Proc. Indian Acad. Sci. (Math. Sci.) (2020) 130:1
https://doi.org/10.1007/s12044-019-0537-1

On $\mathcal{D}$-closed submodules

YILMAZ DURĞUN[1] and SALAHATTIN ÖZDEMIR[2,*]

[1]Department of Mathematics, Çukurova University, Adana, Turkey
[2]Department of Mathematics, Dokuz Eylul University, Izmir, Turkey
*Corresponding author.
E-mail: ydurgun@cu.edu.tr; salahattin.ozdemir@deu.edu.tr

MS received 30 January 2019; accepted 1 May 2019

Abstract. A submodule N of a module M is called $\mathcal{D}$-closed if the socle of M/N is zero. $\mathcal{D}$-closed submodules are similar to $\mathcal{S}$-closed submodules (a generalization of closed submodules) defined through nonsingular modules. First, we describe the smallest proper class (due to Buchsbaum) containing the class of short exact sequences determined by $\mathcal{D}$-closed submodules in terms of that submodule, and show that it coincides with other classes of modules under certain conditions. Second, we study coprojective modules of this class, called edc-flat modules. We give some equivalent conditions for injective modules to be edc-flat for special rings, and for edc-flat modules to be projective (flat) for any ring.

Keywords. Closed submodule; flat module; Dickson torsion theory; semiartinian module; proper class.

Mathematics Subject Classification. 16D40, 18G25.

1. Introduction

With the aim of defining the functor 'Ext' without the use of projectives or injectives, Buchsbaum introduced in [1] certain conditions on a class of monomorphisms. Since in an abelian category any monomorphism induces an epimorphism and a short exact sequence, these conditions lead to the notion of *proper classes* of such sequences (see also, Quillen [21] and Keller [16], these classes are called *exact structures* on an additive category). Well-known examples are the classes of splitting short exact sequences, pure-exact sequences (in the sense of Cohn [5]) and all short exact sequences. The study of proper classes is useful to find characterizations of objects in terms of such proper classes. In this paper, we are only interested in proper classes on module categories.

Let R be an associative ring with identity, and let M be a unitary right R-module. Recall that a submodule N of M is said to be *closed in* M if N has no proper essential extension in M. We also say in this case that N is a closed submodule. See, for example, [8] for general properties of closed submodules. This class of submodules which plays an important role in rings and modules, and relative homological algebra induces a proper class of short exact

© Indian Academy of Sciences
Published online: 03 January 2020

sequences, denoted by $\mathcal{C}losed$ (see, for example, [4, §10]). More recently, several authors have studied different generalizations of closed submodules, one of which is the notion of $\mathcal{S}$-closed submodules defined by Goodearl [12] (see, for example, [7,9,15]). A submodule N of M is called *$\mathcal{S}$-closed* in M if M/N is nonsingular, or equivalently, M/N belongs to the torsion free class of the Goldie torsion theory (the torsion theory generated by all singular modules). Motivated by this fact, we consider modules whose factor modules belong to the torsion free class of Dickson torsion theory (the torsion theory generated by the class of all (semi) simple modules), and we call N a *$\mathcal{D}$-closed* submodule of M or *$\mathcal{D}$-closed* in M if the socle of M/N is zero. As the class $\mathcal{D}$-$\mathcal{C}losed$ of short exact sequences determined by $\mathcal{D}$-closed submodules does not induce a proper class (see Example 4), in the first part of this paper, we investigate the smallest proper class $\langle\mathcal{D}\text{-}\mathcal{C}losed\rangle$ containing $\mathcal{D}$-$\mathcal{C}losed$.

Another important generalization of closed submodules is the notion of neat submodules (a generalization of neat subgroups of abelian groups introduced by Honda [14]) which induces a proper class [22]. A submodule N of M is called *neat* in M if the sequence $\mathrm{Hom}(S, M) \to \mathrm{Hom}(S, M/N) \to 0$ is exact for each simple right R-module S. We denote by $\mathcal{N}eat$ the class of all short exact sequences $0 \to A \xrightarrow{f} B \to C \to 0$ such that $\mathrm{Im}\, f$ is neat in B. Recall that M is said to be *flat* if the functor $M \otimes_R -$ is exact. Note that M is flat if and only if every short exact sequence $0 \to A \to B \to M \to 0$ of R-modules is pure-exact (see, for example, [23, Proposition 3.67]). This relation between the notions of flatness and purity has led to further studies on some classes of modules defined via closed (neat) submodules. Namely, the modules M such that any short exact sequence ending with M is included in $\mathcal{C}losed$ (resp. in $\mathcal{N}eat$), which are called *weakly-flat* (resp., *neat-flat*), have been studied in [3,26]. Also, in the recent paper [6], flat objects related to a proper class (or an exact structure) of an additive category have been studied in a similar manner. Motivated by these studies, in the second part of the paper, we investigate the modules M such that any short exact sequence ending with M is included in the proper class $\langle\mathcal{D}\text{-}\mathcal{C}losed\rangle$, and we call such a module M *extended $\mathcal{D}$-closed flat* (or *edc-flat* for short).

Our paper is organized as follows. We will give in section 2 the axioms for a class of short exact sequences to be a *proper class* on module categories in the sense of Buchsbaum [1].

In section 3, we first describe the smallest proper class $\langle\mathcal{D}\text{-}\mathcal{C}losed\rangle$ which contains $\mathcal{D}$-$\mathcal{C}losed$ in terms of $\mathcal{D}$-closed submodules (Proposition 5), and show that every module with zero socle is projective if and only if $\langle\mathcal{D}\text{-}\mathcal{C}losed\rangle$ coincides with the class $\mathcal{S}plit$ of all splitting short exact sequences (Proposition 7). We know that $\mathcal{D}$-closed submodules are always neat (Remark 8). We prove the converse for essential submodules (Proposition 10). Also, for a ring with $\mathrm{Soc}(R_R) = 0$, we prove that $\langle\mathcal{D}\text{-}\mathcal{C}losed\rangle = \mathcal{C}losed$ if and only if R is a right SI-ring (i.e., a ring over which every singular R-module is injective) (Theorem 12).

In section 4, we show that $\mathrm{Soc}(R_R) = 0$ if and only if edc-flat modules are exactly modules with zero socle (Proposition 15). We prove that if the torsion submodule $t(E)$ of an injective module E is projective, then E is edc-flat; and the converse is true for C-rings (Theorem 17). Also, for C-rings, we prove that all injective modules are edc-flat if and only if the injective hull of each semiartinian module is projective if and only if every semiartinian module embeds in a projective module (Theorem 18). Moreover, we show that a commutative artinian ring R is a QF-ring if and only if every injective R-module is edc-flat, where R is called a *quasi-Frobenius* ring (or *QF-ring* for short) if it is right Noetherian and right self-injective (Corollary 20). Finally, we prove that every module with a zero socle is projective (flat) if and only if every edc-flat module is projective (flat) (Propositions 23 and 24). As a consequence, for a right perfect ring R, we have $\mathcal{S}plit = \langle \mathcal{D}\text{-}\mathcal{C}losed \rangle$ if and only if every finitely generated module with a zero socle is projective.

Throughout, we shall assume that all rings are associative with identity and all modules are unitary right modules. A (proper) submodule N of a module M will be denoted by ($N \lneqq M$) $N \leq M$. By $E(M)$ and $\mathrm{Soc}(M)$, we shall denote the injective hull and the socle of M as usual. By a short exact sequence, we mean a short exact sequence of modules and module homomorphisms. For undefined notions used in the text, we refer the reader to [4,13,24].

2. Proper class

Throughout this section, let $\mathcal{P}$ be a class of short exact sequences of modules and module homomorphisms. If a short exact sequence

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

belongs to $\mathcal{P}$, then f is said to be a $\mathcal{P}$-*monomorphism* and g is said to be a $\mathcal{P}$-*epimorphism*. A short exact sequence $\mathbb{E}$ is determined by each of the monomorphisms f and the epimorphisms g, uniquely up to isomorphism.

DEFINITION 1

The class $\mathcal{P}$ is said to be *proper* if it satisfies the following conditions (see, for example, [24], [17] or [4, §10]):

- (P1) If a short exact sequence $\mathbb{E}$ is in $\mathcal{P}$, then $\mathcal{P}$ contains every short exact sequence isomorphic to $\mathbb{E}$.
- (P2) $\mathcal{P}$ contains all splitting short exact sequences.
- (P3) The composite of two $\mathcal{P}$-monomorphisms (respectively $\mathcal{P}$-epimorphisms) is a $\mathcal{P}$-monomorphism (respectively $\mathcal{P}$-epimorphism) if this composite is defined.
- (P4) If g and f are monomorphisms and $g \circ f$ is a $\mathcal{P}$-monomorphism, then f is a $\mathcal{P}$-monomorphism. If g and f are epimorphisms and $g \circ f$ is a $\mathcal{P}$-epimorphism, then g is a $\mathcal{P}$-epimorphism.

Note that two short exact sequences $0 \to A \to B \to C \to 0$ and $0 \to A \to B' \to C \to 0$ are isomorphic (or equivalent) if the following diagram is commutative, where $1_A : A \to A$ and $1_C : C \to C$ are identity maps and $h : B \to B'$ is a homomorphism (which must be an isomorphism by five lemma):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 1_C} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Throughout the paper, we will identify a class of isomorphic short exact sequences with any of its elements.

Remark 2. The conditions of Definition 1 can be considered for more general settings. For instance, Quillen defined the concept of an *exact category* as an additive category $\mathfrak{C}$ with a class $\mathcal{P}$ of short exact sequences satisfying these conditions. In this case, $\mathcal{P}$ is called an *exact structure* on $\mathfrak{C}$ [16,21].

The intersection of all proper classes containing the class $\mathcal{P}$ is clearly a proper class, denoted by $\langle\mathcal{P}\rangle$. The class $\langle\mathcal{P}\rangle$ is the smallest proper class containing $\mathcal{P}$, called the proper class *generated* by $\mathcal{P}$. A module M is called *$\mathcal{P}$-projective* if it is projective with respect to all short exact sequences in $\mathcal{P}$, and is called *$\mathcal{P}$-coprojective* if every short exact sequence ending with M is in the class $\mathcal{P}$. Note that $\langle\mathcal{P}\rangle$ has the same projective modules as $\mathcal{P}$.

3. Proper classes relative to Dickson torsion theory

A module M is called *semiartinian* if every non-zero homomorphic image of M contains a simple submodule, that is, $\mathrm{Soc}(M/N) \neq 0$ for every submodule $N \subsetneqq M$. The torsion theory $t = (\mathbb{T}_D, \mathbb{F}_D)$ generated by the class of semisimple (or even simple) modules is a hereditary torsion theory, called the *Dickson torsion theory*. Its torsion and torsion free classes are respectively $\mathbb{T}_D = \{M \mid M \text{ is semiartinian}\}$ and $\mathbb{F}_D = \{M \mid \mathrm{Soc}(M) = 0\}$. Note that $\mathbb{T}_D$ is closed under submodules, homomorphic images, direct sums and extensions, while $\mathbb{F}_D$ is closed under submodules, direct products, extensions and injective hulls. For any module M, since any sum of semiartinian submodules of M is semi-artinian, M contains a unique maximal semiartinian submodule, called the *torsion submodule* with respect to this torsion theory and denoted by $t(M)$. Clearly, $\mathrm{Soc}(M) = 0$ if and only if $t(M) = 0$.

The notion of a right C-ring has been introduced in [22]. A ring R is said to be a right *C-ring* if, for every R-module B and for every essential proper submodule A of B, $\mathrm{Soc}(B/A) \neq 0$ or, equivalently, $\mathrm{Soc}(R/I) \neq 0$ for every essential proper right ideal I of R. It is known that a ring R is a right C-ring if and only if every singular R-module is semiartinian (see, for example, [4, 10.10]). Left perfect rings, two-sided hereditary Noetherian rings, and right semiartinian rings are some of the examples of right C-rings.

Recall that a submodule N of a module M is called $\mathcal{D}$-closed in M if $M/N \in \mathbb{F}_D$, that is, $\mathrm{Soc}(M/N) = 0$. Note that $\mathcal{D}$-closed submodules need not be closed, and also closed submodules need not be $\mathcal{D}$-closed in general.

Example 3. Let R be a ring which is not right C-ring. Then there exists an essential proper right ideal I of R such that $\mathrm{Soc}(R/I) = 0$. By considering R as a module over itself, we see that I is a $\mathcal{D}$-closed submodule of R. However, since I is essential in R, it cannot be closed in R. Moreover, the zero submodule is closed in any module M, but it is $\mathcal{D}$-closed in M only if $\mathrm{Soc}(M) = 0$.

The class $\mathcal{D}$-$Closed$ of all short exact sequences $0 \to A \xrightarrow{f} B \to C \to 0$ such that $\mathrm{Im}\, f$ is $\mathcal{D}$-closed in B need not be a proper class in general, as the following example shows.

Example 4. Let M be a module with a non-zero socle. Then the short exact sequence $\mathbb{E} : 0 \to 0 \to M \to M \to 0$ splits, but $\mathbb{E}$ is not $\mathcal{D}$-closed exact sequence since $M/0 \cong M$ has a non-zero socle. Thus, by the condition (P2) of being a proper class, we see that $\mathcal{D}$-$Closed$ does not form a proper class.

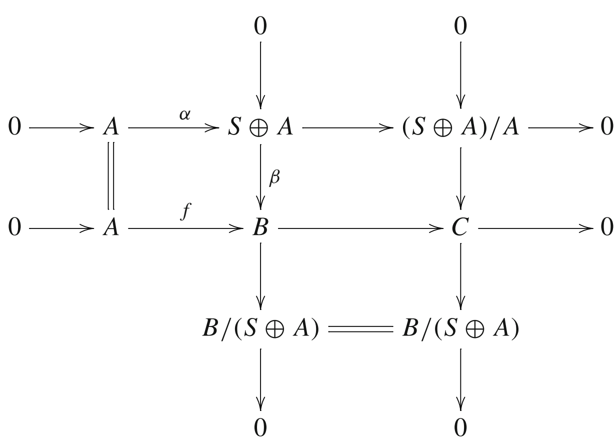
So, we aim to obtain a proper class containing $\mathcal{D}$-*Closed* and for this we define a new concept of *extended* $\mathcal{D}$-*closed* submodules using certain conditions introduced in [17] on a class of monomorphisms. A submodule A of a module B is called *extended* $\mathcal{D}$-*closed* in B if there is a submodule $S \leq B$ such that $S \cap A = 0$ and $\mathrm{Soc}(B/(S \oplus A)) = 0$. Taking $S = 0$, we see that $\mathcal{D}$-closed submodules are extended $\mathcal{D}$-closed, but the converse is not true in general (see Example 4). Denote by $\mathcal{D}\text{-}\overline{Closed}$ the class of all short exact sequences $0 \to A \xrightarrow{f} B \to C \to 0$ such that $\mathrm{Im}\, f$ is extended $\mathcal{D}$-closed in B. This is known to be a proper class (see [17, Theorem 2.1]), and in fact it is the smallest proper class generated by the class $\mathcal{D}$-*Closed*.

Notice that, by considering the class of all short exact sequences ending with a torsion free module instead of the class $\mathcal{D}$-*Closed*, we can prove Propositions 5–7 for any hereditary torsion theory, but we will only give their proofs for Dickson torsion theory which we are interested in.

PROPOSITION 5

$\langle \mathcal{D}\text{-}Closed \rangle = \mathcal{D}\text{-}\overline{Closed}$.

Proof. Since $\mathcal{D}\text{-}Closed \subseteq \mathcal{D}\text{-}\overline{Closed}$, we clearly have $\langle \mathcal{D}\text{-}Closed \rangle \subseteq \mathcal{D}\text{-}\overline{Closed}$. Now, let $0 \to A \xrightarrow{f} B \to C \to 0 \in \mathcal{D}\text{-}\overline{Closed}$. Then there is a submodule $S \leq B$ such that $S \cap A = 0$ and $\mathrm{Soc}(B/(S \oplus A)) = 0$. In the following commutative diagram, which exists because of the inclusion map $(S \oplus A)/A \hookrightarrow B/A \cong C$, α is a $\mathcal{S}plit$-monomorphism, and so a $\langle \mathcal{D}\text{-}Closed \rangle$-monomorphism by (P2) of Definition 1. Moreover, β is a $\langle \mathcal{D}\text{-}Closed \rangle$-monomorphism since $\mathrm{Soc}(B/(S \oplus A)) = 0$. Thus $f = \beta\alpha$ is also a $\langle \mathcal{D}\text{-}Closed \rangle$-monomorphism by (P3) of Definition 1.



□

The following result can be obtained by [4, 10.6], so the proof is omitted.

PROPOSITION 6

A submodule A of a module B is extended $\mathcal{D}$-closed in B if and only if every semiartinian module T is projective with respect to the projection $B \to B/A$, that is, $\mathrm{Hom}(T, B) \to \mathrm{Hom}(T, B/A) \to 0$ *is exact for each semiartinian module T.*

PROPOSITION 7

Every module with zero socle is projective if and only if every extended $\mathcal{D}$-closed exact sequence splits, that is, $\mathcal{S}plit = \mathcal{D}\text{-}\overline{\mathcal{C}losed}$.

Proof.
($\Rightarrow$) Let $0 \to A \xrightarrow{f} B \to C \to 0 \in \mathcal{D}\text{-}\overline{\mathcal{C}losed}$. Then there is a submodule $S \leq B$ such that $S \cap A = 0$ and $\mathrm{Soc}(B/(S \oplus A)) = 0$. Now, consider the diagram in the proof of Proposition 5. Since $B/(S \oplus A)$ is projective by assumption, β is a $\mathcal{S}plit$-monomorphism, and so $f = \beta\alpha$ is also a $\mathcal{S}plit$-monomorphism.
($\Leftarrow$) Let C be a module with zero socle. Consider the short exact sequence $\mathbb{E} : 0 \to K \to P \to C \to 0$ with P projective. Since $\mathrm{Soc}(C) = 0$, $\mathbb{E}$ is a $\mathcal{D}$-closed, and so an extended $\mathcal{D}$-closed short exact sequence. Then $\mathbb{E}$ splits by assumption, and thus C is projective as a direct summand of P. □

Remark 8. In general, we have the inclusions $\mathcal{D}\text{-}\mathcal{C}losed \subseteq \mathcal{D}\text{-}\overline{\mathcal{C}losed} \subseteq \mathcal{N}eat$, where the second inclusion follows by Proposition 6 since every simple module is semiartinian, and $\mathcal{C}losed \subseteq \mathcal{N}eat$ by [25, Proposition 5]. Furthermore, $\mathcal{C}losed = \mathcal{N}eat$ if and only if R is a right C-ring by [11, Theorem 5]. So, it follows that $\mathcal{D}\text{-}\mathcal{C}losed \subseteq \mathcal{C}losed$ for C-rings.

For the converse inclusions, we have the following results.

PROPOSITION 9

Neat submodules are extended $\mathcal{D}$-closed if and only if each semiartinian module T can be represented as $T = S \oplus P$, where S is a semisimple module and P is a projective module.

Proof.

($\Rightarrow$) Let T be a semiartinian module, and let $\mathbb{E} : 0 \to A \to B \to B/A \to 0$ be a neat-exact sequence. Then $\mathbb{E}$ is extended $\mathcal{D}$-closed by assumption. Thus, T is projective with respect to $\mathbb{E}$ by Proposition 6, that is, T is $\mathcal{N}eat$-projective. Hence T is a direct sum of a projective module and a semisimple module by [10, Theorem 2.6].
($\Leftarrow$) By assumption, we see that every semiartinian module T is $\mathcal{N}eat$-projective. Thus, every neat-exact sequence is extended $\mathcal{D}$-closed by Proposition 6. □

PROPOSITION 10

Essential neat submodules are $\mathcal{D}$-closed.

Proof. Let A be an essential neat submodule of B. Suppose on the contrary, $\mathrm{Soc}(B/A) \neq 0$. Then there exists a simple submodule S/A of B/A, where $S \leq B$ and A is a maximal submodule of S. Since A is essential in B, it follows that A is essential in S. But A is neat in B, and so it is a direct summand of S, which is a contradiction. □

COROLLARY 11

For an essential submodule A of a module B, we have A is neat in B if and only if A is $\mathcal{D}$-closed in B.

Recall that a ring R is called a right *SI-ring* if every singular R-module is injective. Note that R is a right SI-ring if and only if it is right nonsingular and every singular right R-module is semisimple (see [12]).

Theorem 12. *Let R be a ring with* $\mathrm{Soc}(R_R) = 0$. *Then $\mathcal{D}$-$\overline{Closed} = Closed$ if and only if R is a right SI-ring.*

Proof.

(⇒) Let A be a singular module. First, we will show that R is a right C-ring. Assume that $\mathrm{Soc}(R/I) = 0$ for some essential proper right ideal I of R. Then I is a $\mathcal{D}$-closed submodule of R, and so it is closed in R by assumption, which contradicts the essentiality of I in R. Thus R is a right C-ring, that is, each singular module is semiartinian. So A is semiartinian. Therefore, by Remark 8 and Proposition 9, A can be written as a direct sum of a projective module and a semisimple module. Since $\mathrm{Soc}(R_R) = 0$, it follows that A is semisimple. Now, it suffices to show that R is right nonsingular. Suppose not, then it has a nonzero singular submodule, which is also semiartinian as R is a C-ring. This contradicts the fact that $\mathrm{Soc}(R_R) = 0$. Hence R is a right SI-ring.
(⇐) Since a right SI-ring is a right C-ring, we have $\mathcal{D}$-$\overline{Closed} \subseteq Closed$ by Remark 8. Now, we show that every semiartinian module is singular, which implies that it is semisimple. Let M be a semiartinian module which is not singular. Then M has a nonsingular quotient, say A. Since semiartinian modules are closed under quotient, A is also semiartinian. Therefore, A has a simple submodule, say S. Since A is nonsingular, S is also nonsingular. It follows that S is projective, because any simple module is either singular or projective. But this is impossible since $\mathrm{Soc}(R_R) = 0$ by assumption. Thus M must be singular, and so semisimple as claimed. Hence the inclusion $Closed \subseteq \mathcal{D}$-$\overline{Closed}$ follows from Proposition 9. □

4. *edc*-Flat modules

Recall that a module M is called *extended $\mathcal{D}$-closed flat* (or *edc-flat* for short) if the kernel of any epimorphism $L \to M$ is extended $\mathcal{D}$-closed in the module L. By Proposition 6, we observe that M is an edc-flat module if and only if, for any epimorphism $L \to M$, the induced map $\mathrm{Hom}(T, L) \to \mathrm{Hom}(T, M)$ is surjective for each semiartinian module T.

The following result is due to [19, Proposition 1.12–1.13].

PROPOSITION 13

The following statements are equivalent for a module M:

(1) *M is edc-flat.*
(2) *There exists an extended $\mathcal{D}$-closed sequence $0 \to K \to F \to M \to 0$ with F projective.*
(3) *There exists an extended $\mathcal{D}$-closed sequence $0 \to K \to F \to M \to 0$ with F edc-flat.*

Remark 14.

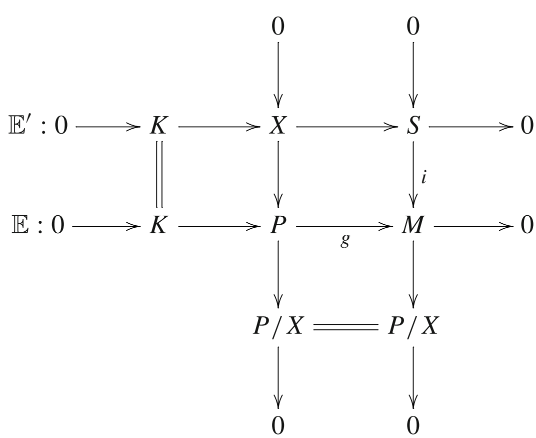
(1) Projective modules and modules with zero socle are edc-flat.
(2) The edc-flat modules are neat-flat, since extended $\mathcal{D}$-closed submodules are neat.
(3) A semiartinian module T is edc-flat if and only if it is projective, by Proposition 6. So, R is a semisimple ring if and only if every right (or left) R-module is edc-flat.
(4) The class of edc-flat modules is closed under direct summands, extensions and finite direct sums.

PROPOSITION 15

For any ring R, $\mathrm{Soc}(R_R) = 0$ if and only if edc-flat modules are exactly modules with zero socle.

Proof.

($\Leftarrow$) Since R is an edc-flat module over itself, by assumption $\mathrm{Soc}(R_R) = 0$.
($\Rightarrow$) Suppose that $\mathrm{Soc}(R_R) = 0$. Modules with zero socle are always edc-flat. Now let M be an edc-flat module. Then there exists an extended $\mathcal{D}$-closed sequence $\mathbb{E} : 0 \to K \to P \to M \to 0$ with P projective by Proposition 13. Assume on the contrary that $\mathrm{Soc}(M) \neq 0$. Then there exists a simple submodule S of M. So we have the following commutative diagram by pullback of g and i:



Since $\mathbb{E} \in \mathcal{D}\text{-}\overline{Closed}$, then $\mathbb{E}' \in \mathcal{D}\text{-}\overline{Closed}$. Thus $\mathbb{E}'$ is a neat-exact sequence, and so it splits since S is simple. It follows that $X = K \oplus Y$ with $Y \cong S$. However, $\mathrm{Soc}(X) = \mathrm{Soc}(K) \oplus \mathrm{Soc}(Y) \subseteq \mathrm{Soc}(P) = \mathrm{Soc}(R_R)P = 0$, since P is projective and $\mathrm{Soc}(R_R) = 0$ by assumption. Hence $\mathrm{Soc}(X) = 0$, which implies that $Y = \mathrm{Soc}(Y) = 0$. This contradiction shows that $\mathrm{Soc}(M) = 0$ as desired. $\square$

PROPOSITION 16

Let $f : N \to M$ be an epimorphism of modules. If M is an edc-flat module, then any semiartinian submodule of M is isomorphic to a semiartinian submodule of N. In particular, the torsion submodule $t(M)$ of M embeds in a projective module.

Proof. Let T be a semiartinian submodule of M and let $\iota : T \to M$ be the inclusion map. Since M is edc-flat, the map $\mathrm{Hom}(T, N) \to \mathrm{Hom}(T, M)$ is surjective. So there is a homomorphism $g : T \to N$ such that $fg = \iota$. Since g is a monomorphism, and semiartinian modules are closed under homomorphic images, it follows that $g(T)$ is a semiartinian submodule of N. The particular case follows by taking an epimorphism $h : P \to M$ with P projective and $T = t(M)$. □

Theorem 17. *Let E be an injective module. If the torsion submodule $t(E)$ of E is projective, then E is an edc-flat module. Furthermore, the converse of the statement holds for right C-rings.*

Proof. Suppose that $t(E)$ is a projective module. Let $0 \to A \to F \xrightarrow{g} E \to 0$ be a short exact sequence with F projective, and let $f : T \to E$ be a homomorphism with T a semiartinian module. We need to find a homomorphism $h : T \to F$ such that $gh = f$. Since T is a semiartinian module, $f(T)$ is also semiartinian, and so it embeds in $t(E)$, that is, there is an inclusion map $\iota : f(T) \to t(E)$. There is also an inclusion map $\iota' : f(T) \to E$. Combining these maps, we obtain the following diagram:

$$\begin{array}{ccccccccc} & & & & T & & \\ & & & & \downarrow{\scriptstyle f'} & & \\ & & & & f(T) & \xrightarrow{\iota} & t(E) \\ & & & & \downarrow{\scriptstyle \iota'} & & \\ 0 \longrightarrow A & \longrightarrow & F & \xrightarrow{g} & E & \longrightarrow & 0 \end{array}$$

By the injectivity of E, there is a homomorphism $v : t(E) \to E$ such that $v\iota = \iota'$ and so by the projectivity of $t(E)$, there is a homomorphism $u : t(E) \to F$ such that $gu = v$. Setting $h = u\iota f' : T \to F$, we obtain that $gh = f$, as desired. Hence, E is an edc-flat module by Proposition 13.

For the converse, suppose that E is edc-flat. Then by Proposition 16, $t(E)$ embeds in a projective module, say in P. Since $\mathrm{Soc}(E/t(E)) = 0$, then $t(E)$ is a $\mathcal{D}$-closed submodule of E, and so it is neat in E. Because neat submodules and closed submodules coincide for a right C-ring, it follows that $t(E)$ is closed in E. Thus, it is an injective module as a direct summand of E. Hence $t(E)$ is a direct summand of the projective module P, which implies that it is projective. □

In the proof of the following result, we use the fact that the injective hull of semiartinian modules is also semiartinian, which is known to be true for C-rings. Indeed, if A is a semiartinian R-module, then $E(A)/A$ is semiartinian, as R is a right C-ring. Moreover, since semiartinian modules are closed under extension, $E(A)$ must be also semiartinian. Note that the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M will be denoted by M^+.

A ring R is called a *right Kasch* ring if each simple R-module embeds in R_R.

Theorem 18. *The following statements are equivalent for a right C-ring R:*

(1) *Injective modules are edc-flat.*
(2) *The injective hull $E(T)$ of each semiartinian module T is projective.*
(3) *Every semiartinian module embeds in a projective module.*
(4) *Every injective module E can be represented as $E = P \oplus N$, where P is a projective semiartinian module and N is a module with zero socle.*
(5) *For every free left R-module F, the character module F^+ is edc-flat.*

In particular, R is a right Kasch ring.

Proof.

(1) $\Rightarrow$ (2) Since T is a semiartinian module, its injective hull $E(T)$ is also semiartinian. Besides, $E(T)$ is an edc-flat module by assumption. Thus $E(T)$ is a projective module by Remark 14(3).
(2) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (4) Let E be an injective module. Since R is a C-ring, then $E = t(E) \oplus N$, where $t(E)$ is the torsion submodule of E and N is a module with a zero socle. By assumption, $t(E)$ embeds in a projective module, say in K. But since $t(E)$ is injective as a direct summand of E, it is also a direct summand of K. It follows that $t(E)$ is projective.
(4) $\Rightarrow$ (5) For every free left R-module F, the character module F^+ is injective by [23, Proposition 3.54]. Then by assumption $F^+ = P \oplus N$, where P is a projective semiartinian module and N is a module with zero socle. Since P and N are edc-flat modules, by Remark 14(1), it follows that F^+ is edc-flat by Remark 14(5).
(5) $\Rightarrow$ (1) Let E be an injective module. Then there is a free left R-module F and an epimorphism $F \to E^+$ from which we obtain an exact sequence $0 \to E^{++} \to F^+$. Since E is injective and $E \leq E^{++}$, E is a direct summand of F^+. Thus E is edc-flat since F^+ is edc-flat by assumption.

Finally, since edc-flat modules are neat-flat, it follows by (1) that injective modules are neat-flat. Thus, R is a right Kasch ring by [2, Theorem 4.9] □

For a ring R, a right R-module is projective if and only if it is injective exactly when R is a QF-ring (see [18, Theorem 15.9]).

COROLLARY 19

The following statements hold for a right C-ring R:

(1) *If R_R is semiartinian, then every injective module is edc-flat if and only if R is a QF-ring.*
(2) *If* $\mathrm{Soc}(R_R) = 0$, *then every injective module is edc-flat if and only if R is semisimple.*

Proof.

(1) Assume that R_R is semiartinian. Then every non-zero R-module has a non-zero socle. Now, if every injective module is edc-flat, then every injective module is projective by Theorem 18(4). Thus, R is a QF ring. Conversely, if R is a QF-ring, then all injective modules are projective, and so they are edc-flat.

(2) If $\mathrm{Soc}(R_R) = 0$, then $\mathrm{Soc}(P) = \mathrm{Soc}(R_R)P = 0$ for every projective module P. Now assume that every injective module is edc-flat. Since every submodule of P has a zero socle, then every semiartinian module is zero by Theorem 18(3). In particular, every simple module is semiartinian, and so it is a zero module, which is projective. Thus R is semisimple. Conversely, if R is semisimple, then all injective modules are projective, and so they are edc-flat. □

Because a commutative Artinian ring is both semiartinian and a right C-ring, Corollary 19(1) gives the following result.

COROLLARY 20

Let R be a commutative Artinian ring. Then, R is a QF-ring if and only if every injective module is edc-flat.

Note that if the injective hull of a finitely generated module is projective, then it is also finitely generated by [20, Lemma 3.70].

COROLLARY 21

Let R be a right C-ring. If R satisfies one of the equivalent conditions of Theorem 18, then injective hulls of finitely generated semiartinian modules are finitely generated.

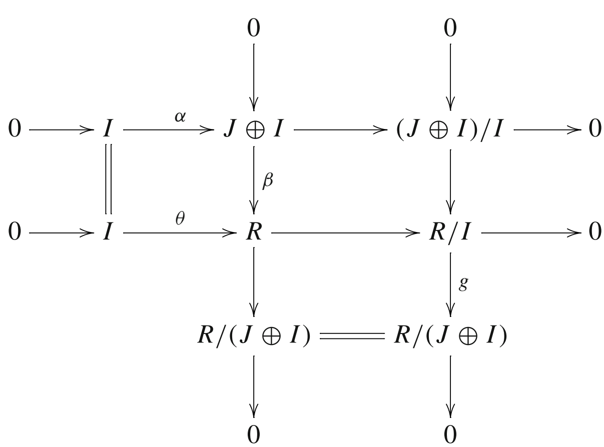
We know that projective modules are edc-flat. For the converse, we have the following results.

PROPOSITION 22

Every cyclic module with zero socle is projective if and only if every cyclic edc-flat module is projective.

Proof.

($\Rightarrow$) Let M be a cyclic edc-flat module. Then $M \cong R/I$ for some right ideal I of R, and so I is an extended $\mathcal{D}$-closed ideal of R. Therefore, there is a right ideal J of R such that $J \cap I = 0$ and $\mathrm{Soc}(R/(J \oplus I)) = 0$. Now, we have the following commutative diagram:



Since $\mathrm{Soc}(R/(J \oplus I)) = 0$, then $R/(J \oplus I)$ is projective by assumption. So β is a $\mathcal{S}plit$-monomorphism. Moreover, since α is a $\mathcal{S}plit$-monomorphism, $\theta = \beta\alpha$ is also a $\mathcal{S}plit$-monomorphism. Thus R/I is projective as a direct summand of R.
($\Leftarrow$) It follows by the fact that every module with zero socle is edc-flat. □

PROPOSITION 23

The following statements are equivalent:

(1) *Every module with zero socle is projective.*
(2) *Every edc-flat module is projective.*
(3) *Every $\mathcal{D}$-closed submodule of a projective module is direct summand.*

Proof.

(1) $\Rightarrow$ (2) Let C be an edc-flat module. Then there is an extended $\mathcal{D}$-closed exact sequence $0 \to A \xrightarrow{f} B \to C \to 0$ with B projective. So, there is a submodule $S \leq B$ such that $S \cap A = 0$ and $\mathrm{Soc}(B/(S \oplus A)) = 0$. Now, consider the diagram in the proof of Proposition 5. Since $B/(S \oplus A)$ is projective by assumption, β is a splitting monomorphism, and so $f = \beta\alpha$ is also a splitting monomorphism. Thus C is projective as it is isomorphic to a direct summand of B.
(2) $\Rightarrow$ (3) Let B be a projective module and let A be a $\mathcal{D}$-closed submodule of B. Since we have an extended $\mathcal{D}$-closed exact sequence $\mathbb{E} : 0 \to A \to B \to C \to 0$ with B projective, it follows by Proposition 13 that C is edc-flat. Then C is projective by assumption, and so the sequence $\mathbb{E}$ splits. Thus A is a direct summand of B.
(3) $\Rightarrow$ (1) Assume that C is a module with a zero socle. Then there is a $\mathcal{D}$-closed exact sequence $0 \to A \to B \to C \to 0$ with B projective. So, by assumption, A is a direct summand of B. In fact, $B \cong A \oplus C$, which implies that C is projective. □

PROPOSITION 24

The following statements are equivalent:

(1) *Every extended $\mathcal{D}$-closed submodule is pure.*
(2) *Every edc-flat module is flat.*
(3) *Every module with zero socle is flat.*
(4) *Every finitely generated module with zero socle is flat.*

Proof. The implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are clear. First, we recall that if every finitely generated submodule of a module is flat, then it is also flat by [23, Proposition 3.48]. Hence, every module with zero socle is flat by assumption. Let $0 \to A \xrightarrow{f} B \to C \to 0$ be an extended $\mathcal{D}$-closed exact sequence. Then there is a submodule $S \leq B$ such that $S \cap A = 0$ and $\mathrm{Soc}(B/(S \oplus A)) = 0$. Now, consider the diagram in the proof of Proposition 5. Since $B/(S \oplus A)$ is flat, β is a pure-monomorphism, and thus $f = \beta\alpha$ is also a pure-monomorphism. □

It is well known that a ring R is right perfect if and only if every flat module is projective. So, by Propositions 7 and 24, we have the following result.

COROLLARY 25

Let R be a right perfect ring. Then $\mathcal{S}plit = \mathcal{D}\text{-}\overline{\mathcal{C}losed}$ *if and only if every finitely generated module with zero socle is projective.*

Acknowledgement

The authors would like to thank the referee for the review of this manuscript.

References

[1] Buchsbaum D A, A note on homology in categories, *Ann. Math.* **69(1)** (1959) 66–74
[2] Büyükaşık E and Durğun Y, Absolutely s-pure modules and neat-flat modules, *Comm. Algebra* **43(2)** (2015) 384–399
[3] Büyükaşık E and Durğun Y, Neat-flat modules, *Comm. Algebra* **44(1)** (2016) 416–428
[4] Clark J, Lomp C, Vanaja N and Wisbauer R, Lifting Modules (2006) (Basel: Birkhäuser Verlag)
[5] Cohn P M, On the free product of associative rings, *Math. Z.* **71** (1959) 380–398
[6] Crivei S and Keskin Tütüncü D, Relatively divisible and relatively flat objects in exact categories, https://arxiv.org/pdf/1810.11637.pdf
[7] Crivei S and Şahinkaya S, Modules whose closed submodules with essential socle are direct summands, *Taiwanese J. Math.* **18(4)** (2014) 989–1002
[8] Dung N V, Huynh D V, Smith P F and Wisbauer R, Extending Modules, Pitman Research Notes in Mathematical Series 313 (1994) (Harlow: Longman Scientific & Technical)
[9] Durğun Y and Özdemir S, On $\mathcal{S}$-closed submodules, *J. Korean Math. Soc.* **54(4)** (2017) 1281–1299
[10] Fuchs L, Neat submodules over integral domains, *Period. Math. Hungar.* **64(2)** (2012) 131–143
[11] Generalov A I, On weak and w-high purity in the category of modules, *Math. USSR, Sb.* **34** (1978) 345–356, translated from Russian from *Mat. Sb., N. Ser.* **105(147)** 389–402
[12] Goodearl K R, Singular torsion and the splitting properties, 124 (1972) (Providence, RI: American Mathematical Society)
[13] Goodearl K R, Ring Theory (1976) (New York-Basel: Marcel Dekker Inc.)
[14] Honda K, Realism in the theory of abelian groups I, *Comment. Math. Univ. St. Paul.* **5** (1956) 37–75
[15] Kara Y and Tercan A, When some complement of a z-closed submodule is a summand, *Comm. Algebra* **46(7)** (2018) 3071–3078
[16] Keller B, Chain complexes and stable categories, *Manuscripta Math.* **67** (1990) 379–417
[17] Kepka T, On one class of purities, *Comment. Math. Univ. Carolinae* **14** (1973) 139–154
[18] Lam T Y, Lectures on modules and rings, Graduate Texts in Mathematics (1999) (New York: Springer-Verlag)
[19] Mišina A P and Skornjakov L A, Abelian Groups and Modules, Algebra, Logic and Applications, AMS Translational Series: 2, 107 (1976) (Providence, RI)
[20] Nicholson W K and Yousif M F, Quasi-Frobenius rings, Cambridge Tracts in Mathematics 158 (2003) (Cambridge University Press)
[21] Quillen D, Higher algebraic K-theory: I, Lecture Notes in Mathematics, vol. 341 (1973) (Berlin: Springer) pp. 85–147
[22] Renault G, Étude de certains anneaux liés aux sous-modules compléments d'un A-module, *C. R. Acad. Sci. Paris* **259** (1964) 4203–4205
[23] Rotman J J, An introduction to homological algebra, 2nd edition, Universitext (2009) (New York: Springer)
[24] Sklyarenko E G, Relative homological algebra in the category of modules, *Usp. Mat. Nauk.* **33(3)** (1978) 85–120

[25] Stenström B T, High submodules and purity, *Ark. Mat.* **7** (1967) 173–176
[26] Zöschinger H, Schwach-Flache Moduln, *Comm. Algebra* **41(12)** (2013) 4393–4407

COMMUNICATING EDITOR: B Sury