C}_G(u)_q$ is abelian. Since $|y^G| = q$, we may assume that $y^{g_2} \in \mathbf{C}_G(u)_p$ for some $g_2 \in G$. As $x \in \mathbf{C}_G(u)_q$, we have $[x, y^{g_2}] = 1$, leading to $m = pq$, the final contradiction. □

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