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BLO estimates for Marcinkiewicz integrals associated with Schrödinger operators

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Abstract. Let $L = -\Delta + V$ be a Schrödinger operator, where Δ is the Laplacian operator on $\mathbb{R}^d$, while the nonnegative potential V belongs to the reverse Hölder class $B_q (q \geq 1)$. In this paper, we will show that Marcinkiewicz integrals associated with Schrödinger operator are bounded from BMO_L to BLO_L, when $V \in B_d$.

Keywords. Marcinkiewicz integral; Schrödinger operator; BLO estimate.

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1. Introduction

Let $\mathbb{R}^d$, $d \geq 3$ be the d-dimensional Euclidean space and S^{d-1} be the unit sphere in $\mathbb{R}^d$ equipped with normalized Lebesgue measure $\mathrm{d}\sigma = \mathrm{d}\sigma(x)$. Let $\Omega \in L^s(S^{d-1})$, $s \geq 1$ be a homogeneous function of degree zero on $\mathbb{R}^d$ and satisfy

$$\int_{S^{d-1}} \Omega(x)\mathrm{d}\sigma(x) = 0. \tag{1.1}$$

The Marcinkiewicz integral operator μ is defined by

$$\mu f(x) = \left(\int_0^\infty \left| \int_{|x-y|<t} \frac{\Omega(x-y)}{|x-y|^{d-1}} f(y)\mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^3} \right)^{1/2}.$$

The above operator was introduced by Stein in [12] as an extension of the notion of Marcinkiewicz integral from one dimension to higher dimensions. Meanwhile, Stein [12] showed that if $\Omega \in \mathrm{Lip}_\alpha(S^{d-1})$ for some $0 < \alpha \leq 1$, then μ is a bounded operator on $L^p(\mathbb{R}^d)$ for $1 < p \leq 2$, and from $L^1(\mathbb{R}^d)$ to weak $L^1(\mathbb{R}^d)$.

On the other hand, the study on the Schrödinger operator $L = -\Delta + V$ recently attracted much attention. In particular, Shen [11] considered L^p estimates for Schrödinger operator L with certain potentials which include Schrödinger–Riesz transforms $R_j^L = \frac{\partial}{\partial x_j} L^{\frac{1}{2}}$, $j = 1, 2, \ldots, d$. When $V \in B_{d/2}$ with $d \geq 3$, Fefferman [7], Shen [11] and Zhong [14] obtained

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some basic results on L and the boundedness of Riesz transform $\nabla L^{-1/2}$ on $L^p(\mathbb{R}^d)$ with some $p \in (1, +\infty)$. Dziubański and Zienkiewicz [4] originally characterized the Hardy space $H_L^1(\mathbb{R}^n)$ associated with L in terms of atoms, the maximal function defined by the semigroup $\{e^{-tL}\}_{t>0}$ and the Riesz transforms $\nabla L^{-1/2}$. Dziubański and Zienkiewicz [4] also characterized the above Hardy space $H_L^1(\mathbb{R}^d)$ via certain localized maximal function associated to the auxiliary function determined by the potential V. Recently, Dong and Liu [3] proved that the Schrödinger–Riesz transforms R_j^L are bounded on $\mathrm{BMO}_L(\mathbb{R}^d)$.

Similar to the classical Marcinkiewicz integral operator μ, Marcinkiewicz integral operators μ_j^L associated with Schrödinger operators L introduced by Gao and Tang [9] are defined by

$$\mu_j^L f(x) = \left(\int_0^\infty \left|\int_{|x-y|<t} K_j^L(x, y) f(y) \mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2},$$

where $K_j^L(x, y) = \widetilde{K_j^L}(x, y)|x-y|$ and $\widetilde{K_j^L}(x, y)$ is the kernel of $R_j^L = \frac{\partial}{\partial x_j} L^{\frac{1}{2}}$, $j = 1, 2, \ldots, d$. In particular, when $V \equiv 0$, $K_j^\Delta(x, y) = \widetilde{K_j^\Delta}(x, y)|x-y| = \frac{(x_j - y_j)/|x-y|}{|x-y|^{d-1}}$ and $\widetilde{K_j^\Delta}(x, y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j} \Delta^{\frac{1}{2}}$, $j = 1, 2, \ldots, d$. In this paper, we write $K_j(x, y) = K_j^\Delta(x, y)$ and

$$\mu_j f(x) = \left(\int_0^\infty \left|\int_{|x-y|<t} K_j(x, y) f(y) \mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}.$$

Obviously, μ_j are the classical Marcinkiewicz integral operators. Therefore, it will be interesting to study the properties of μ_j^L.

It is well known that the kernels $K_j(x, y)$ of the classical Marcinkiewicz integral operators μ_j satisfy the following inequalities:

$$|K_j(x, y)| \leq \frac{C}{|x-y|^{d-1}}, \tag{1.2}$$

$$|K_j(x+h, y) - K_j(x, y)| \leq \frac{C|h|}{|x-y|^d}, \tag{1.3}$$

and the cancellation condition

$$\int_{|y-x|=1} K_j(x, y) \mathrm{d}x = 0. \tag{1.4}$$

Let $1 < q < \infty$. A non-zero, non-negative locally L^q integrable function $V(x)$ is said to belong to B_q, if there exists constant $C > 0$ such that the reverse Hölder's inequality

$$\left(\frac{1}{|B|}\int_B V(x)^q \mathrm{d}x\right)^{\frac{1}{q}} \leq C\left(\frac{1}{|B|}\int_B V(x) \mathrm{d}x\right) \tag{1.5}$$

holds for every ball $B \subset \mathbb{R}^d$ (see [11]). Obviously, $B_{q_2} \subset B_{q_1}$ if $q_2 > q_1$. But it is important that the B_q class has a property of 'self improvement', that is, if $V \in B_q$, then $V \in B_{q+\epsilon}$ for some positive number ϵ. Furthermore, B_d includes non-negative polynomials and some non-smooth functions.

The auxiliary function $\rho(x, V) = \rho(x)$ introduced by Shen [11] is defined by

$$\rho(x) = \sup_{r>0}\left\{r : \frac{1}{r^{d-2}}\int_{B(x,r)} V(y) \mathrm{d}y \leq 1\right\}, \quad x \in \mathbb{R}^d.$$

The auxiliary function determined by the potential V played a key role in several papers such as [4–6, 11] and [10]. It is known that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^d$ (see [11]).

For $V \in B_{d/2}(\mathbb{R}^d)$ with $d \geq 3$, Dziubański *et al.* [6] introduced the BMO-type space $\mathrm{BMO}_L(\mathbb{R}^d)$ associated with L and established the duality between $H^1_L(\mathbb{R}^d)$ and $\mathrm{BMO}_L(\mathbb{R}^d)$, as well as a characterization of $\mathrm{BMO}_L(\mathbb{R}^d)$ in terms of the Carleson measure and $\mathrm{BMO}_L(\mathbb{R}^d)$ estimates for the versions of some classical operators associated with L, including semigroup maximal functions and fractional integral operator I_α with $\alpha \in (0, d)$. Moreover, for $V \in B_d$ with $d \geq 3$, Dong and Liu [3] further established $\mathrm{BMO}_L(\mathbb{R}^d)$ estimates for $\nabla L^{-1/2}$, the Fefferman–Stein decomposition of $\mathrm{BMO}_L(\mathbb{R}^d)$ via $\nabla L^{-1/2}$, and a characterization of $H^1_L(\mathbb{R}^d)$ in terms of the adjoint operators of $\nabla L^{-1/2}$. Yang *et al.* [13] introduced the BMO-type space $\mathrm{BMO}_\rho(\mathbb{R}^d)$ and the BLO-type space $\mathrm{BLO}_\rho(\mathbb{R}^d)$ with an admissible function ρ, and established $\mathrm{BMO}_\rho(\mathbb{R}^d)$ estimates for the localized Riesz transforms, as well as estimates from $\mathrm{BMO}_\rho(\mathbb{R}^d)$ to $\mathrm{BLO}_\rho(\mathbb{R}^d)$ for the corresponding maximal operators. Being analogous to BMO_L, Gao *et al.* [8] also defined BLO-type space BLO_L associated with the Schrödinger operator, which is equivalent to the definition of $\mathrm{BLO}_\rho(\mathbb{R}^d)$ with the admissible function $\rho = \rho(x)$. They also gave a characterization of BLO_L in parallel with that of the BLO (see [1]), and obtained BLO_L estimate for maximal Riesz transform associated with L. The space BLO_L is a subspace of both the classical BLO space and BMO_L.

DEFINITION 1.1 [6]

Let $V \in B_q$ for some $q \geq d/2$. A locally integrable function f is said to belong to BMO_L, if there exists constant $C \geq 0$ such that

$$\frac{1}{|B|}\int_B |f(y) - f_B|\mathrm{d}y \leq C \quad \text{and} \quad \frac{1}{|B_r|}\int_{B_r} |f(y)|\mathrm{d}y \leq C$$

hold for all the balls $B \subset \mathbb{R}^n$ and $B_r = B_r(x) : r \geq \rho(x)$, where $f_B = \frac{1}{|B|}\int_B f(y)\mathrm{d}y$. The smallest constant C will be denoted by $\|f\|_{\mathrm{BMO}_L}$.

In fact, $\|f\|_{\mathrm{BMO}_L}$ is a norm. Moreover, BMO_L is a Banach space. Obviously, $\|f\|_{\mathrm{BMO}} \leq 2\|f\|_{\mathrm{BMO}_L}$ holds for $f \in \mathrm{BMO}_L$. When $V \equiv 0$, BMO_L space becomes the classical BMO space. The properties of BMO_L are as follows (see [6]).

For $p \in [1, +\infty)$, there exists $c = c(p, V) > 0$ such that for every $f \in \mathrm{BMO}_L$,

$$\left(\frac{1}{B}\int_B |f(y) - f_B|^p \mathrm{d}y\right)^{\frac{1}{p}} \leq c\|f\|_{\mathrm{BMO}_L}, \quad \forall B \subset \mathbb{R}^n \tag{1.6}$$

and

$$\left(\frac{1}{B}\int_B |f(y)|^p \mathrm{d}y\right)^{\frac{1}{p}} \leq c\|f\|_{\mathrm{BMO}_L}, \quad \forall B = B_r(x) : r \geq \rho(x). \tag{1.7}$$

In 1980, Coifman and Rochberg [2] introduced the BLO space, which is a subspace of the BMO space. The BLO space is defined by

$$\mathrm{BLO} = \left\{ f \in L_{\mathrm{loc}}(\mathbb{R}^n) : \frac{1}{|B|}\int_B f(x) - \inf_{x\in B} f(x)\mathrm{d}x \leq C \right\}.$$

The BLO-type spaces associated with the Schrödinger operators L introduced in [8] and [13] is defined by the following.

DEFINITION 1.2 [8,13]

Let $V \in B_q$ for some $q \geq d/2$. A locally integrable function f is said to belong to BLO_L, if there exists constant $C \geq 0$ such that

$$\frac{1}{|B|}\int_B f(y) - \inf_B f \mathrm{d}y \leq C \quad \text{and} \quad \frac{1}{|B_r|}\int_{B_r} |f(y)|\mathrm{d}y \leq C$$

hold for all the balls $B \subset \mathbb{R}^n$ and $B_r = B_r(x) : r \geq \rho(x)$, where $f_B = \frac{1}{|B|}\int_B f(y)\mathrm{d}y$. The smallest constant C will be denoted by $\|f\|_{\mathrm{BLO}_L}$.

Remark 1.3.

(a) Obviously, $L^\infty \subsetneqq \mathrm{BLO}_{\mathcal{L}} \subsetneqq \mathrm{BLO}$; $\quad L^\infty \subsetneqq \mathrm{BLO}_L \subsetneqq \mathrm{BMO}_L$;
(b) For a non-negative measurable function $f(y)$, by the inequality

$$\frac{1}{|B|}\int_B (f(y) - \inf_B f(y))\mathrm{d}y \leq C\frac{1}{|B|}\int_B f(y)\mathrm{d}y,$$

we have that $f \in \mathrm{BLO}_L$ if and only if

$$\frac{1}{|B_s|}\int_{B_s} (f(y) - \inf_{B_s} f(y))\mathrm{d}y \leq C \quad \text{and} \quad \frac{1}{|B_r|}\int_{B_r} f(y)\mathrm{d}y \leq C$$

hold for the balls $B_s = B(x, s)$ and $B_r = B(x, r) : s < \rho(x) \leq r$.

Gao and Tang [9] showed that the operators μ_j^L are bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, bounded from $L^1(\mathbb{R}^d)$ to weak $L^1(\mathbb{R}^d)$, bounded on BMO_L and bounded from $H_L^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$. Our purpose in this paper is to promote BMO_L estimate to BLO_L estimate for the operators μ_j^L.

Our main result is stated as follows.

Theorem 1.4. *Let $V \in B_d$. The Marcinkiewicz integral operators $\mu_j^L (j = 1, 2, \ldots, n)$ associated with the Schrödinger operator are bounded from BMO_L to BLO_L. Furthermore, for any $f \in \mathrm{BMO}_L$, there exists constant $C > 0$ such that*

$$\|\mu_j^L f\|_{\mathrm{BLO}_L} \leq C\|f\|_{\mathrm{BMO}_L}.$$

In what follows, C always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \leq CB$. For any set $E \subset \mathbb{R}^d$, χ_E denotes its characteristic function. For $x \in \mathbb{R}^d$ and $r > 0$, $B(x, r)$ denotes the ball centered at x and having radius r.

2. Proof of Theorem 1.4

First, we need the following lemmas found in [11] and [9].

Lemma 2.1 [11]. *Let $V \in B_d$. There exist $l_0, C > 0$ such that*

$$\frac{1}{C}\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-l_0} \leq \frac{\rho(y)}{\rho(x)} \leq C\left(1 + \frac{|x-y|}{\rho(x)}\right)^{l_0/(l_0+1)}. \tag{2.1}$$

In particular, $\rho(x) \sim \rho(y)$ *if* $|x-y| < C\rho(x)$.

Lemma 2.2 [11]. *Let* $V \in B_d$. *For any* $l > 0$, *there exist* $C_l, C > 0$ *such that*

$$|K_j^L(x,y)| \le C_l \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-l} \frac{1}{|x-y|^{d-1}} \tag{2.2}$$

and

$$|K_j^L(x,y) - Kj(x,y)| \le \frac{C}{|x-y|^{d-1}} \left(\frac{|x-y|}{\rho(x)}\right). \tag{2.3}$$

Lemma 2.3 [9]. *Let* $V \in B_d$. *The Marcinkiewicz integral operators* μ_j^L *are bounded on* $L^p(\mathbb{R}^d)$ *for* $1 < p < \infty$, *and bounded from* $L^1(\mathbb{R}^d)$ *to weak* $L^1(\mathbb{R}^d)$.

Lemma 2.4 [9]. *Let* $V \in B_d$. *The Marcinkiewicz integral operators* μ_j^L *are bounded on* $\mathrm{BMO}_L(\mathbb{R}^d)$.

Proof of Theorem 1.4. Let $f \in \mathrm{BMO}_L(\mathbb{R}^d)$ and fix a ball $B = B(x_0, r)$. First, we suppose that $r \ge \rho(x_0)$, and write

$$f = f\chi_{B^*} + f\chi_{(B^*)^c} = f_1 + f_2,$$

where B^* denotes the ball with the same center and twice the radius of B. For Lemma 2.3, we have

$$\begin{aligned}\frac{1}{|B|}\int_B \mu_j^L f_1(x)\mathrm{d}x &\le \left(\frac{1}{|B|}\int_B |\mu_j^L f_1(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\ &\le \left(\frac{C}{|B|}\int_{B^*} |f(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\ &\le C\|f\|_{\mathrm{BMO}_L}.\end{aligned}$$

Let $x \in B$. The inequality (2.2) in [9] gives

$$\mu_j^L f_2(x) \le C\|f\|_{\mathrm{BMO}_L}$$

and

$$\frac{1}{|B|}\int_B \mu_j^L f_2(x)\mathrm{d}x \le C\|f\|_{\mathrm{BMO}_L}.$$

So, we have

$$\frac{1}{|B|}\int_B \mu_j^L f(x)\mathrm{d}x \le C\|f\|_{\mathrm{BMO}_L}, \quad r \ge \rho(x_0).$$

If $r < \rho(x_0)$, we set

$$f = f\chi_{\bar{B}} + f\chi_{(\bar{B})^c} = f_1^* + f_2^*,$$

where $\bar{B} = B(x_0, 2\rho(x_0))$. Note that $\rho(x) \sim \rho(x_0)$ for any $x \in B$. Similar to the inequality (2.2) in [9], we have

$$\frac{1}{|B|}\int_B \mu_j^L f_2^*(x)\mathrm{d}x \le C\|f\|_{\mathrm{BMO}_L}.$$

To complete the proof of Theorem 1.4 by Remark 1.3(b), we only need to prove that there exists a positive constant C such that

$$\frac{1}{|B|}\int_B \mu_j^L f_1^*(x)\mathrm{d}x \le C\|f\|_{\mathrm{BMO}_L} + \inf_{x\in B}\mu_j^L f(x).$$

It is obvious that

$$\frac{1}{|B|}\int_B \mu_j^L f_1^*(x)\mathrm{d}x \le \frac{1}{|B|}\int_B |\mu_j^L f_1^*(x) - \mu_j f_1^*(x)|\mathrm{d}x + \frac{1}{|B|}\int_B \mu_j f_1^*(x)\mathrm{d}x.$$

Let $x \in B$ and $B_{x,k} = B(x, 2^{2-k}\rho(x_0))$, $k = 0, 1, 2, \ldots$. Then $\rho(x) \sim \rho(x_0)$. It is clear that $|f_{B_{x,0}}| \le C\|f\|_{\mathrm{BMO}_{\mathcal{L}}}$. Because

$$|f_{B_{x,k}} - f_{B_{x,k-1}}| \le C\|f\|_{\mathrm{BMO}},$$

we have

$$|f_{B_{x,k}}| \le C(k+1)\|f\|_{\mathrm{BMO}_L}.$$

It follows that

$$\int_{B_{x,k}} |f(x)|\mathrm{d}x \le C(k+1)|B_{x,k}|\|f\|_{\mathrm{BMO}_L}. \tag{2.4}$$

Making use of (2.3) and (2.4), we get

$$\begin{aligned}
&|\mu_j^L f_1^*(x) - \mu_j f_1^*(x)| \\
&\le \left(\int_0^\infty \left|\int_{|x-y|<\min t, 4\rho(x_0)} |K_j^L(x,y) - K_j(x,y)||f(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t}\right)^{1/2} \\
&\le C\left(\int_0^{4\rho(x_0)} \left|\int_{|x-y|<t} |K_j^L(x,y) - K_j(x,y)||f(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t}\right)^{1/2} \\
&\quad + C\left(\int_{4\rho(x_0)}^\infty \left|\int_{|x-y|<4\rho(x_0)} |K_j^L(x,y) - K_j(x,y)||f(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t}\right)^{1/2} \\
&\le C\left(\int_0^{4\rho(x_0)} \left|\rho(x_0)^{-1}\int_{|x-y|<t} \frac{|f(y)|}{|x-y|^{d-2}}\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t}\right)^{1/2} \\
&\quad + C\left(\int_{4\rho(x_0)}^\infty \left|\int_{|x-y|<4\rho(x_0)} \frac{|f(y)|}{|x-y|^{d-1}}\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t}\right)^{1/2} \\
&\le C\left(\int_0^{4\rho(x_0)} \left|\rho(x_0)^{-1}\sum_{k=-\infty}^{0} (2^k t)^{2-d}\int_{|x-y|<2^k t} |f(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t}\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&+C\left(\int_{4\rho(x_0)}^{\infty}\Big|\sum_{k=-\infty}^{2}(2^k\rho(x_0))^{1-d}\int_{|x-y|<2^k\rho(x_0)}\frac{|f(y)|}{|x-y|^{d-1}}\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t}\right)^{1/2}\\
&\leq C\|f\|_{\mathrm{BMO}_L}.
\end{aligned}
$$

Thus, we have

$$\frac{1}{|B|}\int_B|\mu_j^L f_1^*(x)-\mu_j f_1^*(x)|\mathrm{d}x\leq C\|f\|_{\mathrm{BMO}_L}. \tag{2.5}$$

It remains to show that

$$\frac{1}{|B|}\int_B\mu_j f_1^*(x)\mathrm{d}x\leq C\|f\|_{\mathrm{BMO}_L}+\inf_{x\in B}\mu_j^L f(x). \tag{2.6}$$

Let $B_k^*=B(x_0,2^{1-k}\rho(x_0))$, $k=0,1,2,\ldots,k_0$, where k_0 satisfies $2^{-k_0-1}\rho(x_0)\leq r<2^{-k_0}\rho(x_0)$. Set

$$
\begin{aligned}
f_1^*&=(f_1^*-f_{B_{k_0}^*})\chi_{2B_{k_0}^*}+(f_1^*-f_{B_{k_0}^*})\chi_{(2B_{k_0}^*)^c}+f_{B_{k_0}^*}\\
&=f_{1,1}+f_{1,2}+f_{1,3}.
\end{aligned}
$$

A trivial computation for $\mu_j(1)$ leads to

$$\mu_j f_{1,3}=0. \tag{2.7}$$

By Hölder's inequality and $L^2(\mathbb{R}^d)$ bounedeness of μ_j, we have

$$
\begin{aligned}
\frac{1}{|B|}\int_B\mu_j f_{1,1}(x)\mathrm{d}x&\leq\left(\frac{1}{|B|}\int_B|\mu_j f_{1,1}(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}\\
&\leq\left(\frac{C}{|2B_{k_0}^*|}\int_{2B_{k_0}^*}|f_1^*(x)-f_{B_{k_0}^*}|^2\mathrm{d}x\right)^{\frac{1}{2}}\\
&\leq C\|f\|_{\mathrm{BMO}}.
\end{aligned}
$$

Since $\mu_j^L f(x)\geq 0$, $\inf_{x\in B}\mu_j^L f(x)$ exists. Thus, for any $\epsilon>0$, there exists a point $x_\epsilon\in B$ such that

$$\mu_j^L f(x_\epsilon)\leq\inf_{x\in B}\mu_j^L f(x)+\epsilon. \tag{2.8}$$

For all $x\in B$, we have

$$\mu_j(f_{1,2})(x)\leq|\mu_j(f_{1,2})(x)-\mu_j(f_{1,2})(x_\epsilon)|+\mu_j(f_{1,2})(x_\epsilon). \tag{2.9}$$

First, we need to prove the following inequality:

$$|\mu_j(f_{1,2})(x)-\mu_j(f_{1,2})(x_\epsilon)|\leq C\|f\|_{\mathrm{BMO}_L} \tag{2.10}$$

holds for any $x \in B$. In fact, we need only to prove that for any $x \in B$,

$$|\mu_j(f_{1,2})(x) - \mu_j(f_{1,2})(x_0)| \le C\|f\|_{\mathrm{BMO}_L}. \tag{2.11}$$

For any $x \in B$, we have

$$\begin{aligned}
&|\mu_j(f_{1,2})(x) - \mu_j(f_{1,2})(x_0)| \\
&\le \left(\int_0^\infty \left|\int_{|x-y|<t,|x_0-y|<t} |K_j(x_0,y) - K_j(x,y)||f_{1,2}(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\quad + \left(\int_0^\infty \left|\int_{|x_0-y|\le t<|x-y|} |K_j(x_0,y)||f_{1,2}(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\quad + \left(\int_0^\infty \left|\int_{|x-y|\le t<|x_0-y|} |K_j(x,y)||f_{1,2}(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\doteq I_1 + I_2 + I_3.
\end{aligned}$$

For I_1, by $|K_j(x_0,y) - K_j(x,y)| \le \frac{C|x_0-x|}{|x-y|^d}$ and Minkowski's inequality, we get

$$\begin{aligned}
I_1 &\le \int_{\mathbb{R}^d} \frac{C|x_0-x|}{|x-y|^d}|f_{1,2}(y)|\mathrm{d}y \left(\int_{|x-y|<t,|x_0-y|<t} \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\le Cr\int_{\mathbb{R}^d} \frac{|f_{1,2}(y)|}{|x-y|^{d+1}}\mathrm{d}y \\
&\le C\sum_{k=0}^{k_0-1} \frac{2^{k-k_0}}{|B_k^*|}\int_{B_k^*}(|f(y) - f_{B_k^*}| + |f_{B_k^*} - f_{B_{k_0}^*}|)\mathrm{d}y \\
&\le C\sum_{k=0}^{k_0-1}(k_0-k+1)2^{k-k_0}\|f\|_{\mathrm{BMO}} \\
&\le C\|f\|_{\mathrm{BMO}_L}.
\end{aligned}$$

Since the estimates for I_2 and I_3 follow along similar lines, we only consider I_2. Since $|x_0 - x| \sim |x-y|$, by Minkowski's inequality and the integral meaning theorem, we have

$$\begin{aligned}
I_2 &\le C\int_{\mathbb{R}^d} \frac{|f_{1,2}(y)|}{|x-y|^{d-1}}\mathrm{d}y \left(\int_{|x_0-y|<t<|x-y|} \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\le Cr^{1/2}\int_{\mathbb{R}^d} \frac{|f_{1,2}(y)|}{|x-y|^{d+1/2}}\mathrm{d}y \\
&\le C\sum_{k=0}^{k_0-1} \frac{2^{(k-k_0)/2}}{|B_k^*|}\int_{B_k^*}(|f(y) - f(B_k^*)| + |f(B_k^*) - f(B_{k_0}^*)|)\mathrm{d}y \\
&\le C\sum_{k=0}^{k_0-1}(k_0-k+1)2^{(k-k_0)/2}\|f\|_{\mathrm{BMO}}.
\end{aligned}$$

Now, we need only to prove that

$$\mu_j(f_{1,2})(x_\epsilon) \le C\|f\|_{\mathrm{BMO}_\mathcal{L}} + \mu_j^L f(x_\epsilon). \tag{2.12}$$

On the other hand, by the definition of μ_j and the cancellation condition of their kernels and $|x_0 - y| \sim |x_\epsilon - y|$, we obtain

$$
\begin{aligned}
&\mu_j(f_{1,2})(x_\epsilon)\\
&= \left(\int_0^\infty \left|\int_{|y-x_\epsilon|<t} K_j(x_\epsilon, y)(f_1^*(y) - f_{B_{k_0}^*})\chi_{(2B_{k_0}^*)^c}(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}\\
&= \left(\int_{3r}^\infty \left|\int_{|y-x_\epsilon|<t} K_j(x_\epsilon, y)(f_1^*(y) - f_{B_{k_0}^*})\chi_{(2B_{k_0}^*)^c}(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}\\
&\le \left(\int_{3r}^\infty \left|\int_{|y-x_\epsilon|<t} K_j(x_\epsilon, y)(f_1^*(y) - f_{B_{k_0}^*})\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}\\
&\quad + \left(\int_{3r}^\infty \left|\int_{|y-x_\epsilon|<t} K_j(x_\epsilon, y)(f_1^*(y) - f_{B_{k_0}^*})\chi_{2B_{k_0}^*}(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}\\
&\le \mu_j(f_1^*)(x_\epsilon)\\
&\quad + \left(\int_{3r}^\infty \left|\int_{|y-x_\epsilon|<t} K_j(x_\epsilon, y)(f(y) - f_{B_{k_0}^*})\chi_{2B_{k_0}^*}(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}\\
&\le \mu_j(f_1^*)(x_\epsilon) + C\left(\int_{3r}^\infty \left|\int_{|y-x_\epsilon|<\min\{t,8r\}} \frac{|f(y) - f_{B_{k_0}^*}|}{|y-x_\epsilon|^{d-1}}\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}\\
&\le \mu_j(f_1^*)(x_\epsilon)\\
&\quad + C\left(\int_{3r}^\infty \frac{\mathrm{d}t}{t^3}\right)^{1/2} \sum_{k=-\infty}^{3} (2^k r)^{1-d} \int_{2^{k-1}r<|y-x_\epsilon|<2^k r} |f(y) - f_{B_{k_0}^*}|\mathrm{d}y\\
&\le \mu_j(f_1^*)(x_\epsilon) + C\|f\|_{\mathrm{BMO}}.
\end{aligned}
$$

By Minkowski's inequality, similar to (2.2) in [9] and (2.5), we have

$$
\begin{aligned}
&\mu_j(f_1^*)(x_\epsilon)\\
&= \left(\int_0^\infty \left|\int_{|y-x_\epsilon|<t} K_j(x_\epsilon, y) f_1^*(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}\\
&\quad - \left(\int_0^\infty \left|\int_{|y-x_\epsilon|<t} K_j^L(x_\epsilon, y) f(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} + \mu_j^L f(x_\epsilon)\\
&\le \left(\int_0^\infty \left|\int_{|y-x_\epsilon|<t} [K_j(x_\epsilon, y) f_1^*(y) - K_j^L(x_\epsilon, y) f(y)]\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}\\
&\quad + \mu_j^L f(x_\epsilon)\\
&\le \left(\int_0^\infty \left|\int_{|y-x_\epsilon|<t} (K_j(x_\epsilon, y) - K_j^L(x_\epsilon, y)) f_1^*(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}\\
&\quad + \left(\int_0^\infty \left|\int_{|y-x_\epsilon|<t} K_j^L(x_\epsilon, y) f_2^*(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} + \mu_j^L f(x_\epsilon)\\
&\le C\|f\|_{\mathrm{BMO}_{\mathcal{L}}} + \mu_j^L f(x_\epsilon).
\end{aligned}
$$

Hence (2.12) is obtained. By (2.7), (2.8), (2.9) ,(2.10) and (2.12), we have

$$\frac{1}{|B|}\int_B \mu_j f_1^*(x)\mathrm{d}x \leq C\|f\|_{\mathrm{BMO}_L} + \inf_B \mu_j^L f(x) + \epsilon.$$

By the arbitrariness of ϵ, we get (2.6). Thus, it is follows that

$$\|\mu_j^L f\|_{\mathrm{BLO}_L} \leq C\|f\|_{\mathrm{BMO}_L}.$$

Thus, Theorem 1.4 is proved. □

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