Proc. Indian Acad. Sci. (Math. Sci.) (2020) 130:4
https://doi.org/10.1007/s12044-019-0528-2

A note on the weak law of large numbers of Kolmogorov and Feller

FUQIANG MA[1], YU MIAO[2,*] and JIANYONG MU[3]

[1]College of Mathematics and Statistics, Pingdingshan University, Pingdingshan 467000, Henan Province, China
[2]College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, Henan Province, China
[3]School of Mathematics and Statistics, Wuhan University, Wuhan 430072, Hubei Province, China
*Corresponding author.
Email: mafq0314@126.com; yumiao728@gmail.com; jianyongmu@163.com

MS received 26 March 2018; revised 16 January 2019; accepted 25 June 2019

Abstract. In this paper, we establish the weak laws of large numbers for the negative quadrant-dependent random sequences which extend the classic Kolmogorov–Feller weak law of large numbers. In addition, the moment convergence for the negative quadrant-dependent random sequences are also given.

Keywords. NQD sequence; weak law of large numbers; moment convergence.

Mathematics Subject Classification. 60F15, 60G50.

1. Introduction

The following celebrated Kolmogorov–Feller weak law of large numbers provides a necessary and sufficient condition for a sequence of independent identically distributed (i.i.d.) random variables.

Theorem 1.1 **[2].** *Let $\{X, X_i, i \geq 1\}$ be a sequence of i.i.d. random variables with partial sums $S_n = X_1 + \cdots + X_n$. Then*

$$\frac{S_n - n\mathbb{E}X1\{|X| \leq n\}}{n} \xrightarrow{\mathbb{P}} 0, \quad as \ \ n \to \infty$$

if and only if

$$x\mathbb{P}(|X| > x) \to 0, \quad as \ \ x \to \infty. \tag{1.1}$$

Klass and Teicher [7] extended the above Kolmogorov–Feller weak law of large numbers as follows.

© Indian Academy of Sciences
Published online: 03 January 2020

Theorem 1.2 **[7].** *Let $\{X, X_i, i \geq 1\}$ be i.i.d. random variables and let $\{b_n, n \geq 1\}$ be constants such that $0 < b_n \uparrow \infty$ and either*

$$b_n/n \downarrow 0, \quad b_n/n^{1/2} \to \infty \quad \textit{and} \quad \sum_{j=1}^{n} \left(\frac{b_j}{j}\right)^2 = O\left(\frac{b_n^2}{n}\right)$$

or

$$b_n/n \uparrow .$$

Then

$$\frac{S_n - n\mathbb{E}X1\{|X| \leq b_n\}}{b_n} \xrightarrow{\mathbb{P}} 0, \quad \textit{as} \;\; n \to \infty$$

if and only if

$$n\mathbb{P}(|X| > b_n) \to 0, \quad \textit{as} \;\; n \to \infty.$$

Gut [4] showed that if $b_n = n^r h(n)$, where $r \geq 1$ is a fixed number and $h(\cdot)$ is a slowly varying function, then the same conclusions hold. Recently, Kruglov [8] extended the Kolmogorov–Feller weak law of large numbers from i.i.d. case to negatively associated random variables. Chandra [1] extended the works of Kruglov [8] to asymptotically almost negatively associated sequences which are strictly weaker than negatively associated sequences. In this paper, we shall study further the Kolmogorov–Feller weak law of large numbers for the case of negative quadrant-dependent random sequence.

Let us recall the concept of negative quadrant-dependent (NQD) random variables, which was introduced by Lehmann [9]. Two random variables X and Y are said to be NQD if for any $x, y \in \mathbb{R}$,

$$\mathbb{P}(X \leq x, Y \leq y) \leq \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y).$$

A sequence of random variables $\{X_n, n \geq 1\}$ is said to be pairwise NQD if, for all $i, j \in N$, $i \neq j$, X_i and X_j are NQD. The pairwise NQD structure is a very important kind of dependence structures. Many known types of dependence structure have developed on the basis of this notion. A finite sequence $\{X_1, \ldots, X_n\}$ of random variables is said to be negatively associated (NA) if for any disjoint subsets A, B of $\{1, \ldots, n\}$ and any real coordinate-wise nondecreasing functions f on $\mathbb{R}^{|A|}$ and g on $\mathbb{R}^{|B|}$,

$$\mathrm{Cov}(f(X_k, k \in A), g(X_k, k \in B)) \leq 0 \tag{1.2}$$

whenever the covariance exists, where $|A|$ denotes the cardinality of A. An infinite family of random variables is NA if every finite subfamily is NA. The concept of NA sequence was introduced by Joag-Dev and Proschan [6], and it is easy to see that a sequence of NA random variables is a pairwise NQD sequence. So the pairwise NQD structure is more comprehensive than the NA structure, and it is very significant to study probabilistic properties of this wider pairwise NQD structure. Many precise results about the different dependent forms for the law of large numbers have been established. For example, Miao *et al.* [12] obtained the L^p-convergence theorem and Marcinkiewicz–Zygmund strong law of large numbers for the sequence of negatively associated random variables. Miao *et al.* [13] studied the rate of convergence in the strong law of large numbers for martingales. Hoffmann-Jørgensen *et al.* [5] established the upper and lower bounds of a Kolmogorov-type law of the logarithm for random arrays. Miao *et al.* [11] gave an analogue for Marcinkiewicz–Zygmund strong law of negatively associated random variables. Zhang *et al.* [15] studied the complete convergence for weighted sums of mixingale sequences and gave some statistical applications.

This paper is organized as follows. In section 2, we introduce a weakly dominated condition for the random sequence $\{X_n, n \geq 1\}$, and state the main results. The proofs of our results will be obtained in section 3. Throughout this paper, the symbol C represents positive constants whose values may change from one place to another.

2. Main results

Before giving our main results, we introduce some dominated conditions. First, let us recall the following weakly dominated (WD) condition, where weak refers to the fact that domination is distributional. The random variables $\{X_n, n \geq 1\}$ are said to be *uniformly dominated* by a random variable X if there exists a random variable X, such that

$$\mathbb{P}(|X_n| > x) \leq \mathbb{P}(|X| > x) \tag{2.1}$$

for all $x > 0$ and $n \geq 1$. In [3], Gut introduced a weakly mean dominated condition. The random variables $\{X_n, n \geq 1\}$ are said to be weakly mean dominated (WMD) by the random variable X, where X is possibly defined on a different space if for some $C > 0$,

$$\frac{1}{n}\sum_{k=1}^{n}\mathbb{P}(|X_k| > x) \leq C\mathbb{P}(|X| > x) \tag{2.2}$$

for all $x > 0$, all $n \geq 1$.

It is clear that if X dominates the sequence $\{X_n, n \geq 1\}$ in the WD-sense, then it also dominates the sequence in the WMD-sense (with $C = 1$). Furthermore, Gut [3] gave an example to show that the condition (2.2) (by taking $C = 1$) is weaker than the above condition (2.1).

Theorem 2.1. *Let $\{X_i, i \geq 1\}$ be a sequence of pairwise NQD random variables satisfying the following WMD condition*:

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{P}(|X_i| > x) \leq C\mathbb{P}(|X| > x).$$

Assume that $\{b_n, n \geq 1\}$ is a non-decreasing sequence of positive constants such that

$$\sum_{j=1}^{n}\frac{b_j^2}{j^2\log^2 j} = O\left(\frac{b_n^2}{n\log^2 n}\right) \tag{2.3}$$

and the random variable X satisfies

$$\lim_{n\to\infty} n\mathbb{P}(|X| > b_n/\log n) = 0. \tag{2.4}$$

Then we have

$$\frac{1}{b_n}\max_{1\le k\le n}\left|S_k - \sum_{i=1}^{k}\mathbb{E}\left(X_i 1(|X_i| \le b_n/\log n)\right)\right| \xrightarrow{\mathbb{P}} 0. \tag{2.5}$$

Remark 2.1. Without loss of generality, we can assume

$$a_1 := \frac{b_1^2}{\log^2 1} = 0.$$

By the condition (2.3), it follows that there exists a positive constant C such that for all n large enough, we have

$$\sum_{j=1}^{n}\frac{b_j^2}{j^2\log^2 j} \le C\frac{b_n^2}{n\log^2 n}.$$

From this inequality, we can get

$$b_2^2 \le C\sum_{j=1}^{n}\frac{b_2^2}{j^2\log^2 j} \le C\sum_{j=1}^{n}\frac{b_j^2}{j^2\log^2 j} \le C\frac{b_n^2}{n\log^2 n}$$

and

$$\infty \leftarrow \sum_{j=2}^{n}\frac{b_2^2}{j} \le C\sum_{j=1}^{n}\frac{b_j^2}{j^2\log^2 j} \le C\frac{b_n^2}{n\log^2 n}$$

which implies that

$$\lim_{n\to\infty} b_n \to \infty \text{ and } \lim_{n\to\infty}\frac{b_n^2}{n\log^2 n} \to \infty. \tag{2.6}$$

Remark 2.2. Assume that $\{b_n^*, n \ge 1\}$ is a non-decreasing sequence of positive constants such that

$$\sum_{k=1}^{n}\frac{b_k^{*2}}{k^2} \le C\frac{b_n^{*2}}{n}$$

for all $n \ge 1$ and for some constant $C > 0$. Then Kruglov [8] showed that many sequences of normalizing constants meet this condition, in particular, $b_n^* = n^r h(n)$, $n \ge 1$ with

$r > 1/2$, where $h(\cdot)$ is a slowly varying function. Hence we take $h(x) = \log^{-1} x$, $r = 1$, i.e., $b_n^* = n/\log n$, then there exists a constant $C > 0$ such that

$$\sum_{k=1}^{n} \frac{k^2}{k^2 \log^2 k} \le C \frac{n}{\log^2 n}.$$

COROLLARY 2.1

Let $\{X_i, i \ge 1\}$ be a sequence of pairwise NQD random variables satisfying the following WMD condition:

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{P}(|X_i| > x) \le C\mathbb{P}(|X| > x).$$

If $\mathbb{E}\left(|X| \log^+ |X|\right) < \infty$, then we have

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left(\max_{1\le k\le n} |S_k - \mathbb{E}S_k|\right) = 0. \tag{2.7}$$

Theorem 2.2. *Let $\{X_i, i \ge 1\}$ be a sequence of pairwise NQD random variables satisfying the following WMD condition*:

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{P}(|X_i| > x) \le C\mathbb{P}(|X| > x).$$

Assume that $\{b_n, n \ge 1\}$ is a non-decreasing sequence of positive constants such that

$$\sum_{j=1}^{n} \frac{b_j^2}{j^2} = O\left(\frac{b_n^2}{n}\right) \tag{2.8}$$

and the random variable X satisfies

$$\lim_{n\to\infty} n\mathbb{P}(|X| > b_n) = 0. \tag{2.9}$$

Then we have

$$\frac{1}{b_n} \left| S_n - \sum_{i=1}^{n} \mathbb{E}(X_i 1(|X_i| \le b_n)) \right| \xrightarrow{\mathbb{P}} 0. \tag{2.10}$$

In addition, if $\{X_i, i \ge 1\}$ is a sequence of pairwise identically distributed NQD random variables with $\mathbb{E}|X_1| < \infty$, then we have

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}\,|S_n - n\mathbb{E}X_1| = 0.$$

3. Proofs of main results

Lemma 3.1 [9]. *Let* $\{X_n, n \geq 1\}$ *be a sequence of pairwise NQD random variables. Let* $\{f_n, n \geq 1\}$ *be a sequence of increasing functions. Then* $\{f_n(X_n), n \geq 1\}$ *is a sequence of pairwise NQD random variables.*

Lemma 3.2 [14]. *Let* $\{X_n, n \geq 1\}$ *be a sequence of pairwise NQD random variables with mean zero and* $\mathbb{E}X_n^2 < \infty$, *and* $T_j(k) = \sum_{i=j+1}^{j+k} X_j$, $j \geq 0$. *Then*

$$\mathbb{E}(T_j(k))^2 \leq C \sum_{i=j+1}^{j+k} \mathbb{E}X_i^2, \quad \mathbb{E} \max_{1 \leq k \leq n} (T_j(k))^2 \leq C \log^2 n \sum_{i=j+1}^{j+n} \mathbb{E}X_i^2.$$

Lemma 3.3. *Let* $\{X_n, n \geq 1\}$ *be a sequence of random variables satisfying WMD condition with mean dominating random variable* X, *i.e., for some* $C > 0$,

$$\frac{1}{n} \sum_{k=1}^{n} \mathbb{P}(|X_k| > x) \leq C\mathbb{P}(|X| > x). \tag{3.1}$$

Let $p > 0$ *and for some* $M > 0$,

$$X_i^{'} = X_i 1_{\{|X_i| \leq M\}}, \; X_i^{''} = X_i 1_{\{|X_i| > M\}}$$

and

$$X^{'} = X 1_{\{|X| \leq M\}}, \; X^{''} = X 1_{\{|X| > M\}}.$$

Then we have

(1) *if* $\mathbb{E}|X|^p < \infty$, *then* $n^{-1} \sum_{k=1}^{n} \mathbb{E}|X_k|^p \leq M\mathbb{E}|X|^p$,
(2) $n^{-1} \sum_{k=1}^{n} \mathbb{E}|X_k^{'}|^p \leq C(\mathbb{E}|X^{'}|^p + M^p\mathbb{P}(|X| > M))$ *for any* $M > 0$,
(3) $n^{-1} \sum_{k=1}^{n} \mathbb{E}|X_k^{''}|^p \leq C\mathbb{E}|X^{''}|^p$.

Proof. We only give the proof of (2) and the others can be obtained by similar methods. By the Fubini's theorem, for any random variable Y with $\mathbb{E}|Y|^p < \infty$, we have

$$\mathbb{E}|Y|^p = p \int_0^{\infty} y^{p-1} \mathbb{P}(|Y| > y) \mathrm{d}y.$$

Hence from this formula, we can get

$$\begin{aligned} & n^{-1} \sum_{k=1}^{n} \mathbb{E}|X_k^{'}|^p \\ & = n^{-1} \sum_{k=1}^{n} p \int_0^{\infty} y^{p-1} \mathbb{P}(|X_k^{'}| > y) \mathrm{d}y \\ & = n^{-1} \sum_{k=1}^{n} p \int_0^{M} y^{p-1} \mathbb{P}(y < |X_k| \leq M) \mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&= n^{-1}\sum_{k=1}^{n} p\int_0^M y^{p-1}\left[\mathbb{P}(|X_k|>y)-\mathbb{P}(|X_k|>M)\right]\mathrm{d}y \\
&= n^{-1}\sum_{k=1}^{n} p\int_0^M y^{p-1}\mathbb{P}(|X_k|>y)\mathrm{d}y - n^{-1}\sum_{k=1}^{n} M^p\mathbb{P}(|X_k|>M) \\
&\le Cp\int_0^M y^{p-1}\mathbb{P}(|X|>y)\mathrm{d}y.
\end{aligned}
$$

By the same calculation, we have

$$
p\int_0^M y^{p-1}\mathbb{P}(|X|>y)\mathrm{d}y = \mathbb{E}|X^{'}|^p + M^p\mathbb{P}(|X|>M),
$$

which implies (2). □

Proof of Theorem 2.1. For $n \ge 1$, denote $a_n := b_n/\log n$ and for any $1 \le i \le n$, let us define

$$
Y_{ni} = -a_n 1(X_i < -a_n) + X_i 1(|X_i| \le a_n) + a_n 1(X_i > a_n)
$$

and

$$
Z_{ni} = (X_i + a_n)1(X_i < -a_n) + (X_i - a_n)1(X_i > a_n).
$$

Then from Lemma 3.1, it follows that $\{Y_{ni}, 1 \le i \le n, n \ge 1\}$ and $\{Z_{ni}, 1 \le i \le n, n \ge 1\}$ are both pairwise NQD, and

$$
\sum_{i=1}^{n} X_i = \sum_{i=1}^{n}(Y_{ni} + Z_{ni}).
$$

Furthermore, let us define $X_{ni} = X_i 1(|X_i| \le a_n)$, then the limit (2.5) holds if we show

$$
\lim_{n\to\infty}\frac{1}{b_n}\max_{1\le k\le n}\left|\sum_{i=1}^{k}(Y_{ni}-\mathbb{E}Y_{ni})\right| = 0 \text{ in probability,} \tag{3.2}
$$

$$
\lim_{n\to\infty}\frac{1}{b_n}\max_{1\le k\le n}\left|\sum_{i=1}^{k}(X_i - Y_{ni})\right| = 0 \text{ in probability} \tag{3.3}
$$

and

$$
\lim_{n\to\infty}\frac{1}{b_n}\max_{1\le k\le n}\left|\sum_{i=1}^{k}(\mathbb{E}X_{ni}-\mathbb{E}Y_{ni})\right| = 0. \tag{3.4}
$$

From Lemma 3.2, we have

$$
\frac{1}{b_n^2}\mathbb{E}\max_{1\le k\le n}\left|\sum_{i=1}^{k}(Y_{ni}-\mathbb{E}Y_{ni})\right|^2 \le \frac{C\log^2 n}{b_n^2}\sum_{i=1}^{n}\mathbb{E}(Y_{ni}-\mathbb{E}Y_{ni})^2
$$

$$\leq \frac{C\log^2 n}{b_n^2}\sum_{i=1}^{n}\mathbb{E}[X_i^2 1(|X_i|\leq a_n)] + \frac{Ca_n^2\log^2 n}{b_n^2}\sum_{i=1}^{n}\mathbb{P}(|X_i|\geq a_n). \tag{3.5}$$

It is easy to check that

$$\begin{aligned}\sum_{i=1}^{n}\mathbb{E}[X_i^2 1(|X_i|\leq a_n)] &= \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}[X_i^2 1(a_{j-1}<|X_i|\leq a_j)]\\ &\leq \sum_{i=1}^{n}\sum_{j=1}^{n}a_j^2\mathbb{P}[a_{j-1}<|X_i|\leq a_j]\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}a_j^2[\mathbb{P}(|X_i|>a_{j-1})-\mathbb{P}(|X_i|>a_j)]\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n-1}(a_{j+1}^2-a_j^2)\mathbb{P}(|X_i|>a_j)\\ &\quad+\sum_{i=1}^{n}[a_1^2\mathbb{P}(|X_i|>0)-a_n^2\mathbb{P}(|X_i|>a_n)]. \end{aligned} \tag{3.6}$$

From WMD condition, we have

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n-1}(a_{j+1}^2-a_j^2)\mathbb{P}(|X_i|>a_j)\\ &\quad\leq Cn\sum_{j=1}^{n-1}(a_{j+1}^2-a_j^2)\mathbb{P}(|X|>a_j)\\ &\quad= Cn\sum_{j=1}^{n-1}\frac{1}{j}(a_{j+1}^2-a_j^2)j\mathbb{P}(|X|>a_j).\end{aligned}$$

By the condition (2.4), we know that for any $\varepsilon > 0$, there exists a positive constant N_0 such that for all $j \geq N_0$,

$$j\mathbb{P}(|X|>a_j)\leq\varepsilon.$$

Moreover, from the condition (2.3), we have

$$\begin{aligned}\sum_{j=N_0}^{n-1}\frac{1}{j}(a_{j+1}^2-a_j^2) &= \sum_{j=N_0}^{n-1}\left(\frac{1}{j}-\frac{1}{j+1}\right)a_{j+1}^2+\sum_{j=N_0}^{n-1}\frac{a_{j+1}^2}{j+1}-\sum_{j=N_0}^{n-1}\frac{a_j^2}{j}\\ &= \sum_{j=N_0}^{n-1}\frac{a_{j+1}^2}{j(j+1)}+\frac{a_n^2}{n}-\frac{a_{N_0}^2}{N_0}\end{aligned}$$

$$\leq C \sum_{j=N_0+1}^{n} \frac{b_j^2}{j^2 \log^2 j} + \frac{b_n^2}{n \log^2 n} \leq C \frac{b_n^2}{n \log^2 n}.$$

Now from Remark 2.1 and (3.6), we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\log^2 n}{b_n^2} \sum_{i=1}^{n} \mathbb{E}[X_i^2 1(|X_i| \leq a_n)] \\
&\leq \lim_{n\to\infty} \frac{Cn \log^2 n}{b_n^2} \sum_{j=1}^{n-1} \frac{1}{j}(a_{j+1}^2 - a_j^2) j \mathbb{P}(|X| > a_j) \\
&\leq \lim_{n\to\infty} \frac{Cn \log^2 n}{b_n^2} \left[\sum_{j=1}^{N_0-1} (a_{j+1}^2 - a_j^2) + \sum_{j=N_0}^{n-1} \frac{1}{j}(a_{j+1}^2 - a_j^2)\varepsilon \right] \leq C\varepsilon. \qquad (3.7)
\end{aligned}$$

Furthermore, by the WMD condition and the condition (2.4), we have

$$\frac{a_n^2 \log^2 n}{b_n^2} \sum_{i=1}^{n} \mathbb{P}(|X_i| \geq a_n) \leq n\mathbb{P}(|X| \geq b_n / \log n) \to 0. \qquad (3.8)$$

Hence from (3.5), (3.7) and (3.8), the claim (3.2) holds.

Since for any $r > 0$, we have

$$\begin{aligned}
\left\{ \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} (X_i - Y_{ni}) \right| > r \right\} &\subset \left\{ \sum_{i=1}^{n} |(X_i - Y_{ni})| > r \right\} \\
&\subset \left\{ \bigcup_{i=1}^{n} \{|X_i| > a_n\} \right\}
\end{aligned}$$

then the claim (3.3) holds, by using the following fact:

$$\mathbb{P}\left(\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} (X_i - Y_{ni}) \right| > r \right) \leq \sum_{i=1}^{n} \mathbb{P}(|X_i| > a_n) \leq n\mathbb{P}(|X| > a_n) \to 0.$$

Furthermore, it is easy to see that

$$\frac{1}{b_n} \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} (\mathbb{E}X_{ni} - \mathbb{E}Y_{ni}) \right| \leq \frac{a_n}{b_n} \sum_{i=1}^{n} \mathbb{P}(|X_i| > a_n) \leq \frac{Cn}{\log n} \mathbb{P}(|X| > a_n) \to 0$$

which implies (3.4). □

Proof of Corollary 2.1. From the condition $\mathbb{E}\left(|X| \log^+ |X|\right) < \infty$, we have

$$\begin{aligned}
&\lim_{n\to\infty} n\mathbb{P}(|X| > n/\log n) \\
&\leq \lim_{n\to\infty} \mathbb{E}(|X| \log^+ |X| 1(|X| \log^+ |X| > n/2)) = 0. \qquad (3.9)
\end{aligned}$$

From Remark 2.2, the condition (2.3) holds by taking $b_n = n$. Hence from Theorem 2.1, we have

$$\frac{1}{n}\max_{1\le k\le n}\left|S_k - \sum_{i=1}^{k}\mathbb{E}\left(X_i 1(|X_i|\le n/\log n)\right)\right| \xrightarrow{\mathbb{P}} 0. \tag{3.10}$$

Let us define

$$X'_{i,n} = X_i 1(|X_i|\le n/\log n),\quad X''_{i,n} = X_i 1(|X_i| > n/\log n)$$

and

$$S'_{k,n} = \sum_{i=1}^{k} X'_{i,n},\quad S''_{k,n} = \sum_{i=1}^{k} X''_{i,n}.$$

So in order to obtain the claim (2.7), it is enough to show

$$\frac{1}{n}\Big(\max_{1\le k\le n}|\mathbb{E}S_k - \mathbb{E}S'_k|\Big) = \frac{1}{n}\Big(\max_{1\le k\le n}|\mathbb{E}S''_k|\Big) \to 0 \tag{3.11}$$

and

$$\mathbb{E}V_n := \frac{1}{n}\mathbb{E}\Big(\max_{1\le k\le n}|S_k - \mathbb{E}S'_k|\Big) \to 0. \tag{3.12}$$

From Lemma 3.3, we have

$$\frac{1}{n}\Big(\max_{1\le k\le n}|\mathbb{E}S''_k|\Big) \le \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}|X''_{k,n}| \le C\mathbb{E}\left(|X|1(|X| > n/\log n)\right),$$

then we can get (3.11) by using the condition $\mathbb{E}(|X|\log^+|X|) < \infty$. Next we shall prove the claim (3.12). For any $\varepsilon > 0$, we have

$$\mathbb{E}|V_n| = \mathbb{E}|V_n|1(V_n > \varepsilon) + \mathbb{E}|V_n|1(V_n \le \varepsilon) \le \mathbb{E}|V_n|1(V_n > \varepsilon) + \varepsilon. \tag{3.13}$$

From the definition of V_n, we have

$$\begin{aligned}
&\mathbb{E}|V_n|1(V_n > \varepsilon)\\
&\quad\le \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}(|X_k - \mathbb{E}X'_{k,n}|1(V_n > \varepsilon))\\
&\quad\le \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}(|X'_{k,n} - \mathbb{E}X'_{k,n}|1(V_n > \varepsilon)) + \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}(|X''_{k,n}|1(V_n > \varepsilon))\\
&\quad\le \frac{2}{n}\sum_{k=1}^{n}\mathbb{E}(|X'_{k,n}|1(V_n > \varepsilon)) + \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}(|X''_{k,n}|1(V_n > \varepsilon)).
\end{aligned} \tag{3.14}$$

For any $M > 0$, from Lemma 3.3, we get

$$\begin{aligned}
&\frac{2}{n}\sum_{k=1}^{n}\mathbb{E}(|X'_{k,n}|1(V_n > \varepsilon)) \\
&= \frac{2}{n}\sum_{k=1}^{n}\mathbb{E}(|X'_{k,n}|1(V_n > \varepsilon, |X'_{k,n}| > M)) \\
&\quad + \frac{2}{n}\sum_{k=1}^{n}\mathbb{E}(|X'_{k,n}|1(V_n > \varepsilon, |X'_{k,n}| \le M)) \\
&\le \frac{2}{n}\sum_{k=1}^{n}\mathbb{E}(|X'_{k,n}|1(|X'_{k,n}| > M)) + 2M\mathbb{P}(V_n > \varepsilon) \\
&\le \frac{2}{n}\sum_{k=1}^{n}\mathbb{E}(|X_k|1(|X_k| > M)) + 2M\mathbb{P}(V_n > \varepsilon) \\
&\le 2C\mathbb{E}(|X|1(|X| > M)) + CM\mathbb{P}(|X| > M) + 2M\mathbb{P}(V_n > \varepsilon). \qquad (3.15)
\end{aligned}$$

In addition, for the second term in (3.14), by Lemma 3.3, we have

$$\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}(|X''_{k,n}|1(V_n > \varepsilon)) \le \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}|X''_{k,n}| \le C\mathbb{E}(|X|1(|X| > n/\log n)). \qquad (3.16)$$

Hence from (3.13)–(3.16) and the arbitrariness of M, we have

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\Big(\max_{1\le k\le n}|S_k - \mathbb{E}S'_k|\Big) = 0.$$

□

Proof of Theorem 2.2. By similar proofs as in Theorem 2.1 and Corollary 2.1, the desired results can be obtained. □

4. Further remarks

In this section, we give some examples to show that our main results are significative. Let $\{Z_n, n \ge 1\}$ be a sequence of i.i.d. $N(0, 1)$ random variables, then $\{Z_n - Z_{n+1}, n \ge 1\}$ is a sequence of identically distributed $N(0, 2)$ random variables with the distribution function H. Furthermore, let $\{F_n, n \ge 1\}$ be a sequence of (right-continuous) distribution functions, and for any $n \ge 1$, define

$$F_n^{-1}(t) = \inf\{x \in [-\infty, +\infty]; F_n(x) \ge t\}, \quad 0 \le t \le 1.$$

Li *et al.* [10] proved that $\{X_n, n \ge 1\}$ is a sequence of pairwise NQD random variables with distribution functions $\{F_n, n \ge 1\}$, where $X_n := F_n^{-1}(H(Z_n - Z_{n+1}))$, for any $n \ge 1$. Moreover, if the distribution functions $\{F_n, n \ge 1\}$ are all continuous, then $\{X_n, n \ge 1\}$

can be chosen so that for all $k \geq 1$, $\{X_n, n \geq 1\}$ is not a sequence of independent random variables.

Now, let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables with distribution functions

$$F_n(x) = \begin{cases} 1 - \mathrm{e}^{-nx}, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad \text{for} \quad n \geq 1,$$

and let X be a random variable with distribution function

$$F(x) = \begin{cases} 1 - \frac{x}{\mathrm{e}^x - 1}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Then it is easy to see that

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{P}(|X_i| > x) = \frac{x}{\mathrm{e}^x - 1}\frac{1 - \mathrm{e}^{-xn}}{nx} = \frac{1 - \mathrm{e}^{-xn}}{nx}\mathbb{P}(|X| > x) \leq \mathbb{P}(|X| > x).$$

Let $b_n = n^p$ for some $p > \frac{1}{2}$, then we have

$$\sum_{j=1}^{n}\frac{b_j^2}{j^2\log^2 j} = O\left(\frac{b_n^2}{n\log^2 n}\right)$$

and

$$\lim_{n\to\infty} n\mathbb{P}(|X| > b_n/\log n) = 0.$$

Hence we have by Theorem 2.1,

$$\frac{1}{n^p}\max_{1\leq k\leq n}\left|S_k - \sum_{i=1}^{k}\mathbb{E}\left(X_i 1(|X_i| \leq n^p/\log n)\right)\right| \xrightarrow{\mathbb{P}} 0. \tag{4.1}$$

Furthermore, it is easy to see that $EX^2 < \infty$, then from Corollary 2.1, we have

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\Big(\max_{1\leq k\leq n}|S_k - \mathbb{E}S_k|\Big) = 0.$$

Acknowledgements

This work is supported by IRTSTHN (14IRTSTHN023), NSFC (11471104, 11971154).

References

[1] Chandra T K, On an extension of the weak law of large numbers of Kolmogorov and Feller, *Stoch. Anal. Appl.* **32(3)** (2014) 421–426

[2] Feller W, An introduction to probability theory and its applications, vol. II, second edition (1971) (New York: John Wiley & Sons Inc.)

[3] Gut A, Complete convergence for arrays, *Period. Math. Hungar.* **25** (1992) 51–75

[4] Gut A, An extension of the Kolmogorov–Feller weak law of large numbers with an application to the St. Petersburg game, *J. Theoret. Probab.* **17(3)** (2004) 769–779

[5] Hoffmann-Jørgensen J, Miao Y, Li X C and Xu S F, Kolmogorov type law of the logarithm for arrays, *J. Theoret. Probab.* **29(1)** (2016) 32–47

[6] Joag-Dev K and Proschan F, Negative association of random variables, with applications, *Ann. Statist.* **11(1)** (1983) 286–295

[7] Klass M and Teicher H, Iterated logarithm laws for asymmetric random variables barely with or without finite mean, *Ann. Probab.* **5(6)** (1997) 861–874

[8] Kruglov V M, A generalization of weak law of large numbers, *Stoch. Anal. Appl.* **29(4)** (2011) 674–683

[9] Lehmann E L, Some concepts of dependence, *Ann. Math. Statist.* **37(5)** (1966) 1137–1153

[10] Li D L, Rosalsky A and Volodin A I, On the strong law of large numbers for sequences of pairwise negative quadrant dependent random variables, *Bull. Inst. Math. Acad. Sin. (N.S.)* **1(2)** (2006) 281–305

[11] Miao Y, Mu J Y and Xu J, An analogue for Marcinkiewicz-Zygmund strong law of negatively associated random variables, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **111(3)** (2017) 697–705

[12] Miao Y, Xu W F, Chen S S and Adler A, Some limit theorems for negatively associated random variables, *Proc. Indian Acad. Sci. (Math. Sci.)* **124(3)** (2014) 447–456

[13] Miao Y, Yang G Y and Stoica G, On the rate of convergence in the strong law of large numbers for martingales, *Stochastics* **87(2)** (2015) 185–198

[14] Wu Q Y, Convergence properties of pairwise NQD random sequences, *Acta Math. Sinica (Chinese Ser.)* **45(3)** (2002) 617–624

[15] Zhang L, Miao Y, Mu J Y and Xu J, Complete convergence for weighted sums of mixingale sequences and statistical applications, *Comm. Statist. Theory Methods* **46(21)** (2017) 10692–10701

COMMUNICATING EDITOR: Parameswaran Sankaran