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Signs of Fourier coefficients of cusp form at sum of two squares

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Abstract. In this article, we investigate the sign changes of the sequence of coefficients at sum of two squares where the coefficients are the Fourier coefficients of the normalized Hecke eigen cusp form for the full modular group. We provide the quantitative result for the number of sign changes of the sequence in a small interval.

Keywords. Cusp form; Fourier coefficients; sign change; asymptotic behaviour; Rankin–Selberg L-function.

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1. Introduction

Let $S_k(\Gamma)$ be the space of holomorphic cusp forms of even integral weight k for the full modular group $\Gamma = SL(2, \mathbb{Z})$. Suppose that $f(z)$ is an eigenfunction of all Hecke operators belonging to $S_k(\Gamma)$. Then the Hecke eigenform $f(z)$ has the following Fourier expansion at the cusp ∞:

$$f(z) = \sum_{n=1}^{\infty} a(n) \mathrm{e}^{2\pi i n z},$$

where we have normalized $f(z)$ such that $a(1) = 1$. Instead of $a(n)$, one often considers the normalized Fourier coefficients

$$\lambda(n) = \frac{a(n)}{n^{\frac{k-1}{2}}}.$$

Here $\lambda(n)$ are real and satisfy the multiplicative property that

$$\lambda(m)\lambda(n) = \sum_{d|(m,n)} \lambda\left(\frac{mn}{d^2}\right), \tag{1.1}$$

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where m and n are any positive integers. It also satisfies the celebrated Deligne's bound [2] that

$$\lambda(n) \leq d(n) \ll_{\epsilon} n^{\epsilon}, \tag{1.2}$$

where $d(n)$ is the number of divisors of n and ϵ is any arbitrarily small positive constant.

Sign changes of Fourier coefficients $a(n)$ for $n \geq 1$ of cusp forms in one or several variables have been studied in various aspects. It is known that, if the Fourier coefficients of a cusp form are real then they change signs infinitely often [6]. Further, many quantitative results for the number of sign changes for the sequence of the Fourier coefficients have been established. The sign changes of the subsequence of the Fourier coefficients at prime numbers was first studied by Ram Murty [14] (see also the work of Kohnen and Sengupta [9]). Later, Meher *et al.* [13] studied the problem for the subsequence $\{a(n^j)\}_{n\geq 1}$ $(j = 2, 3, 4)$. In 2014, Kohnen and Martin [7] proved that the subsequence $\{a(p^{jn})\}_{n\geq 0}$ has infinitely many sign changes for almost all primes p and $j \in \mathbb{N}$.

In this article, we are mainly interested in studying the sign changes of the subsequence $\{\lambda(n_k)\}_{n_k\geq 1}$, where n_k can be written as a sum of two squares i.e., $n_k = c^2 + d^2$ for some integers c and d. In our main theorem, we obtain the behaviour of infinitely many sign changes of the sequence. In fact, we provide the lower bound for the number of sign changes in the interval $(x, 2x]$, for sufficiently large x. For more details on problems related to sign changes of Fourier coefficients, we refer to [3,8,12].

2. Main theorem

Let $f \in S_k(\Gamma)$ be a normalized Hecke eigenform of even integral weight k for the full modular group and $\lambda(n)$ denotes its n-th normalized Fourier coefficient. We state our main result.

Theorem 2.1. *The sequence* $\{\lambda(c^2 + d^2)\}_{c,d\geq 1}$ *has infinitely many sign changes. Moreover, the sequence changes its signs at least* $x^{1/8-2\epsilon}$ *times in the interval* $(x, 2x]$ *for sufficiently large* x, *where* ϵ *is arbitrarily small positive constant.*

3. Background set up

To get the sign change at the sum of two squares, one needs to consider the partial sums

$$S_1(x) := \sum_{n=c^2+d^2\leq x} \lambda(c^2 + d^2),$$

$$S_2(x) := \sum_{n=c^2+d^2\leq x} \lambda^2(c^2 + d^2)$$

for $x \geq 1$ and $(c, d) \in \mathbb{Z}^2$. Also, one needs to find the upper bound of $S_1(x)$ and the approximate behaviour of $S_2(x)$. As we are just interested in the sign change of the coefficients at sum of two squares, one can use $r_2(n)$ (the number of ways n can be written as sum of two squares) as the weighted characteristic function of the sum of two squares. Also, $r_2(n)$ is always non-negative, so to consider the sign change of the Fourier coefficients at sum of two squares, it is enough to consider the following sums:

$$S(x)=\sum_{n\le x}\lambda(n)r_2(n),$$
$$S_f(x)=\sum_{n\le x}\lambda^2(n)r_2(n),$$

as multiplication by $r_2(n)$ does not affect the sign of $\lambda(n)$ and restricts the sum to all those n, which can be written as the sum of two squares. In number theory, the function $r_2(n)$ has received much attention. It is well-known that $\theta^2(z)=1+\sum_{n=1}^{\infty}r_2(n)\mathrm{e}^{2\pi inz}$ is a modular form of weight 1 for $\Gamma_0(4)$ with character χ_{-4}, where $\theta(z)=1+2\sum_{n=1}^{\infty}\mathrm{e}^{2\pi in^2z}$ is the classical theta function. A formula for $r_2(n)$ is given by

$$r_2(n)=4\sum_{d|n}\chi_{-4}(d),$$

see for example [4]. We set $r(n):=\frac{1}{4}r_2(n)=\sum_{d|n}\chi_{-4}(d)$. So for any prime p, we have

$$r(p)=1+\chi_{-4}(p),\quad r(p^2)=1+\chi_{-4}(p)+\chi_{-4}(p^2), \tag{3.1}$$

and so on. We define

$$L(s):=\sum_{n=1}^{\infty}\frac{\lambda^2(n)r(n)}{n^s}. \tag{3.2}$$

We use the following L-functions associated to f defined by

$$L(s,f\times f):=\sum_{n=1}^{\infty}\frac{\lambda^2(n)}{n^s} \tag{3.3}$$

and

$$L(s,f\times f\times\chi_{-4}):=\sum_{n=1}^{\infty}\frac{\chi_{-4}(n)\lambda^2(n)}{n^s}, \tag{3.4}$$

where $\mathrm{Re}(s)>1$. Note that one can think of $L(s,\chi_{-4}\times f\times f)$ as the convolution L-function associated to f and $g=f\otimes\chi_{-4}=\sum_{n\ge1}a(n)\chi_{-4}(n)\mathrm{e}^{2\pi inz}$. Then the Rankin–Selberg L-functions associated to f and $g=f\otimes\chi_{-4}$ are given by

$$L(s,f\otimes f):=\zeta(2s)L(s,f\times f), \tag{3.5}$$
$$L(s,f\otimes f\otimes\chi_{-4}):=\zeta(2s)L(s,f\times f\times\chi_{-4}). \tag{3.6}$$

These L-functions are well studied, they have analytic continuation and also satisfy functional equation. For details, we refer to [1]. The following lemmas are important to study the average behaviour of $S(x)$ and $S_f(x)$.

Lemma 3.1. *For Re*$(s)>1$, *we have*

$$L(s)=L(s,f\times f)L(s,f\times f\times\chi_{-4})U(s), \tag{3.7}$$

where $U(s)$ *converges absolutely and uniformly in the half-plane* $\mathrm{Re}(s) \geq 1/2+\epsilon$ *for any* $\epsilon > 0$. $L(s, f \times f)$ *and* $L(s, f \times f \times \chi_{-4})$ *are defined as in* (3.3) *and* (3.4) *respectively.*

Proof. The Euler product representation of $L(s, f \times f)$ and $L(s, f \times f \times \chi_{-4})$ can be written in the form

$$L(s, f \times f) = \prod_p \left(1 + \frac{\lambda^2(p)}{p^s} + \frac{\lambda^2(p^2)}{p^{2s}} + \cdots\right) \tag{3.8}$$

and

$$L(s, f \times f \times \chi_{-4}) = \prod_p \left(1 + \frac{\lambda^2(p)\chi_{-4}(p)}{p^s} + \frac{\chi_{-4}(p^2)\lambda^2(p^2)}{p^{2s}} + \cdots\right) \tag{3.9}$$

respectively, for $\mathrm{Re}(s) > 1$. Since the coefficients in (3.2) are multiplicative, after applying (3.1) and (1.1), we have

$$\begin{aligned} L(s) &= \sum_{n=1}^{\infty} \frac{\lambda^2(n)r(n)}{n^s} \\ &= \prod_p \left(1 + \frac{\lambda^2(p)r(p)}{p^s} + \frac{\lambda^2(p^2)r(p^2)}{p^{2s}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{\lambda^2(p)(1+\chi_{-4}(p))}{p^s}\right. \\ &\quad \left. + \frac{\lambda^2(p^2)(1+\chi_{-4}(p)+\chi_{-4}(p^2))}{p^{2s}} + \cdots\right) \end{aligned} \tag{3.10}$$

for $\mathrm{Re}(s) > 1$. Now from (3.8), (3.9) and (3.10), we have, for $\mathrm{Re}(s) > 1$,

$$L(s) = L(s, f \times f)L(s, f \times f \times \chi_{-4})U(s),$$

where

$$U(s) = \prod_p \left(1 - \frac{\chi_{-4}(p)(2\lambda^2(p) - 1)}{p^{2s}} + \cdots\right).$$

It follows from (1.2) that $U(s)$ converges absolutely and uniformly in the half-plane $\mathrm{Re}(s) \geq 1/2 + \epsilon$. This completes the proof. □

Lemma 3.2. *For any* $\epsilon > 0$ *and* $0 \leq \sigma \leq 1$, *we have*

$$\frac{s-1}{s+1} L(\sigma + it, f \otimes f) \ll_{f,\epsilon} (1 + |t|)^{2(1-\sigma)+\epsilon}. \tag{3.11}$$

Proof. The proof involves standard arguments using the Stirling formula for Gamma function in the functional equation of $L(\sigma + it, f \otimes f)$ and the Phargmen–Lindelöf principle. One can refer to page 100, chapter 5 of [5]. □

Lemma 3.3. *For* $1/2 \leq \sigma \leq 3/4$, *we have*

$$\text{(i)} \qquad \int_0^T |L(\sigma + it, f \otimes f)|^2 \mathrm{d}t \ll T^{4-4\sigma} (\log T)^{1+\epsilon} \tag{3.12}$$

and

$$\text{(ii)} \qquad \int_T^{2T} |L(\sigma + it, f \otimes f)|^2 \mathrm{d}t \ll T^{4-4\sigma} (\log T)^{1+\epsilon}. \tag{3.13}$$

Proof. We refer to [11] for a proof. We also mention that the second inequality is valid for Rankin–Selberg convolution of two different forms f and g. The proof goes exactly as in the case $f = g$, so we omit the details here. For more details, we refer to §4 and §5 of [11]. □

Now we state the main proposition, which provides the asymptotic behaviour of $S_f(x)$ and the upper bound for $S(x)$. We use the basic tool of [10] to prove this proposition.

PROPOSITION 3.4

We have

$$S(x) \ll x^{3/4+\epsilon} \tag{3.14}$$

and

$$S_f(x) = Cx + O_{f,\epsilon}(x^{3/4+\epsilon}), \tag{3.15}$$

where C *is a constant and* $\epsilon > 0$ *is arbitrarily small.*

Proof. Define

$$L_j(s) = \begin{cases} L(s, f \otimes \theta^2) & \text{if } j = 1, \\ L(s) & \text{if } j = 2. \end{cases} \tag{3.16}$$

Now by using truncated Perron's formula (cf. [5, Proposition 5.54]), we have

$$S(x) = \sum_{n \leq x} \lambda(n) r_2(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} L_1(s) \frac{x^s}{s} \mathrm{d}s + O\left(\frac{x^{1+\epsilon}}{T}\right) \tag{3.17}$$

and

$$S_f(x) = 4 \sum_{n \leq x} \lambda^2(n) r(n) = \frac{4}{2\pi i} \int_{b-iT}^{b+iT} L_2(s) \frac{x^s}{s} \mathrm{d}s + O\left(\frac{x^{1+\epsilon}}{T}\right), \tag{3.18}$$

where $b = 1+\epsilon$ and $1 \leq T \leq x$ is a parameter to be chosen later. We observe that $L_1(s)$ has an analytic continuation to the whole complex plane as it is a Rankin–Selberg L-function of a cusp form f and θ^2 (see [1] for details on Rankin–Selberg L-function). Also, it follows from Lemma 3.1 and the analytic continuation of Rankin–Selberg L-function (cf. [1]) that $L_2(s)$ can be analytically continued to the half plane $\mathrm{Re}(s) > 1/2$. In this region, $L_2(s)$ has a simple pole at $s = 1$.

Next we move the line of integration to $\mathrm{Re}(s) = 1/2+\epsilon$ and apply the Cauchy residue theorem to obtain

$$\begin{aligned}\sum_{n\leq x}\lambda(n)r_2(n) &= \frac{1}{2\pi i}\left\{\int_{1/2+\epsilon-iT}^{1/2+\epsilon+iT} + \int_{1/2+\epsilon+iT}^{b+iT}\right.\\ &\quad \left. + \int_{b-iT}^{1/2+\epsilon-iT}\right\} L_1(s)\frac{x^s}{s}\mathrm{d}s + O\left(\frac{x^{1+\epsilon}}{T}\right)\\ &= I_1 + I_2 + I_3 + O\left(\frac{x^{1+\epsilon}}{T}\right)\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\sum_{n\leq x}\lambda^2(n)r(n) &= 4 \operatorname*{Res}_{s=1} L(s)\,x + \frac{4}{2\pi i}\left\{\int_{1/2+\epsilon-iT}^{1/2+\epsilon+iT} + \int_{1/2+\epsilon+iT}^{b+iT}\right.\\ &\quad \left. + \int_{b-iT}^{1/2+\epsilon-iT}\right\} L(s)\frac{x^s}{s}\mathrm{d}s + O\left(\frac{x^{1+\epsilon}}{T}\right)\\ &= Cx + J_1 + J_2 + J_3 + O\left(\frac{x^{1+\epsilon}}{T}\right).\end{aligned} \tag{3.20}$$

Here $C = 4 \operatorname*{Res}_{s=1} L(s)$ is a constant. Now we evaluate the integrals in equation (3.19). By using the convexity bound for the Rankin–Selberg L-function Lemma 3.2, we have

$$I_1 \ll x^{1/2+\epsilon}\left(1 + \int_1^T \frac{|L_1(1/2+\epsilon+it)|}{t}\mathrm{d}t\right) \tag{3.21}$$

$$\ll x^{1/2+\epsilon} + x^{1/2+\epsilon}T^{1-\epsilon}. \tag{3.22}$$

To evaluate the horizontal integral, we write $s = \sigma + it$.

$$\begin{aligned}I_2 + I_3 &\ll \int_{1/2+\epsilon}^{1+\epsilon} |L_1(\sigma+iT)|\frac{x^\sigma}{T}\mathrm{d}\sigma\\ &\ll \int_{1/2+\epsilon}^{1+\epsilon} \frac{\max\limits_{1/2+\epsilon<\sigma\leq 1+\epsilon}|L_1(\sigma+iT)x^\sigma|}{T}\mathrm{d}\sigma \quad \ll \frac{x^{1/2+\epsilon}}{T^\epsilon} + \frac{x^{1+\epsilon}}{T^{1+\epsilon}}.\end{aligned} \tag{3.23}$$

So, we have

$$S(x) = O(x^{1/2+\epsilon}T^{1-\epsilon}) + O\left(\frac{x^{1+\epsilon}}{T^{1+\epsilon}}\right). \tag{3.24}$$

Now we evaluate the integrals in equation (3.20). By applying Lemma 3.1 and Cauchy–Schwartz inequality on the first integral J_1, we obtain

$$\begin{aligned} J_1 &\ll x^{1/2+\epsilon}\Bigg[\left(\int_0^T |L(1/2+\epsilon+it, f\otimes f)|^2 \mathrm{d}t\right)^{1/2} \\ &\quad\times\left(\int_0^T \frac{|L(1/2+\epsilon+it, \chi_{-4}\otimes f\otimes f)|^2}{|1/2+\epsilon+it|^2}\mathrm{d}t\right)^{1/2}\Bigg] \\ &\ll x^{1/2+\epsilon}\Bigg[\left(\int_0^T |L(1/2+\epsilon+it, f\otimes f)|^2 \mathrm{d}t\right)^{1/2} \\ &\quad\times\left(1+\int_1^T \frac{|L(1/2+\epsilon+it, \chi_{-4}\otimes f\otimes f)|^2}{t^2}\mathrm{d}t\right)^{1/2}\Bigg]. \end{aligned} \tag{3.25}$$

Here, we use standard argument and Lemma 3.3(ii) in the second integral of (3.25) and obtain

$$\begin{aligned} &\int_1^T \frac{|L(1/2+\epsilon+it, \chi_{-4}\otimes f\otimes f)|^2}{t^2}\mathrm{d}t \\ &\quad\ll \log T \max_{1<T_1\le T}\frac{1}{T_1^2}\int_{T_1/2}^{T_1}|L(1/2+\epsilon+it, \chi_{-4}\otimes f\otimes f)|^2\mathrm{d}t \\ &\quad\ll \log T. \end{aligned} \tag{3.26}$$

By inserting (3.26) in (3.25) and applying Lemma 3.3(i), we obtain

$$J_1 \ll x^{1/2+\epsilon}\, T^{1-2\epsilon}(\log T)^{3/2+\epsilon}. \tag{3.27}$$

Now, we will concentrate on the horizontal integrals J_2 and J_3. Consider $s=\sigma+it$. After applying Lemma 3.2, we get

$$\begin{aligned} J_2+J_3 &\ll \int_{1/2+\epsilon}^{1+\epsilon} |L(\sigma+iT, f\otimes f)||L(\sigma+iT, \chi_{-4}\otimes f\otimes f)|\,\frac{x^\sigma}{T}\mathrm{d}\sigma \\ &\ll \max_{1/2+\epsilon<\sigma\le 1+\epsilon} x^\sigma\, T^{4(1-\sigma)+2\epsilon}\, T^{-1} = \max_{1/2+\epsilon<\sigma\le 1+\epsilon}\left(\frac{x}{T^4}\right)^\sigma T^{3+2\epsilon} \\ &\ll \frac{x^{1+\epsilon}}{T^{1+2\epsilon}} + x^{1/2+\epsilon}\, T^{1-2\epsilon}. \end{aligned} \tag{3.28}$$

Finally from (3.20), (3.27) and (3.28), we get

$$S_f(x) = Cx + O\left(\frac{x^{1+\epsilon}}{T^{1+2\epsilon}}\right) + O(x^{1/2+\epsilon}\, T^{1-2\epsilon}(\log T)^{3/2+\epsilon}). \tag{3.29}$$

Now, we choose $T=x^{1/4}$ in both the cases (3.24) and (3.29) and we obtain

$$S(x) \ll x^{3/4+\epsilon}$$

and

$$S_f(x) = Cx + O(x^{3/4+\epsilon}),$$

which completes the proof of the proposition. □

4. Proof of Theorem 2.1

Now consider $h = h(x) = x^{7/8}$. The proof is by contradiction, so assume that the sequence $\{\lambda(n) : n = c^2 + d^2)\}_{n\geq 1}$ has a constant sign, say positive for all $n \in (x, x + h]$.

Now we apply (1.2) and Proposition 3.4 respectively to obtain

$$\sum_{x<n\leq x+h} \lambda^2(n) r_2(n) = \sum_{x<n\leq x+h} \lambda(n)\lambda(n) r_2(n) \ll x^{\epsilon} \sum_{x<n\leq x+h} \lambda(n) r_2(n) \ll x^{2\epsilon}[(x+h)^{3/4+\epsilon} + x^{3/4+\epsilon}] \ll x^{3/4+2\epsilon}. \tag{4.1}$$

On the other hand, from Proposition 3.4, we get

$$\sum_{x<n\leq x+h} \lambda^2(n) r_2(n) = Ch + O_{f,\epsilon}(x^{3/4+\epsilon}) \gg x^{7/8}. \tag{4.2}$$

Note that each time ϵ may have different value. Now we compare the bounds in (4.1) and (4.2) and arrive at a contradiction. Therefore, the sequence $\{\lambda(n) r_2(n)\}_{n\geq 1}$ has atleast one sign change in the interval $(x, x + h]$. This in particular gives us that the sequence $\{\lambda(c^2 + d^2)\}_{c,d\geq 1}$ has infinitely many sign changes. In fact, there are atleast $x^{1/8-2\epsilon}$ many sign changes in the interval $(x, 2x]$, for sufficiently large x, where ϵ is arbitrarily small.

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