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Class group of the ring of invariants of an exponential map on an affine normal domain

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Abstract. Let k be a field and let B be an affine normal domain over k. Let ϕ be a non-trivial exponential map on B and let $A = B^{\phi}$ be the ring of ϕ-invariants. Since A is factorially closed in B, $A = K \cap B$ where K denotes the field of fractions of A. Hence A is a Krull domain. We investigate here a relation between the class group $\mathrm{Cl}(A)$ of A and the class group $\mathrm{Cl}(B)$ of B. In this direction, we give a sufficient condition for an injective group homomorphism from $\mathrm{Cl}(A)$ to $\mathrm{Cl}(B)$. We also give an example to show that $\mathrm{Cl}(A)$ may not be realized as a subgroup of $\mathrm{Cl}(B)$.

Keywords. Exponential map; ring of invariants; Krull domain; class group; Rees algebra.

Mathematics Subject Classification. Primary: 13N15, 13C20; Secondary: 14R20, 13A30.

1. Introduction

Let k be a field and let B be an affine normal domain over k. Let ϕ be a non-trivial exponential map on B and let $A = B^{\phi}$ be the ring of ϕ-invariants. Since A is factorially closed in B, $A = K \cap B$ where K denotes the field of fractions of A. As a consequence, A is a Krull domain. Moreover, if B is factorial then so also is A. Now suppose k is an algebraically closed field of characteristic zero and $B = k[X, Y, Z]/(XY - f(Z))$, $f(Z) \in k[Z]\backslash k$. Then B is a two-dimensional affine normal domain over k and if $f(Z) \in k[Z]$ is a polynomial of degree $n > 1$, then B is not factorial. Let D be a nonzero locally nilpotent derivation on B such that $D(x) = 0$, $D(z) = x$ (x, z denote images of X, Z respectively in B). Since characteristic of k is zero, D induces a non-trivial exponential map ϕ on B such that $B^{\phi} = \mathrm{Ker}(D)$. It is easy to see that $\mathrm{Ker}(D) = k[x]$ (a polynomial algebra in one variable over k) and hence it is factorial. In view of these results, it is natural to ask the following question:

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Question. Let k be a field and let B be an affine normal domain over k. Let ϕ be a non-trivial exponential map on B and let $A = B^{\phi}$ be the ring of ϕ-invariants. Can one realize $\mathrm{Cl}(A)$ (the class group of A) as a subgroup of $\mathrm{Cl}(B)$?

In this paper, we prove the following theorem which says that the above question has an affirmative answer under an additional hypothesis. More precisely, we prove the following.

Theorem. *Let k be a field and let B be an affine normal domain over k. Let ϕ be a non-trivial exponential map on B and let $A = B^{\phi}$ be the ring of ϕ-invariants. Assume that for every height one prime ideal Q of B, $P = A \cap Q$ is a prime ideal of A of height ≤ 1. Then there exists a canonical group homomorphism $\Psi : \mathrm{Cl}(A) \to \mathrm{Cl}(B)$ which is injective. Moreover, if for every height one prime ideal P of A, $A_P \otimes_A B$ is factorial then Ψ is an isomorphism.*

We also give an example to show that if for some height one prime ideal Q of B the height of $A \cap Q \geq 2$, then (even abstractly) $\mathrm{Cl}(A)$ may not be realized as a subgroup of $\mathrm{Cl}(B)$.

2. Preliminaries

Throughout the paper, we will assume rings to be commutative with identity. For a ring R and a proper ideal I of R, R^* will denote the group of invertible elements of R and $\mathrm{rad}(I)$ will denote the radical ideal of I. $R^{[n]}$ denotes the polynomial algebra in n variables over R.

In this section, we collect some definitions and results for later use.

DEFINITION 2.1

Let B be a domain and let $D : B \to B$ be a derivation. D is said to be locally nilpotent if for every $b \in B$ there exists $n_0 \in \mathbb{N}$ (depending on b) such that $D^n(b) = 0$ for all $n \geq n_0$.

DEFINITION 2.2

Let B be a domain. A locally finite iterative higher derivation on B is a sequence $d_i : B \to B$ $(0 \leq i < \infty)$ satisfying following properties:

(1) $d_0 = id$.
(2) $d_n(a + b) = d_n(a) + d_n(b)$ where $a, b \in B$ for all $n \geq 0$.
(3) $d_n(a.b) = \sum_{i=0}^{n} d_i(a)d_{n-i}(b)$ for all $n \geq 0$.
(4) $d_i d_j = \binom{i+j}{i} d_{i+j}$ for all $i, j \geq 0$.
(5) For $b \in B$, there exists $n_0 \in \mathbb{N}$ (depending on b) such that $d_n(b) = 0$ for all $n \geq n_0$.

DEFINITION 2.3

Let B be a domain and $d_i : B \to B$ $(0 \leq i < \infty)$ be a locally finite iterative higher derivation. A ring morphism $\phi : B \to B[T]$ defined by

$$\phi(b) = \sum d_i(b)T^i$$

is called the *exponential map* (associated to a sequence d_i $(0 \leq i < \infty)$).

DEFINITION 2.4

Let B be a domain and let $\phi : B \to B[T]$ be the exponential map associated to a locally finite iterative higher derivation $d_i (0 \le i < \infty)$. Let $B^{\phi} = \{a \mid a \in B, \phi(a) = a\}$ (equivalently $B^{\phi} = \{a \mid d_i(a) = 0 \,\forall\, i \ge 1\}$). Then B^{ϕ} is called the ring of ϕ-invariants. We say ϕ is non-trivial if $B^{\phi} \neq B$.

Remark 2.5. Let B, ϕ, B^{ϕ} be as above. Since $B^{\phi} = \phi^{-1}(B)$, it is easy to see that B^{ϕ} is factorially and hence algebraically closed in B. As a consequence, $B^{\phi} = K \cap B$, where K denotes the field of fractions of B^{ϕ}.

If $\mathbb{Q} \subset B$, then giving an exponential map ϕ on B is equivalent to giving a locally nilpotent derivation D on B such that $B^{\phi} = \mathrm{Ker}(D)$.

DEFINITION 2.6

Let R be a domain and let P be a prime ideal of R. The n-th symbolic power $P^{(n)}$ of P is defined to be $R \cap P^n R_P$.

DEFINITION 2.7

A domain R is said to be a *Krull domain* if it satisfies the following properties:

(1) Every nonzero, nonunit $x \in R$ belongs to only finitely many height one prime ideals.
(2) For every height one prime ideal P of R, R_P is a discrete valuation ring.
(3) $R = \bigcap_{\mathrm{ht}(P)=1} R_P$.

Remark 2.8. Every Noetherian normal (integrally closed) domain is a Krull domain. Let R be a Krull domain and let x be a nonzero, nonunit element of R. Let $\{P_1, \dots, P_l\}$ be the set of all prime ideals of height one containing x. Let $xR_{P_i} = {P_i}^{n_i} R_{P_i}$. Then $xR = \bigcap_{1 \le i \le l} {P_i}^{(n_i)}$ is the reduced primary decomposition.

Let R be a Krull domain. The (ideal) class group $\mathrm{Cl}(R)$ of R is defined as follows: Let $S = \{[P] | P \in \mathrm{Spec}(R), \mathrm{ht}(P) = 1\}$. Let F be the free Abelian group on the set S. For a nonzero, nonunit $x \in R$, we associate an element $\mathrm{cl}(x) \in F$ as follows:

If $xR = \bigcap_{1 \le i \le l} {P_i}^{(n_i)}$ is the reduced primary decomposition, then

$$\mathrm{cl}(x) = \sum_{1 \le i \le l} n_i [P_i] \in F.$$

Let H be the subgroup of F generated by $\{\mathrm{cl}(x) | x \in R \backslash R^* \cup \{0\}\}$.

DEFINITION 2.9

$\mathrm{Cl}(R) = F/H$.

Remark 2.10. Let K be the field of fractions of R. Let $f \in K^*$. We associate to f an element $\mathrm{cl}(f)$ of H as follows: If $f = x/y,\ x, y \in R$, then we write $\mathrm{cl}(f) = \mathrm{cl}(x) - \mathrm{cl}(y)$. It is easy to see that $\mathrm{cl}(f)$ is a well defined element of H and this association induces a group isomorphism from K^*/R^* to H.

Elements of F are called cycles of codimension one. An element of F of the type $\sum_{1\le i\le l} n_i[P_i]$, $n_i \ge 0$ is called an effective cycle. Let P be a prime ideal of R of height 1 and let n be a positive integer. Let $x \in P^n$. Then it is easy to see that the cycle $\mathrm{cl}(x) - n[P]$ is effective. As a consequence, every element of $\mathrm{Cl}(R) = F/H$ is the image of an effective cycle.

Let $\sum m_j[P_j]$, $m_j > 0$ be an effective cycle. Then the ideal $\bigcap P_j^{(m_j)}$ is a principal ideal if and only if the cycle $\sum m_j[P_j] \in H$.

It easy to see that R is factorial if and only if $\mathrm{Cl}(R) = 0$.

Remark 2.11. Class group can be defined in a more general set up (see [2], pages 130–149 for definition and related results).

DEFINITION 2.12

The Rees algebra of an ideal I in a ring R is defined to be

$$R[IW] = \bigoplus_{n=0}^{\infty} I^n W^n \subseteq R[W].$$

DEFINITION 2.13

Let R be a ring and I be an ideal of R. An element $r \in R$ is said to be integral over I if there exists $a_i \in I^i$ such that

$$r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0.$$

The set $\bar{I} = \{r \in R | r \text{ is integral over } I\}$ is called the integral closure of I.

Remark 2.14. It is well known that the integral closure $\bar{I}$ of I is an ideal of R (see [3], Corollary 1.3.1, page 6).

The proof of the following lemma can be found in [3] (see Proposition 5.2.1, page 96).

Lemma 2.15. *Let R be a ring and I be an ideal of R. Then the integral closure of the Rees algebra $R[IW]$ in $R[W]$ is $\bigoplus_{n=0}^{\infty} \bar{I}^n W^n$.*

We presume that this proposition is well known to experts. However, for the sake of completeness, we give a proof.

PROPOSITION 2.16

Let k be a field of characteristic zero and let $F(X_1, X_2, \ldots, X_{n+1}) \in k[X_1, X_2, \ldots, X_{n+1}]$ be a homogeneous prime element such that $A = k[X_1, X_2, \ldots, X_{n+1}]/(F)$ is normal. Let $m = (x_1, x_2, \ldots, x_{n+1})$ be the maximal ideal of A where x_i denotes the image of X_i in A. Then $mA[mW]$ is a prime ideal of height one and the Rees algebra $A[mW]$ is an affine normal domain.

Proof. It is obvious that $A[mW]$ is affine over k. For the sake of simplicity of notation, we write $B = A[mW]$. It is easy to see that $B/mB \cong \bigoplus_{i=0}^{\infty} \frac{m^i}{m^{i+1}} = gr_m A$.

Claim. mB is a prime ideal of B of height one.

Proof of the Claim. Since m is a maximal ideal of A of height n, $gr_m A$ is an affine k-algebra of dimension n. Let $\bar{x}_i$ denote the image of x_i in m/m^2. Then $gr_m A = k[\overline{x}_1, \ldots, \overline{x}_{n+1}]$ (an affine k-algebra of dimension n). Since $F(x_1, \ldots, x_{n+1}) = 0$ in A and $F(X_1, X_2, \ldots, X_{n+1})$ is homogeneous, we have $F(\overline{x}_1, \ldots, \overline{x}_{n+1}) = 0$ in $gr_m A$. Hence, as F is a prime element, we see that $gr_m A$ is an affine integral domain of dimension n. So $gr_m A \cong B/mB$ is an affine integral domain of dimension $n = \dim B - 1$. Therefore mB is a prime ideal of B of height one.

Since A is normal, $A[W]$ is also normal. Therefore, to prove that Rees algebra $A[mW]$ is normal it is enough to show that $A[mW]$ is integrally closed in $A[W]$. Hence, in view of Lemma 2.15, it is enough to prove that m^l is integrally closed in A for every $l \geq 1$. For sake of simplicity of notation, we write $I = m^l$.

Let $r(\neq 0) \in A$ be such that it is integral over I, i.e.,

$$r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0,$$

where $a_i \in I^i$. It is obvious that $r \in m$. We want to show that $r \in I = m^l$. That is, if s is the greatest integer such that $r \in m^s$, then $l \leq s$.

If not, then $l > s$. Since $gr_m A$ is a domain and $r \in m^s - m^{s+1}$, $r^n \in m^{sn} - m^{sn+1}$. But $r^n = -\sum_{i=1}^{n} a_i r^{n-i}$, where $a_i \in I^i = m^{li}$ and $r^{n-i} \in m^{s(n-i)}$. Now $a_i r^{n-i} \in m^{li+s(n-i)} = m^{(l-s)i+sn}$. As $l - s > 0$ and $i \geq 1$, we have $sn + (l-s)i \geq sn + 1$. Thus, $\sum_{i=1}^{n} a_i r^{n-i} \in m^{sn+1}$; a contradiction to the fact that $r^n \in m^{sn} - m^{sn+1}$. Thus, $l \leq s$. □

COROLLARY 2.17

Let k, A, m, B be as in Proposition 2.16. Assume that $\dim(A) \geq 2$. *Then there exists a short exact sequence of Abelian groups*

$$0 \to \mathbb{Z} \to \mathrm{Cl}(B) \to \mathrm{Cl}(A) \to 0.$$

Proof. Let $mB = Q$ and let $U_1 = \mathrm{Spec}(B) \backslash V(Q)$. By Proposition 2.16, Q is a prime ideal of B of height one. By abuse of notation, we denote $[Q]$ as an element of $\mathrm{Cl}(B)$. By part (c) of p. 133, Proposition 6.5 of [2], there is an exact sequence

$$\mathbb{Z} \to \mathrm{Cl}(B) \to \mathrm{Cl}(U_1) \to 0,$$

where the first map is defined by $1 \to [Q]$.

Claim. $\mathbb{Z} \to \mathrm{Cl}(B)$ is injective.

Proof of the Claim. To prove the claim, it is enough to prove that $Q^{(n)}$ is not a principal ideal (i.e. $n[Q] \neq 0$ in $\mathrm{Cl}(B)$) for any positive integer n.

Suppose $Q^{(n)} = bB$. Since $m^n \subset A \cap Q^{(n)} \subset Q^{(n)} = bB$ and A is factorially closed in $B \subset A[W]$, it follows that $b \in m = A \cap Q$. Let P be a prime ideal of A of height one such that $b \in P$. Since $m \not\subset P$, there exists $c \in m$ such that $c \notin P$. Note that $B_c = A_c^{[1]}$. As a consequence, there exists a prime ideal Q_1 of B of height one such that $P = A \cap Q_1$. Therefore, $b \in Q_1$ which leads to a contradiction as $Q \neq Q_1$.

Let $U_2 = \mathrm{Spec}(A) \backslash V(m)$. It can be shown that $U_1 = U_2 \times \mathbb{A}^1$. Therefore, by p. 134, Proposition 6.6 of [2], $\mathrm{Cl}(U_1) \cong \mathrm{Cl}(U_2)$. Since $\mathrm{ht}(m) \geq 2$, by part (b) of p. 133, Proposition 6.5 of [2], $\mathrm{Cl}(U_2) \cong \mathrm{Cl}(A)$. Thus we get an exact sequence

$$0 \to \mathbb{Z} \to \mathrm{Cl}(B) \to \mathrm{Cl}(A) \to 0. \qquad \square$$

We end this section by stating the following result. Its proof is implicit in the proof of Theorem 3.5 of Bhatwadekar and Dutta (see [1], Theorem 3.5).

Theorem 2.18. *Let R be a Noetherian normal domain containing the field $\mathbb{Q}$ of rationals. Let I be a prime ideal of R of height one. Let $A = \bigoplus_{n \geq 0} I^{(n)} W^n \subset R[W]$ be the symbolic Rees algebra. Let $B = R^{[2]}$. Then there exists a locally nilpotent R-derivation D of B such that A is canonically R-isomorphic to* $\mathrm{Ker}(D)$. *Moreover there exists an R-sequence $\{s, t\}$ such that*

(1) $A_s = R_s^{[1]}$ *and* $A_t = R_t^{[1]}$.
(2) $B_s = A_s^{[1]}$ *and* $B_t = A_t^{[1]}$.

3. Main result

Let B be a domain and let ϕ be a non-trivial exponential map on B. Let $A = B^{\phi}$ be the ring of ϕ-invariants. Recall that $A = K \cap B$, where K denotes the field of fractions of A. As a consequence, if B is a Krull domain then so is A.

Lemma 3.1. *Let $A \hookrightarrow B$ be an extension of Krull domains. Assume that $A = K \cap B$, where K denotes the field of fractions of A. Let P be a prime ideal of A of height one. Then there exists a prime ideal Q of B of height one such that $P = A \cap Q$.*

Proof. First note that, since $A = K \cap B$, $xA = A \cap xB$ for every $x \in A$. Let $x \in P$ be a nonzero element. Then, as $xA = A \cap xB$, x is a nonzero, nonunit element of B. Since B is Krull, $\mathrm{rad}(xB) = \bigcap_{1 \leq i \leq l} Q_i$, where Q_i is a prime ideal of B of height one. Let $P_i = A \cap Q_i$. Since $x \in A$ and $xA = A \cap xB$, we have $\mathrm{rad}(xA) = A \cap \mathrm{rad}(xB) = \bigcap_{1 \leq i \leq l} P_i$. Hence, as $xA \subseteq P$, we have $\bigcap_{1 \leq i \leq l} P_i \subseteq P$. Therefore $P_j \subseteq P$ for some $j \in \{1, 2, \ldots, l\}$. Since $xA \neq 0$, we see that $P_j \neq 0$. Hence $P = P_j = A \cap Q_j$ since P is a prime ideal of A of height one. Thus $Q = Q_j$ is a prime ideal of B of height one such that $P = A \cap Q$. $\square$

Let $A \hookrightarrow B$ be an extension of Krull domains. Let Q be a prime ideal of B of height one such that $P = A \cap Q$ is a prime ideal of height one. Let $PB_Q = Q^r B_Q$. We say that r is the ramification index of P in Q and we write $r = r(P, Q)$.

Lemma 3.2. *Let $A \hookrightarrow B$ be an extension of Krull domains. Let Q be a prime ideal of B of height one such that $P = Q \cap A$ is a prime ideal of A of height one. Let $x \in P$ be a nonzero element. Let $Q^{(n_1)}$ be the Q-primary component of xB and $P^{(n)}$ be the P-primary component of xA. Then $n_1 = rn$, where $r = r(P, Q)$ is the ramification index of P in Q.*

Proof. Let p be a uniformizing parameter of the discrete valuation ring A_P and q be a uniformizing parameter of the discrete valuation ring B_Q. Since $P^{(n)}$ is the P-primary component of xA, $xA_P = p^n A_P$. Since $A \cap Q = P$, $A_P \hookrightarrow B_Q$ is a local homomorphism of discrete valuation rings and $pB_Q = q^r B_Q$, where $r = r(P, Q)$ is the ramification index of P in Q. Therefore, $xB_Q = q^{rn} B_Q$. Hence $n_1 = rn$. □

PROPOSITION 3.3

Let $A \hookrightarrow B$ be an extension of Krull domains such that $A = K \cap B$, where K denotes the field of fractions of A. Assume that for every height one prime ideal Q of B, $A \cap Q$ is a prime ideal of height ≤ 1. Then there exists a (canonical) group homomorphism $\Psi : \mathrm{Cl}(A) \to \mathrm{Cl}(B)$.

Proof. Let $S_A = \{[P] | P \in \mathrm{Spec}(A), \mathrm{ht}(P) = 1\}$ and let $S_B = \{[Q] | Q \in \mathrm{Spec}(B), \mathrm{ht}(Q) = 1\}$. Let F_A and F_B be free abelian groups on S_A and S_B respectively. Let $\Theta : F_A \to F_B$ be a group homomorphism defined as follows:

Let P be a prime ideal of A of height one and let $\{Q_1, \ldots, Q_e\}$ be the set of all prime ideals of B of height one such that $P = A \cap Q_i$. Note that this set is non-empty in view of Lemma 3.1. Define $\Theta([P]) = \sum_{i=1}^{e} r(P, Q_i)[Q_i]$. Note that $\Theta : F_A \to F_B$ is a monomorphism. Let x be a nonzero nonunit element of A. □

Claim. $\Theta(\mathrm{cl}_A(x)) = \mathrm{cl}_B(x)$.

Proof of the Claim. Note that if Q is a prime ideal of B of height one such that $x \in Q$ then $x \in P = A \cap Q$ and hence, by our assumption, P is a prime ideal of height one. Let $P^{(n)}$ be the P-primary component of xA. Let $\{Q_1, Q_2, \ldots, Q_e\}$ be the set of all prime ideals of B of height one such that $P = A \cap Q_i$. Let $Q_i^{(n_i)}$ be the Q_i-primary component of xB. Then, by Lemma 3.2, we know that $n_i = nr(P, Q_i)$. Therefore, the element $\sum_{i=1}^{e} n_i[Q_i] = n \sum_{i=1}^{e} r(P, Q_i)[Q_i] = n\Theta([P])$. From the above discussion, it follows very easily that $\Theta(\mathrm{cl}_A(x)) = \mathrm{cl}_B(x)$.

Hence Θ induces a group homomorphism $\Psi : \mathrm{Cl}(A) \to \mathrm{Cl}(B)$. □

Remark 3.4. Let $A, B, F_A, F_B, \Theta, \Psi$ be as in Proposition 3.3. Then, though Θ is a monomorphism, the induced group homomorphism Ψ need not be injective as the following example shows.

Example 3.5. Let k be a field of characteristic zero and let $B = k[T_1, T_2]/({T_1}^4 + {T_2}^2 - 1)$. It is easy to see that B is a Dedekind domain. Let $x = {t_1}^2$, $y = t_1 t_2 \in B$ and let $A = k[x, y]$ be a k-subalgebra of B. Then B is an integral extension of A and hence A is an affine k-domain of dimension one. Since $y^2 = x - x^3$, it follows that $A \simeq k[W_1, W_2]/(W_2^2 - W_1 + W_1^3)$. Therefore, A is also a Dedekind domain. Let $m = (x, y)A$ be the maximal ideal of A. It is easy to see that m is not a principal ideal of A but $mB = ({t_1}^2, t_1 t_2)B = t_1 B$.

Thus $[m] \notin H_A$ but $\Theta([m]) \in H_B$ and hence Ψ is not injective.

Theorem 3.6. *Let k be a field and let B be an affine normal domain over k. Let ϕ be a non-trivial exponential map on B and let $A = B^{\phi}$ be the ring of ϕ-invariants. Assume* *that for every height one prime ideal Q of B, $P = A \cap Q$ is a prime ideal of A of height ≤ 1. Then there exists a canonical group homomorphism $\Psi : \mathrm{Cl}(A) \to \mathrm{Cl}(B)$ which is injective. Moreover, if for every height one prime ideal P of A, $A_P \otimes_A B$ is factorial, then Ψ is an isomorphism.*

Proof. Note that since A is the ring of ϕ-invariants, A is factorially closed in B and hence $A = K \cap B$, where K is the field of fractions of A. Therefore, A is also a Krull domain. By the hypothesis, for every height one prime ideal Q of B, $A \cap Q$ is a prime ideal of height ≤ 1. Therefore, $A \hookrightarrow B$ is an extension of Krull domains satisfying all the conditions of Proposition 3.3.

Let F_A,F_B, Θ and Ψ be as in Proposition 3.3. We want to show that Ψ is injective.

Let H_A be the subgroup of F_A generated by $\{\mathrm{cl}_A(a) | a \in A \backslash A^* \cup \{0\}\}$ and H_B be the subgroup of F_B generated by $\{\mathrm{cl}_B(b) | b \in B \backslash B^* \cup \{0\}\}$.

Note that Ψ is injective if $\Theta^{-1}(H_B) = H_A$.

From the proof of Proposition 3.3, it is obvious that $\Theta(H_A) \subseteq H_B$. Let $\eta \in F_A$ be such that $\Theta(\eta) \in H_B$. We want to show $\eta \in H_A$. Note that, in view of Remark 2.10, without loss of generality, we may assume that η is an effective cycle, say $\eta = \sum_{i=1}^{l} n_i[P_i]$, where $n_i > 0$. Therefore, $\Theta(\eta) \in F_B$ is also an effective cycle.

Now suppose $\Theta(\eta) \in H_B$. Since $\Theta(\eta)$ is effective, $\Theta(\eta) \in H_B$ implies that there exists $b \in B \backslash B^* \cup \{0\}$ such that $\Theta(\eta) = \mathrm{cl}_B(b)$.

Claim. $b \in A$.

Proof of the Claim. Note that, since $\mathrm{cl}_B(b) = \Theta(\eta) = \sum_{i=1}^{l} n_i \Theta([P_i])$, where $n_i > 0$, a prime ideal Q of B of height one contains b if and only if $A \cap Q = P_i$ for some $i \in \{1, 2, \ldots, l\}$. As a consequence, if Q' is a prime ideal of B of height one such that $A \cap Q' = 0$, then $b \notin Q'$. Hence $b \in (K \otimes_A B)^*$. But as A is the ring of ϕ-invariants, $K^* = (K \otimes_A B)^*$. Therefore, $b \in K \cap B = A$. Thus the claim is proved.

Note that $\Theta(\eta) = \mathrm{cl}_B(b) = \Theta(\mathrm{cl}_A(b))$. Therefore, as $\Theta : F_A \to F_B$ is injective, we have $\eta = \mathrm{cl}_A(b) \in H_A$.

Now assume that $A_P \otimes_A B$ is factorial for every height one prime ideal P of A.

Since $K \otimes_A B = K^{[1]}$, $\mathrm{Cl}(K \otimes_A B) = 0$. Therefore, $\mathrm{Cl}(B)$ is generated by images of $[Q]$, where Q is a prime ideal of B of height one such that $P = A \cap Q$ is a prime ideal of A of height one.

Claim. Q is the only prime ideal of B of height one such that $P = A \cap Q$. Moreover, $r(P, Q) = 1$.

Proof of the Claim. Let $B' = A_P \otimes_A B$. By the hypothesis, B' is a factorial domain having a non-trivial exponential map ϕ_P such that A_P is the ring of ϕ_P-invariants. Therefore, PB' is a prime ideal of B' and $PA_P = A_P \cap PB'$. Hence the claim follows.

In view of the above discussion, it follows that Ψ is surjective. □

COROLLARY 3.7

Let k be a field and B be an affine normal domain over k of dimension two. Let ϕ be a non-trivial exponential map on B and let A be the ring of ϕ-invariants. Then there exists a canonical group homomorphism $\Psi : \mathrm{Cl}(A) \to \mathrm{Cl}(B)$ which is injective.

Remark 3.8. Let A and B be as in Theorem 3.6 and let $x, y \in A$. Then, since A is factorially closed in B, if $\{x, y\}$ is a B-regular sequence, then $\{x, y\}$ is also a A-regular sequence. Since A and B are Krull domains, the condition that for every height one prime ideal Q of B, $P = A \cap Q$ is a prime ideal of height ≤ 1 is equivalent to the condition that if $\{x, y\}$ is a A-regular sequence then $\{x, y\}$ is also a B-regular sequence.

COROLLARY 3.9

Let R, I, A, B, D, s, t be as in Theorem 2.18. Then $\mathrm{Cl}(A) \cong \mathrm{Cl}(B)$.

Proof. Since $\{s, t\}$ is a R-regular sequence and $B = R^{[2]}$, $\{s, t\}$ is also a B-regular sequence. Therefore, as $A = \mathrm{Ker}(D)$, $\{s, t\}$ is also a A-regular sequence.

Let Q be a prime ideal of B of height one and $P = A \cap Q$. As $\{s, t\}$ is a B-regular sequence and height of Q is one, Q can not contain both the elements s and t. Suppose $s \notin Q$, then $s \notin P$. Since $B_s = A_s^{[1]}$ and $PA_s = A_s \cap QB_s$, P is a prime ideal of A of height ≤ 1. Therefore, in view of Theorem 3.6, there exists a monomorphism $\Psi : \mathrm{Cl}(A) \to \mathrm{Cl}(B)$.

To show that Ψ is an isomorphism we need to show that $A_P \otimes_A B$ is factorial for every height one prime ideal P of A. Let P be a prime ideal of A of height one. Note that A_P is a discrete valuation ring. As before, we see that, since A is a Krull domain and $\{s, t\}$ is A-regular, P can not contain both elements s and t. Without loss of generality, we may assume that $t \notin P$. Then $B_t = A_t^{[1]}$ and PA_t is a prime ideal of A_t of height one. Therefore $A_P \otimes_A B = A_P^{[1]}$. Since the discrete valuation ring A_P is factorial, $A_P \otimes_A B$ is also factorial. Hence Ψ is an isomorphism. □

Remark 3.10. Let $R = \mathbb{C}[X, Y, Z]/(Y^2Z - X^3 + XZ^2)$. Then there exists a prime ideal I of R of height one such that the symbolic Rees algebra $A = \bigoplus_{n \geq 0} I^{(n)} W^n \subset R[W]$ is not even Noetherian (see [1], Example 3.6). Note that A is a Krull domain and $\mathrm{Cl}(A) \cong \mathrm{Cl}(R^{[2]}) \cong \mathrm{Cl}(R)$.

Let k be a field and B be an affine normal domain over k. Let ϕ be a non-trivial exponential map and let A be the ring of ϕ-invariants. If there exists a prime ideal Q of B of height one such that $P = A \cap Q$ is a prime ideal of height ≥ 2, then $\mathrm{Cl}(A)$ may not be realized as a subgroup of $\mathrm{Cl}(B)$ even abstractly as the following example shows.

PROPOSITION 3.11

Let k be a field of characteristic zero and let $A = k[X, Y, Z]/(XY - Z^2)$. Let $m = (x, y, z)$ be the maximal ideal of A and let B be the Rees algebra $A[mW] \subseteq A[W]$. Then $\mathrm{Cl}(A) = \mathbb{Z}/(2)$ and $\mathrm{Cl}(B) = \mathbb{Z}$.

Proof. It is well known that A is a normal domain which is not factorial. In fact, if $P = (x, z)$, then P is a prime ideal of A of height one such that P is not principal but $P^{(2)} = xA$. Moreover, $\mathrm{Cl}(A)$ is a cyclic group generated by the image of $[P]$ and hence $\mathrm{Cl}(A) = \mathbb{Z}/(2)$.

Since $XY - Z^2$ is a homogeneous prime element, by Proposition 2.16, $B = A[mW] \subset A[W]$ is an affine normal domain over k (of dimension 3). Moreover, $Q = mB$ is a prime ideal of B of height one. Note that $m = A \cap Q$.

Let $f = xW, g = yW, h = zW \in B$. Then $fgh \notin Q$. Since $xy = z^2$, $yf = xg$, $zg = yh$, $zf = xh$ and $Q = (x, y, z)B$, it follows that $QB_Q = xB_Q = yB_Q = zB_Q$.

Note that $B_y = A_y[W]$ and $PA_y = zA_y$ is the only prime ideal of A_y of height one which contains x. Therefore, there exists only one prime ideal Q_1 of B of height one such that $y \notin Q_1, x \in Q_1$. It is easy to see that $P = A \cap Q_1$ and the ramification index $r(P, Q_1) = 1$. Hence it follows that if Q' is a prime ideal of B of height one such that $x \in Q'$, then either $Q' = Q$ or Q_1. Moreover, since $QB_Q = xB_Q$, $xA = P^{(2)}$ and $r(P, Q_1) = 1$, we see that $\mathrm{cl}_B(x) = [Q] + 2[Q_1]$.

Since $B_x = A_x[W]$ and A_x is factorial, B_x is also factorial. Therefore, as $\mathrm{cl}_B(x) = [Q] + 2[Q_1]$, we see that $\mathrm{Cl}(B)$ is a cyclic group generated by the image of $[Q_1]$.

Claim. $Q_1^{(n)}$ is not a principal ideal of B for any positive integer n.

Proof of the Claim. Suppose $Q_1^{(n)} = bB$ for some $n \geq 1$. Since $P = A \cap Q_1$ and $r(P, Q_1) = 1$, it follows that $z^n \in Q_1^{(n)} = bB$. Since $B \subset A[W]$, A is factorially closed in B. Therefore, $b \in A \cap Q_1 = P$ and hence $b \in Q$, a contradiction as $bB = Q_1^{(n)}$ and $Q_1 \neq Q$. Thus the claim is proved.

In view of this Claim, it follows that $\mathrm{Cl}(B) = \mathbb{Z}$. □

Example 3.12. Let A, m, B be as in Proposition 3.11. Let $D_1 : A[W] \to A[W]$ be a locally nilpotent A-derivation such that $D_1(W) = 1$. Since $B = A[f, g, h] \subset A[W]$ and $D_1(f) = x$, $D_1(g) = y$, $D_1(h) = z$, D_1 gives rise to a locally nilpotent derivation D on B such that $A = \mathrm{Ker}(D)$. By Proposition 3.11, $\mathrm{Cl}(A) = \mathbb{Z}/(2)$ and $\mathrm{Cl}(B) = \mathbb{Z}$. Hence $\mathrm{Cl}(A)$ can not be realized as a subgroup of $\mathrm{Cl}(B)$ even abstractly.

Remark 3.13. Let k be a field of characteristic zero and let $A = k[X, Y, Z, W]/(XY - ZW)$. Let $m = (x, y, z, w)$ be the maximal ideal of A and let $B = A[mW] \subset A[W]$. By Corollary 2.17, we have

$$0 \to \mathbb{Z} \to \mathrm{Cl}(B) \to \mathrm{Cl}(A) \to 0.$$

It is known that $\mathrm{Cl}(A) = \mathbb{Z}$ and therefore, $\mathrm{Cl}(B) = \mathbb{Z} \oplus \mathbb{Z}$. Thus $\mathrm{Cl}(A)$ can be thought as a subgroup of $\mathrm{Cl}(B)$ even though there is no canonical group homomorphism from $\mathrm{Cl}(A)$ to $\mathrm{Cl}(B)$.

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