Infinitely many solutions for the stationary fractional $p$-Kirchhoff problems in $\mathbb{R}^N$

EBUBEKIR AKKOYUNLU$^1$ and RABIL AYAZOGLU$^{2,3,*}$

1Vocational School of Social Sciences, Bayburt University, Bayburt, Turkey
2Faculty of Education, Bayburt University, Bayburt, Turkey
3Institute of Mathematics and Mechanics of ANAS, Baku, Azerbaijan
*Corresponding author.
E-mail: eakkoyunlu@bayburt.edu.tr; rabilmashiyev@gmail.com; rayazoglu@bayburt.edu.tr

MS received 24 August 2018; revised 28 February 2019; accepted 18 March 2019

Abstract. In the present paper, we investigate the existence of multiple solutions for the nonhomogeneous fractional $p$-Kirchhoff equation

$$
M \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \, dx \, dy + \int_{\mathbb{R}^N} V(x) \, |u|^p \, dx \right) \times ((-\Delta)_p^s u + V(x) |u|^{p-2} u) = f(x, u) \text{ in } \mathbb{R}^N,
$$

where $(-\Delta)_p^s$ is the fractional $p$-Laplacian operator, $0 < s < 1 < p < \infty$ with $sp < N$, $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a nonnegative, continuous and increasing Kirchhoff function, the nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that obeys some conditions which will be stated later and $V \in C(\mathbb{R}^N, \mathbb{R}_0^+)$ is a non-negative potential function. We first establish the Bartsch–Pankov–Wang type compact embedding theorem for the fractional Sobolev spaces. Then multiplicity results are obtained by using the variational method, $(S_+)$ mapping theory and Krasnoselskii’s genus theory.

Keywords. Kirchhoff equation; fractional $p$-Laplacian; variational methods; Krasnoselskii’s genus; infinitely many solutions.

Mathematics Subject Classification. 35R11, 35A15, 35J60, 47G20.

1. Introduction and preliminaries

In this paper, we investigate the existence of multiple solutions for fractional $p$-Kirchhoff equations. More precisely, we consider

$$
M \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \, dx \, dy + \int_{\mathbb{R}^N} V(x) \, |u|^p \, dx \right) \times ((-\Delta)_p^s u + V(x) |u|^{p-2} u) = f(x, u) \text{ in } \mathbb{R}^N,
$$

© Indian Academy of Sciences
Published online: 27 July 2019
where \(0 < s < 1 < p < \infty\) with \(sp < N\), \(M : \mathbb{R}_0^+ \to \mathbb{R}_0^+\) is a nonnegative, continuous and nondecreasing function, the nonlinearity \(f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is a Carathéodory function that obeys some conditions which will be stated later, \(V \in C(\mathbb{R}^N, \mathbb{R}^+)^\times\) is a non-negative potential function and \((-\Delta)^p\) is the fractional \(p\)-Laplace operator which (up to normalization factors) may be defined along any \(\varphi \in C_0(\mathbb{R}^N)\) as

\[
(-\Delta)^p \varphi(x) = 2 \lim_{\delta \to 0^+} \int_{\mathbb{R}^N} \frac{|\varphi(x) - \varphi(y)|^{p-2} (\varphi(x) - \varphi(y))}{|x - y|^{N + ps}} \, dy
\]

for \(x \in \mathbb{R}^N\), where \(B_\varepsilon(x) := \{ y \in \mathbb{R}^N : |x - y| < \varepsilon\}\).

There are some fundamental results that concern the fractional \(p\)-Laplacian, such as the nonlinear \(p\)-fractional Kirchhoff and Schrödinger–Kirchhoff equations in [1,4,11,13,21,22,26] and references therein. On the other hand, we remark that (1.1) is a fractional version of the well-known \(p\)-Laplacian, given by \(\text{div}(|\nabla u|^{p-2}\nabla u)\), that is associated with the Sobolev space \(W^{1,p}(\mathbb{R}^N)\).

When \(p = 2\) and \(M \equiv 1\), equation (P) becomes the fractional Laplacian equation

\[
(-\Delta)^s u + V(x)u = f(x, u) \quad \text{in} \quad \mathbb{R}^N.
\]

We would like to quote some important and interesting results for the problems involving the fractional Laplacian \((-\Delta)^s\), \(0 < s < 1\) (see [14–17,30]).

Problem (P) is a generalization of a model, the so-called Kirchhoff equation, introduced by Kirchhoff. To be more precise, Kirchhoff established a model given by the equation

\[
\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx \right) \frac{\partial^2 u}{\partial x^2} = f(x, u),
\]

where \(P_0\), \(h\), \(E\), \(L\) are constants, which extends the classical D’Alambert’s wave equation by considering the effects of the changes in the length of the strings during vibrations. A distinguishing feature of the Kirchhoff equation (K) is that the equation contains a nonlocal coefficient \(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx\), which depends on the average \(\frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx\) of the kinetic energy \(\frac{1}{2} \left| \frac{\partial u}{\partial x} \right|^2\) on \([0, L]\), \(f\) is the area of the cross section, and hence the equation is no longer a pointwise identity.

Pucci et al. [21] investigated the existence of multiple solutions for the following non-homogeneous fractional \(p\)-Laplacian equation of the Schrödinger–Kirchhoff type

\[
M \left( \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N + ps}} \, dx \, dy \right) ((-\Delta)^p u + V(x)|u|^{p-2}u) = f(x, u) + g(x) \quad \text{in} \quad \mathbb{R}^N,
\]

where \(f\), \(g\) functions obey some conditions. The Kirchhoff function \(M\) and the potential \(V\) satisfies the following assumptions:

\((M_1)\): \(M \in C(\mathbb{R}_0^+)\) satisfies \(\inf_{t \in \mathbb{R}_0^+} M(t) \geq a > 0\), where \(a > 0\) is a constant;

\((M_2)\): There exists \(\theta \in [1, N/(N - sp))\) such that

\[
\theta \dot{M}(t) = \theta \int_0^t M(\tau) \, d\tau \geq M(t)t \quad \text{for any} \quad t \in \mathbb{R}_0^+;
\]
(V)\(_1\): \(V \in C(\mathbb{R}^N)\) is bounded from below;
(V)\(_2\): There exists \(r > 0\) such that
\[
\lim_{|y| \to \infty} |\{ x \in B_r(y) : V(x) \leq L \}| = 0 \quad \text{for any } L > 0.
\]

Conditions (V)\(_1\) and (V)\(_2\) which are weaker than the coercivity assumption \(V(x) \to +\infty\) as \(|x| \to +\infty\) (see the following Remark 1.1), was originally introduced by Bartsch and Wang in [3] to overcome the lack of compactness. In [3], the conditions (V)\(_1\) and (V)\(_2\) were used to investigate the existence and multiplicity of solutions to the nonlinear Schrödinger equations, see for example [2] for further discussions.

Remark 1.1. Let \(V\) be a zig-zag function with respect to \(|x|\) defined by
\[
V(x) = n \sin [\pi |x| - \pi (n - 1)] - 1, \quad n - 1 \leq |x| \leq n, \quad n \in \mathbb{N}.
\]

It is easy to check that \(V\) satisfies (V)\(_1\) and (V)\(_2\) in [21], but \(V\) does not satisfy the condition \(V(x) \to +\infty\) as \(|x| \to +\infty\).

In [23], Piersanti and Pucci dealt with critical \(p\)-fractional Hardy Schrödinger Kirchhoff type, that is,

\[
M(||u||_{W^s_p(\mathbb{R}^N)}^p)((-\Delta)^s_p u + V(x)|u|^{p-2}u) - \gamma \frac{|u|^{p-2}u}{|x|^{ps}} = \lambda f(x,u) + g(x,u) + K(x)(u^+)^{p_s^*-1} \quad \text{in } \mathbb{R}^N,
\]

where \(\lambda\) and \(\gamma\) are real parameters, \(0 < s < 1 < p < \infty\) such that \(sp < N\) and \(u^+ = \max\{u, 0\}\). The exponent \(p_s^* = Np/(N - sp)\) is critical in the sense of Sobolev, while the nonlinear terms \(f\) and \(g\) are subcritical. In this paper, the weights \(K\) and \(V\) satisfy

\([K_1]\): \(K \geq 0\) a.e. in \(\mathbb{R}^N\) and \(K \in L^\infty(\mathbb{R}^N)\);

\([V_1]\): \(V \in C(\mathbb{R}^N)\) and \(V(x) \geq V_0 > 0\) for all \(x \in \mathbb{R}^N\), where \(V_0\) is a positive constant;

while the main Kirchhoff function \(M\) verifies the condition.

\([M_1]\): \(M : \mathbb{R}^+_0 \to \mathbb{R}^+_0\) is a nonnegative continuous function and there exists \(\theta \in [1, N/(N - sp)]\) such that
\[
\theta M(t) = \theta \int_0^t M(\tau) d\tau \geq M(t)t \quad \text{for any } t \in \mathbb{R}^+_0.
\]

The existence theorems of nonnegative entire solutions of stationary critical \(p\)-fractional Hardy Schrödinger Kirchhoff equations are presented by Piersanti and Pucci [23].

In [25], Song and Shi studied a class of degenerate \(p\)-fractional Kirchhoff equations

\[
M(||u||_{W^s_p(\mathbb{R}^N)}^p)((-\Delta)^s_p u + V(x)|u|^{p-2}u) = \lambda f(x,u) + \gamma \frac{|u|^{p_s^*(\alpha)-2}u}{|x|^{\alpha}} \quad \text{in } \mathbb{R}^N,
\]

with critical Hardy–Sobolev nonlinearities. In this paper, the \(f\) function obeys some conditions. \(V\) and \(M\) functions satisfy the following assumptions:

\([V]\): \(V : \mathbb{R}^N \to \mathbb{R}^+\) is a continuous function and there exists \(V_0 > 0\) such that \(\inf_{t \in \mathbb{R}^N} V(t) \geq V_0 > 0\);
(M): $M : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ is assumed to be continuous and satisfy the following assumptions.

(M$_1$): There exists $\theta \in [1, p_s^*(\alpha)/p)$, $p_s^*(\alpha) = p(N - \alpha)/(N - ps)$ such that $\theta \dot{M}(t) \geq M(t)t$ for all $t \in \mathbb{R}^+_0$, where $\dot{M}(t) = \int_0^t M(\tau)\,d\tau$;

(M$_2$): For any $\tau > 0$, there exists $m = m(\tau) > 0$ such that $M(t) \geq m$ for all $t \geq \tau$;

(M$_3$): There exists $m_0 > 0$ such that $M(t) \geq m_0 t^{\theta-1}$ for all $t \in [0, 1]$.

Notice that the original meaning of the Kirchhoff function $M$ in the equations (1.3),(1.5) and (1.6) should be an increasing function. Then

$$\theta \dot{M}(t) < \int_0^t M(t)\,ds = M(t)t \quad \text{for all } t \in \mathbb{R}^+_0,$$

and therefore, condition (1.4) cannot be satisfied. The condition (1.4) imposed on $M$ is far away from the physical sense of the original Kirchhoff equation.

Some interesting topics concern the fractional Laplacian, such as the nonlinear fractional Schrödinger–Kirchhoff equations (see [7,10,20,23,28]).

Motivated by the above work, by using the variational approach, $(S_+)$ mapping theory and Krasnoselskii’s genus theory, we show that the existence of multiple solutions for Schrödinger–Kirchhoff type equation involves the fractional $p$-Laplacian in $\mathbb{R}^N$ under some weaker assumptions. We also establish a Bartsch–Pankov–Wang type compact embedding theorem for fractional Sobolev space.

We use the following assumptions:

(V$_0$): $(V$ is coercive type potential): $V \in C(\mathbb{R}^N, \mathbb{R}^+)$, $\inf_{x \in \mathbb{R}^N} V(x) = V^- > 0$ and

$$\lim_{|x| \to +\infty} V(x) = +\infty;$$

(M$_0$): $(\text{Polynomial growth condition})$: $M : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ is a nonnegative, continuous and increasing Kirchhoff function

$$a_0 t^{\beta-1} \leq M(t) \leq a_1 t^{\alpha-1},$$

for all $t \in \mathbb{R}^+_0$ with $1 < \beta \leq \alpha < \infty$ and $a_0, a_1$ are constants such that $0 < a_0 \leq a_1 < \infty$;

(f$_1$): $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$|f(x, t)| \leq \sum_{i=1}^m b_i(x) |t|^\gamma_i - 1 \quad \text{for a.e } x \in \mathbb{R}^N \text{ and for all } t \in \mathbb{R},$$

where $b_i \geq 0, b_i \in L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), 1 < p < \gamma_i < \beta p$, and there are constants $s_i$ such that $s_i < p_s^*$ and

$$\frac{1}{r_i} + \frac{\gamma_i}{s_i} = 1;$$

(f$_2$): there exist a nonzero measure open set $G \subset \mathbb{R}^N$ and three constants $b_0, \delta > 0$, and $1 < \gamma_0 < p$ such that

$$F(x, t) \geq b_0 |t|^{\gamma_0}, \forall (x, t) \in G \times [-\delta, \delta],$$

where $F(x, t) = \int_0^t f(x, s)\,ds$;

(f$_3$): $f$ is an odd function according to $t$, that is, $f(x, t) = -f(x, -t)$ $\forall (x, t) \in \mathbb{R}^N \times \mathbb{R}$. 

For any real $s > 0$ and for any $p \in [1, \infty)$, we define the fractional Sobolev spaces $W^{s,p}(\mathbb{R}^N)$. In the literature, the fractional Sobolev-type spaces are also called Aronszajn, Gagliardo or Slobodeckij spaces, by the name of those who introduced them, almost simultaneously.

We start by fixing the fractional exponent $s$ in $(0, 1)$. For any $p \in [1, +\infty)$, we define $W^{s,p}(\mathbb{R}^N)$ as follows:

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : [u]_{s,p} = \left( \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy \right)^{1/p} < \infty \right\},$$

is the so-called Gagliardo (semi) norm of $u$ and $W^{s,p}(\mathbb{R}^N)$ is equipped with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \|u\|_{s,p} = \left( \|u\|_{L^p(\mathbb{R}^N)}^p + \|u\|_{L^p(\mathbb{R}^N)}^p \right)^{1/p} = \left( [u]_{s,p}^p + \|u\|_{L^p(\mathbb{R}^N)}^p \right)^{1/p}.$$ 

As it is well-known, $W^{s,p}(\mathbb{R}^N) = (W^{s,p}(\mathbb{R}^N), \|\cdot\|_{s,p})$ is a uniformly convex Banach space.

Let $W(\mathbb{R}^N)$ denote the completion of $C_0^\infty(\mathbb{R}^N)$, with respect to the norm

$$\|u\|_{W(\mathbb{R}^N)} = ([u]_{s,p} + \|u\|_{L^p(\mathbb{R}^N)})^{1/p}, \quad \|u\|_{L^p(\mathbb{R}^N)} = \int_{\mathbb{R}^N} V(x)|u|^p \, dx,$$

where $V: \mathbb{R}^N \rightarrow \mathbb{R}^+$ is a potential function. Clearly the definition makes sense since every $\varphi \in C_0^\infty(\mathbb{R}^N)$ has a finite Gagliardo norm as well as a finite norm $\|\varphi\|_{L^p(\mathbb{R}^N)}$. Indeed, $L^p(\mathbb{R}^N), V) = (L^p(\mathbb{R}^N), V), \|\cdot\|_{L^p(\mathbb{R}^N)}$ is a uniformly convex Banach space, thanks to (V0).

By standard arguments, it is clear that $W(\mathbb{R}^N)$ is a uniformly convex Banach space.

DEFINITION 1.1

We call that $u \in W(\mathbb{R}^N)$ is a weak solution of (P), if

$$M(\|u\|_{W(\mathbb{R}^N)})B_\varphi(u) = \int_{\mathbb{R}^N} f(x, u)\varphi(x) \, dx \quad \text{for any } \varphi \in W(\mathbb{R}^N),$$

where

$$B_\varphi(u) = \iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))K(x, y) \, dx \, dy$$

$$+ \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\varphi \, dx,$$

and $K(x, y) = |x - y|^{-(N + ps)}$.

Let $0 < s < 1 < p < \infty$ be real numbers with $sp < N$, and let $p_s^*$ be the fractional Sobolev critical exponent defined by $p_s^* = Np/(N - sp)$.

Lemma 1A [9,21]. Let $(V)_1$ and $(V)_2$ hold. If $v \in [p, p_s^*]$, then the embeddings $W(\mathbb{R}^N) \hookrightarrow W^{s,p}(\mathbb{R}^N) \hookrightarrow L^v(\mathbb{R}^N)$ are continuous, with $\min\{1, V^-\} \|u\|_{s,p} \leq \|u\|_{W(\mathbb{R}^N)}^p$ for all $u \in W(\mathbb{R}^N)$. In particular, there exists a constant $C_v > 0$ such that $\|u\|_v \leq C_v \|u\|_{W(\mathbb{R}^N)}$ for all $u \in W(\mathbb{R}^N)$. If $v \in [1, p_s^*)$, then the embedding $W(\mathbb{R}^N) \hookrightarrow L^v(B_R)$ is compact for any $R > 0$. 
2. Proof of the main result

The following Bartsch–Pankov–Wang type new compact embedding (see [2,3]) which will be proved by us play a crucial role in our subsequent arguments.

**Theorem 2.1.** Let \((V_0)\) hold true. Then the embedding \(W(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)\) with \(\nu \in [p, p^*_s)\) is compact.

**Proof.** For \(R > 0\), we denote by \(B_R := \{x \in \mathbb{R}^N : |x| < R\}\) the open ball in \(\mathbb{R}^N\) with center 0 radius \(R\) and, \(|B_R| := \text{meas}(B_R)\) and \(B^c_R = \mathbb{R}^N \setminus B_R\). We first consider the case \(\nu = p\). Let \(Y = L^p(\mathbb{R}^N)\) and denote \(W(\Omega), Y(\Omega)\) the spaces of functions \(u \in W(\mathbb{R}^N), u \in Y\) restricted onto \(\Omega \subset \mathbb{R}^N\) respectively.

Firstly, by Lemma 1A, we claim that the embedding \(W(B_R) \hookrightarrow Y(B_R)\) is compact. Assume that \(\{|u_n|\}\) is a bounded sequence in \(W(B_R)\). Then \(\{|u_n|\}\) is bounded in \(W(B_R)\). By compactness of the imbedding theorem in the bounded domain \(B_R\), it follows that there exist \(u \in Y(B_R)\) and a subsequence \(\{|u_{n_k}|\}\) of \(\{|u_n|\}\) such that

\[\|u_{n_k} - u\|_{p, B_R} \to 0\]

as \(k \to \infty\). Without loss of generality, we let

\[\|u_{n} - u\|_{p, B_R} \to 0\]

as \(n \to \infty\). By \((V_0)\), we let \(\inf_{x \in B^c_R} V(x) = V^-_R > 0\) for any \(R > 0\) and obviously, \(V^-_R \to +\infty\) as \(R \to +\infty\). We need to show that \(W(B^c_R) \hookrightarrow Y(B^c_R)\), i.e. there exists a constant \(C(R) > 0\) independent of function \(u\) such that

\[\|u\|_{Y(B^c_R)} \leq C(R)\|u\|_{W(\mathbb{R}^N)}\]

We get

\[\int_{B^c_R} |u|^p \, dx \leq \frac{1}{V^-_R} \int_{\mathbb{R}^N} V(x) |u|^p \, dx \leq \frac{1}{V^-_R} \|u\|_{W(\mathbb{R}^N)}^p,\]

which implies

\[\|u\|_{Y(B^c_R)} \leq \frac{1}{V^-_R} \|u\|_{W(\mathbb{R}^N)}\]

Secondly, we claim

\[\lim_{R \to +\infty} \sup_{u \in W(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_{Y(B^c_R)}}{\|u\|_{W(\mathbb{R}^N)}} = 0.\]  (2.1)

In the following, we show that \(u_n \to u\) in \(Y\). Since \(W(\mathbb{R}^N)\) is a reflexive Banach space and \(\{|u_n|\}\) is bounded in \(W(\mathbb{R}^N)\), we can assume (up to a sequence) that \(u_n \to u\) in \(W(\mathbb{R}^N)\) and

\[\|u_n\|_{W(\mathbb{R}^N)} \leq C_0\]  (2.2)
for some constant $C_0 > 0$. By (2.1) and (2.2), we know that for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ so large that

$$
\|u_n\|_{Y(\mathbb{B}_c R_\varepsilon)} \leq C_0^{-1} \varepsilon \|u_n\|_{W(\mathbb{R}^N)} \leq \frac{\varepsilon}{3}, \quad n = 1, 2, \ldots.
$$

(2.3)

From continuousness of measure, we have

$$
\lim_{R \to \infty} \|u\|_{Y(\mathbb{B}_c R_\varepsilon)} = 0,
$$

and thus there exists maybe a new $R_\varepsilon > 0$ such that (2.3) holds and additionally,

$$
\|u\|_{Y(\mathbb{B}_c R_\varepsilon)} \leq \frac{\varepsilon}{3}.
$$

(2.4)

Since $W(\mathbb{B}_c R_\varepsilon) \hookrightarrow \hookrightarrow L^p(\mathbb{B}_c R_\varepsilon)$ is compact for any $R_\varepsilon > 0$, we have

$$
\lim_{n \to \infty} \|u_n - u\|_{L^p(\mathbb{B}_c R_\varepsilon)} = 0.
$$

Thus, there exists $n_\varepsilon$, when $n \geq n_\varepsilon$,

$$
\|u_n - u\|_{Y(\mathbb{B}_c R_\varepsilon)} \leq \frac{\varepsilon}{3}.
$$

(2.5)

Using (2.3), (2.4) and (2.5), given an $\varepsilon$, we may find $R_\varepsilon$ and then $n_\varepsilon$ such that

$$
\|u_n - u\|_Y \leq \|u_n\|_{Y(\mathbb{R}^N \setminus \mathbb{B}_c R_\varepsilon)} + \|u\|_{Y(\mathbb{R}^N \setminus \mathbb{B}_c R_\varepsilon)} + \|u_n - u\|_{Y(\mathbb{B}_c R_\varepsilon)} \\
\quad \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
$$

which shows that $\{u_n\}$ is convergent in $Y$, and $W(\mathbb{R}^N) \hookrightarrow \hookrightarrow Y = L^p(\mathbb{R}^N)$. This implies $u_n \to u$ in $L^p(\mathbb{R}^N)$. For $p < \nu < p^*$, choose $\lambda \in (0, 1)$ satisfying

$$
\frac{1}{\nu} = \frac{\lambda}{p} + \frac{1 - \lambda}{p^*}.
$$

Then by Hölder inequality and Lemma 1A, we get

$$
\|u_n - u\|_{L^{\nu}(\mathbb{R}^N)} \leq \|u_n - u\|_{L^{\nu}(\mathbb{R}^N)}^{\lambda - \nu} \|u_n - u\|_{L^{\nu}(\mathbb{R}^N)}^{\nu - \lambda} \\
\quad \leq C\nu \|u_n - u\|_{L^{\nu}(\mathbb{R}^N)}^{\lambda - \nu} \|u_n - u\|_{L^{\nu}(\mathbb{R}^N)}^{\nu - \lambda} \\
\quad \leq (C\nu) \|u_n - u\|_{L^{\nu}(\mathbb{R}^N)}^{\lambda - \nu} \to 0 \text{ as } n \to \infty,
$$

since $\{u_n\}$ is bounded in $W(\mathbb{R}^N)$, which shows that $\{u_n\}$ is convergent in $L^\nu(\mathbb{R}^N)$, and $W(\mathbb{R}^N) \hookrightarrow \hookrightarrow L^\nu(\mathbb{R}^N)$. The proof of Theorem 2.1 is completed. \qed

For $u \in W(\mathbb{R}^N)$, we define

$$
I(u) = \frac{1}{p} \hat{M}(\|u\|_{W(\mathbb{R}^N)}) - \int_{\mathbb{R}^N} F(x, u)dx := J(u) - \Psi(u),
$$
where \( F(x, t) = \int_0^t f(x, s)ds \) and \( \hat{M}(t) = \int_0^t M(s)ds \) for all \( t \in \mathbb{R}^+ \). Obviously, the energy functional \( I : W(\mathbb{R}^N) \to \mathbb{R} \) associated to problem (P) is well defined.

On the one hand, if \((V_0)\) and \((M_0)\) hold, then the functional \( J : W(\mathbb{R}^N) \to \mathbb{R} \) is well defined and of class \( C^1(\mathbb{R}^N) \). Moreover, the derivative of \( J \) is
\[
\langle J'(u), v \rangle = M(\|u\|_{W(\mathbb{R}^N)}) B(u)
\]
for any \( u, v \in W(\mathbb{R}^N) \) and \( J \) is weakly lower semi-continuous in \( W(\mathbb{R}^N) \) (see, for example, [23,25]). On the other hand, if \((V_0)\) and \((f_1)\) hold, then the functional \( \Psi : W(\mathbb{R}^N) \to \mathbb{R} \) is well defined and of class \( C^1(\mathbb{R}^N) \) and
\[
\langle \Psi'(u), v \rangle = \int_{\mathbb{R}^N} f(x, u) v(x) dx.
\]
Thus, if \( v_n \to v \) weakly in \( W(\mathbb{R}^N) \), then \( \langle \Psi'(u), v_n \rangle \to \langle \Psi'(u), v \rangle \) as \( n \to \infty \) and the functional is weakly continuous in \( W(\mathbb{R}^N) \).

In a standard way, it can be shown that \( I \in C^1(\mathbb{R}^N) \) and that the critical points of \( I \) are solutions of \((P)\). Moreover, the derivative of \( I \) is given by
\[
\langle I'(u), v \rangle = M(\|u\|_{W(\mathbb{R}^N)}) B_v(u)
\]
\[
- \int_{\mathbb{R}^N} f(x, u) v(x) dx := \langle J'(u), v \rangle - \langle \Psi'(u), v \rangle,
\]
for any \( u, v \in W(\mathbb{R}^N) \).

The main result of this paper is the following Theorem 2.2.

**Theorem 2.2.** Suppose \((V_0)\), \((M_0)\), \((f_1)\), \((f_2)\) and \((f_3)\) hold. Then problem (P) has infinitely many pairs of nontrivial weak solutions \( \{ \pm u_k : k = 1, 2, \ldots \} \) with \( I(\pm u_k) < 0 \).

**Before proving Theorem 2.2,** we first give some auxiliary lemmas.

**Lemma 2.1.** Suppose \((V_0)\), \((M_0)\) and \((f_1)\) hold. Then \( I \) is coercive and bounded from below.

**Proof.** For any \( u \in W(\mathbb{R}^N) \), by \((M_0)\), \((f_1)\), Lemma 1A and the Hölder inequality, we have
\[
I(u) = \frac{1}{p} \hat{M}(\|u\|_{W(\mathbb{R}^N)}) - \int_{\mathbb{R}^N} F(x, u) dx
\]
\[
\geq \frac{a_0}{p} \int_0^{\|u\|_{W(\mathbb{R}^N)}} \tau^{p-1} d\tau - \sum_{i=1}^m \frac{1}{\gamma_i} \int_{\mathbb{R}^N} b_i(x)|u|^{\gamma_i} dx
\]
\[
\geq \frac{a_0}{p\beta} \|u\|_{W(\mathbb{R}^N)}^{\beta p} - \sum_{i=1}^m \frac{1}{\gamma_i} \|b_i\|_{L^p(\mathbb{R}^N)} \|u\|_{s_i(\mathbb{R}^N)}^{\gamma_i}
\]
\[
\geq \frac{a_0}{p\beta} \|u\|_{W(\mathbb{R}^N)}^{\beta p} - \sum_{i=1}^m C_b C_{\gamma_i} \|u\|_{W(\mathbb{R}^N)}^{\gamma_i},
\]
(2.6)
where \( C_b, C_{\gamma_i} \) are positive constants. Since \( 1 < \gamma_i \leq \beta p, i = 1, 2, \ldots , m \), we have \( I(u) \to \infty \) as \( \|u\|_{W(\mathbb{R}^N)} \to \infty \). Hence, \( I \) is coercive and bounded from below. The proof of Lemma 2.1 is completed. \( \square \)
We prove the following properties about the derivative operator of $J$. We denote $L = J' : W(\mathbb{R}^N) \to W^*(\mathbb{R}^N)$.

**Lemma 2.2.** Suppose (M0) hold, then

(i) $L : W(\mathbb{R}^N) \to W^*(\mathbb{R}^N)$ is a continuous, bounded and strictly monotone operator;

(ii) $L$ is a mapping of type $(S_+)$, i.e. if $u_n \to u$ in $W(\mathbb{R}^N)$ and

\[ \lim_{n \to +\infty} \langle L(u_n) - L(u), u_n - u \rangle \leq 0, \]

then $u_n \to u$ in $W(\mathbb{R}^N)$;

(iii) $L : W(\mathbb{R}^N) \to W^*(\mathbb{R}^N)$ is a homeomorphism.

**Proof.**

(i) It is obvious that $L$ is continuous and bounded since $M$ is continuous. For any $u, v \in W(\mathbb{R}^N)$ with $u \neq v$, without loss of generality, we may assume that $\|u\|_{W(\mathbb{R}^N)} \geq \|v\|_{W(\mathbb{R}^N)}$ (otherwise, by changing the role of $u$ and $v$ in the following proof). Then we have

\[ M(\|u\|_{W(\mathbb{R}^N)})^p \geq M(\|v\|_{W(\mathbb{R}^N)})^p \tag{2.7} \]

since $M$ is a monotone function. From Cauchy’s inequality, we can write

\[
(u(x) - u(y))(v(x) - v(y)) \leq |u(x) - u(y)| |v(x) - v(y)| \\
\leq \frac{|u(x) - u(y)|^2 + |v(x) - v(y)|^2}{2}. \tag{2.8}
\]

Using (2.8), we obtain

\[
\iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^p K(x, y) dx dy \\
- \iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y)) K(x, y) dx dy \\
\geq \frac{1}{2} \iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^{p-2}|u(x) - u(y)|^2 - |v(x) - v(y)|^2 K(x, y) dx dy \tag{2.9}
\]

and

\[
\iint_{\mathbb{R}^{2N}} |v(x) - v(y)|^{p-2}(v(x) - v(y))(u(x) - u(y)) K(x, y) dx dy \\
- \int_{\mathbb{R}^N \times \mathbb{R}^N} |v(x) - v(y)|^p K(x, y) dx dy \\
\geq \frac{1}{2} \iint_{\mathbb{R}^{2N}} |v(x) - v(y)|^{p-2}|v(x) - v(y)|^2 - |u(x) - u(y)|^2 K(x, y) dx dy. \tag{2.10}
\]
By using Young’s inequality, we can write
\[
\int_\mathbb{R}^2 \left| u(x) - u(y) \right|^{p-2} \left| v(x) - v(y) \right|^2 K(x, y) \, dx \, dy \\
\leq \frac{p-2}{p} \int_\mathbb{R}^2 \left| u(x) - u(y) \right|^p K(x, y) \, dx \, dy \\
+ \frac{2}{p} \int_\mathbb{R}^2 \left| v(x) - v(y) \right|^p K(x, y) \, dx \, dy,
\]
(2.11)
and
\[
\int_\mathbb{R}^2 \left| v(x) - v(y) \right|^{p-2} \left| u(x) - u(y) \right|^2 K(x, y) \, dx \, dy \\
\leq \frac{p-2}{p} \int_\mathbb{R}^2 \left| v(x) - v(y) \right|^p K(x, y) \, dx \, dy \\
+ \frac{2}{p} \int_\mathbb{R}^2 \left| u(x) - u(y) \right|^p K(x, y) \, dx \, dy.
\]
(2.12)
Therefore, by (2.11) and (2.12), we can write
\[
\int_\mathbb{R}^2 \left| u(x) - u(y) \right|^{p-2} \left| v(x) - v(y) \right|^2 K(x, y) \, dx \, dy \\
+ \int_\mathbb{R}^2 \left| v(x) - v(y) \right|^{p-2} \left| u(x) - u(y) \right|^2 K(x, y) \, dx \, dy \\
\leq \int_\mathbb{R}^2 \left| u(x) - u(y) \right|^p K(x, y) \, dx \, dy \\
+ \int_\mathbb{R}^2 \left| v(x) - v(y) \right|^p K(x, y) \, dx \, dy.
\]
(2.13)
Using (2.7), (2.9), (2.10) and (2.13), we obtain
\[
\langle L(u) - L(v), u - v \rangle \\
= \langle L(u), u \rangle - \langle L(u), v \rangle + \langle L(v), v \rangle - \langle L(v), u \rangle \\
= M(\|u\|_{W^{1,p}(\mathbb{R}^N)}) \left[ \int_\mathbb{R}^2 \left| u(x) - u(y) \right|^p K(x, y) \, dx \, dy + \int_{\mathbb{R}^N} V(x) |u|^p \, dx \right] \\
- M(\|u\|_{W^{1,p}(\mathbb{R}^N)}) \left[ \int_\mathbb{R}^2 \left| u(x) - u(y) \right|^{p-2} |v(x)| \, dx \, dy \right]
\]
\[-u(y)(v(x) - v(y))K(x, y)dx\,dy\]
\[-\int_{\mathbb{R}^N} V(x)|u|^{p-2}uv\,dx\]
\[+ M(\|v\|^p_{W(\mathbb{R}^N)}) \left[ \int_{\mathbb{R}^2N} |v(x) - v(y)|^p K(x, y)dx\,dy + \int_{\mathbb{R}^N} V(x)|v|^p\,dx \right]\]
\[-M(\|v\|^p_{W(\mathbb{R}^N)}) \left[ \int_{\mathbb{R}^2N} |v(x) - v(y)|^{p-2}(v(x) - v(y))dx\,dy \right]\]
\[+ \int_{\mathbb{R}^N} V(x)|v|^{p-2}uv\,dx\]
\[\geq \frac{1}{2} M(\|u\|^p_{W(\mathbb{R}^N)}) \left[ \int_{\mathbb{R}^2N} |u(x) - u(y)|^{p-2}(|u(x) - u(y)|^2 \right.
- |v(x) - v(y)|^2)K(x, y)dx\,dy
\[+ \int_{\mathbb{R}^N} V(x)(|u|^{p-2} - |v|^{p-2})(|u|^2 - |v|^2)\,dx \right]\]
\[-\frac{1}{2} M(\|v\|^p_{W(\mathbb{R}^N)}) \left[ \int_{\mathbb{R}^2N} |v(x) - v(y)|^{p-2}(|u(x) - u(y)|^2 \right.
- |v(x) - v(y)|^2)K(x, y)dx\,dy
\[+ \int_{\mathbb{R}^N} V(x)(|v|^{p-2} - |u|^{p-2})(|v|^2 - |u|^2)\,dx \right] \geq 0,
\](2.14)
i.e. \(L\) is a monotone. In fact, \(L\) is strictly monotone. Indeed, if \((L(u) - L(v), u - v) = 0\),
from (2.14), we have
\[
\int_{\mathbb{R}^2N} (|u(x) - u(y)|^{p-2} - |v(x) - v(y)|^{p-2})(|u(x) - u(y)|^2
- |v(x) - v(y)|^2)K(x, y)dx\,dy
\[+ \int_{\mathbb{R}^N} V(x)(|u|^{p-2} - |v|^{p-2})(|u|^2 - |v|^2)\,dx = 0,
\]
so \(|u(x) - u(y)| = |v(x) - v(y)|\) and \(|u| = |v|\). Thus, we get

\[
\langle L(u) - L(v), u - v \rangle = \langle L(u), u - v \rangle - \langle L(v), u - v \rangle
\]

\[
= M(\|u\|_{W^2}^p) \left[ \int_{\mathbb{R}^{2N}} |u(x) - u(y)|^{p-2}(u(x) - u(y)) \right.
\]

\[
- |v(x) - v(y)|^2 K(x, y) \text{d}x \text{d}y
\]

\[
+ \int_{\mathbb{R}^N} V(x)|u|^{p-2}(u - v)^2 \text{d}x = 0,
\]

i.e. \(u \equiv v\), which is contrary with \(u \neq v\). So, \(\langle L(u) - L(v), u - v \rangle > 0\). Then we can say that \(L\) is strictly a monotone operator in \(W(\mathbb{R}^N)\).

(ii) According (i), if \(u_n \rightharpoonup u\) and \(\limsup_{n \to \infty} \langle L(u_n) - L(u), u_n - u \rangle \leq 0\), we have \(\lim_{n \to +\infty} \langle L(u_n) - L(u), u_n - u \rangle = 0\). In view of (2.14), \(u_n(x) - u_n(y) \to u(x) - u(y)\) and \(u_n \to u\) in \(\mathbb{R}^N\), so we get a subsequence satisfying \(u_n \to u\), a.e. \(x, y \in \mathbb{R}^N\) (by Theorem 2.1). Since \(\{u_n\}\) is bounded in \(W(\mathbb{R}^N)\) and by using condition \((M_0)\), for sufficiently large \(n\), we have

\[
M(\|u_n\|_{W^2}^p) \geq a_0\|u_n\|_{W^2}^{p(\beta - 1)} \geq c_2^* > 0,
\]

for some positive constant \(c_2^*\). From Fatou’s lemma,

\[
\liminf_{n \to \infty} \left( \int_{\mathbb{R}^{2N}} |u_n(x) - u_n(y)|^p K(x, y) \text{d}x \text{d}y + \int_{\mathbb{R}^N} V(x)|u_n|^p \text{d}x \right)
\]

\[
\geq \int_{\mathbb{R}^{2N}} |u(x) - u(y)|^p K(x, y) \text{d}x \text{d}y + \int_{\mathbb{R}^N} V(x)|u|^p \text{d}x.
\]

(2.15)

From \(u_n \to u\), we have

\[
\lim_{n \to \infty} \langle L(u_n), u_n - u \rangle = \lim_{n \to \infty} \langle L(u_n) - L(u), u_n - u \rangle.
\]

Using (2.15) and Young’s inequality, we get

\[
\langle L(u_n), u_n - u \rangle
\]

\[
= M(\|u_n\|_{W^2}^p) \left[ \int_{\mathbb{R}^{2N}} |u_n(x) - u_n(y)|^p K(x, y) \text{d}x \text{d}y + \int_{\mathbb{R}^N} V(x)|u_n|^p \text{d}x \right]
\]

\[
- M(\|u_n\|_{W^2}^p) \left[ \int_{\mathbb{R}^{2N}} |u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(u(x)
\]
\[-u(y)K(x, y)dx\,dy - \int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n\,dx\]

\[\geq M(\|u_n\|_{W(\mathbb{R}^N)^p}) \left[ \iint_{\mathbb{R}^{2N}} |u_n(x) - u_n(y)|^p K(x, y)\,dx\,dy \right.\]

\[\left. - \iint_{\mathbb{R}^{2N}} |u_n(x) - u_n(y)|^{p-1}|u(x) - u(y)|K(x, y)\,dx\,dy \right]\]

\[+ M(\|u_n\|_{W(\mathbb{R}^N)^p}) \left( \int_{\mathbb{R}^N} V(x)|u_n|^p\,dx - \int_{\mathbb{R}^N} V(x)|u_n|^{p-1}\,dx \right)\]

\[\geq M(\|u_n\|_{W(\mathbb{R}^N)^p}) \left[ \iint_{\mathbb{R}^{2N}} |u_n(x) - u_n(y)|^p K(x, y)\,dx\,dy \right.\]

\[\left. - \iint_{\mathbb{R}^{2N}} \left( \frac{p-1}{p} |u_n(x) - u_n(y)|^p + \frac{1}{p} |u(x) - u(y)|^p \right) K(x, y)\,dx\,dy \right]\]

\[+ M(\|u_n\|_{W(\mathbb{R}^N)^p}) \left( \int_{\mathbb{R}^N} V(x)|u_n|^p\,dx - \int_{\mathbb{R}^N} V(x)\left( \frac{p-1}{p} |u_n(x)|^p + \frac{1}{p} |u(x)|^p \right)\,dx \right)\]

\[\geq \frac{c_2^p}{p} \left[ \iint_{\mathbb{R}^{2N}} |u_n(x) - u_n(y)|^p K(x, y)\,dx\,dy + \int_{\mathbb{R}^N} V(x)|u_n|^p\,dx \right.\]

\[\left. - \iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^p K(x, y)\,dx\,dy - \int_{\mathbb{R}^N} V(x)|u|^p\,dx \right]. \tag{2.17}\]

According to (2.16) and (2.17), we get

\[\lim_{n \to +\infty} \iint_{\mathbb{R}^{2N}} |u_n(x) - u_n(y)|^p K(x, y)\,dx\,dy \]

\[= \iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^p K(x, y)\,dx\,dy, \tag{2.18}\]

and

\[\lim_{n \to +\infty} \int_{\mathbb{R}^N} V(x)|u_n(x)|^p\,dx = \int_{\mathbb{R}^N} V(x)|u(x)|^p\,dx.\]

From (2.18), it follows that the integrals of the functions family \{\|u_n(x) - u_n(y)|^p\} possess absolutely equicontinuity on \(\mathbb{R}^N\) (see [18, Chapter 6, section 3]). Since

\[(|u_n(x) - u_n(y)| - |u(x) - u(y)|)^p \leq C (|u_n(x) - u_n(y)|^p + |u(x) - u(y)|^p),\]
the integrals of the family \(\{(u_n(x) - u_n(y)) - |u(x) - u(y)|^p\}\) are also absolutely equicontinuous on \(\mathbb{R}^N\) and therefore,

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^{2N}} (|u_n(x) - u_n(y)| - |u(x) - u(y)|^p) K(x, y) \, dx \, dy = 0.
\]

Similarly,

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} V(x)|u_n(x) - u(x)|^p \, dx = 0.
\]

Then we have

\[
\lim_{n \to +\infty} \left( \int_{\mathbb{R}^{2N}} (|u_n(x) - u(x)| - |u_n(y) - u(y)|^p) K(x, y) \, dx \, dy + \int_{\mathbb{R}^N} V(x)|u_n(x) - u(x)|^p \, dx \right) = 0.
\]

Therefore, \(u_n \to u\) in \(W(\mathbb{R}^N)\), i.e. \(L\) is an operator of type \((S_+)\).

(iii) It is clear that \(L\) is an injection since \(L\) is a strictly monotone operator in \(W(\mathbb{R}^N)\). Since

\[
\lim_{\|u\|_{W(\mathbb{R}^N)} \to +\infty} \frac{\langle L(u), u \rangle}{\|u\|_{W(\mathbb{R}^N)}^p} = \lim_{\|u\|_{W(\mathbb{R}^N)} \to +\infty} M(\|u\|_{W(\mathbb{R}^N)}^p) B_u(u) \geq a_0 \lim_{\|u\|_{W(\mathbb{R}^N)} \to +\infty} \|u\|_{W(\mathbb{R}^N)}^{p-1} = +\infty,
\]

\(L\) is coercive, and thus \(L\) is a surjection in view of the Minty–Browder theorem (see [29, Theorem 26A]). Therefore, \(L\) has an inverse mapping \(L^{-1} : W^*(\mathbb{R}^N) \to W(\mathbb{R}^N)\).

Therefore, the continuity of \(L^{-1}\) is sufficient to ensure that \(L\) is a homeomorphism. If \(f_n, f \in W^*(\mathbb{R}^N)\) such that \(f_n \to f\) in \(W^*(\mathbb{R}^N)\). Let \(u_n = L^{-1}(f_n)\) and \(u = L^{-1}(f)\), then \(L(u_n) = f_n\) and \(L(u) = f\). So \(\{u_n\}\) is bounded in \(W(\mathbb{R}^N)\). Without loss of generality, we can assume that \(u_n \to u_0\). Since \(f_n \to f\),

\[
\lim_{n \to +\infty} \langle L(u_n) - L(u_0), u_n - u_0 \rangle = \lim_{n \to +\infty} \langle f_n, u_n - u_0 \rangle = 0.
\]

Since \(L\) is of type \((S_+)\), \(u_n \to u_0\), we conclude that \(u_n \to u\) so \(L\) is continuous. Therefore, the functional \(L\) is a homeomorphism. The proof of Lemma 2.2 is completed.

\[\square\]

**Lemma 2.3.** Suppose \((V_0)\), \((M_0)\) and \((f_1)\) hold. Then \(I\) satisfies the Palais–Smale \((PS)\) condition.

**Proof.** Let us assume that there exists a sequence \(\{u_n\}\) in \(W(\mathbb{R}^N)\) such that

\[
I(u_n) \to c \in \mathbb{R} \text{ and } I'(u_n) \to 0 \text{ in } W^*(\mathbb{R}^N) \text{ as } n \to \infty.
\]

(2.19)
From (2.19), we have \( |I(u_n)| \leq c \). Combining this fact with (2.6) implies that

\[
c \geq I(u_n) \geq \frac{a_0}{p^\beta} \|u_n\|_{W(R^N)}^{\beta p} - \sum_{i=1}^{m} \frac{C_b C_{\gamma_i}}{\gamma_i} \|u_n\|_{W(R^N)}^{\gamma_i} \geq c_1^*
\]

for any \( c_1^* > 0 \). Because \( 1 < \gamma_i \leq p^\beta, i = 1, 2, \ldots, m \), we obtain that \( \{u_n\} \) is bounded in \( W(R^N) \). Going, if necessary, to a subsequence, thanks to Theorem 2.1, we have

\[
\begin{align*}
&u_n \rightharpoonup u \text{ in } W(R^N), \\
u_n \rightarrow u \text{ in } L^{s_i}(R^N), p \leq s_i < p^*_i, \\
u_n(x) \rightarrow u(x) \text{ a.e. in } x \in R^N.
\end{align*}
\]

Now, using \((f_1)\), the Hölder inequality and Lemma 1A, we get

\[
\langle \Psi'(u_n) - \Psi'(u), u_n - u \rangle = \int_{R^N} |(f(x, u_n)) - f(x, u)(u_n - u)| \, dx
\]

\[
\leq \sum_{i=1}^{m} \int_{R^N} b_i(x) (|u_n|^{\gamma_i} - |u|^{\gamma_i}) |u_n - u| \, dx
\]

\[
\leq \sum_{i=1}^{m} \|b_i\|_{\gamma_i, R^N} (\|u_n\|_{\gamma_i, R^N}^{\gamma_i-1} + \|u\|_{\gamma_i, R^N}^{\gamma_i-1}) \|u_n - u\|_{\gamma_i, R^N}
\]

\[
\leq \sum_{i=1}^{m} C_b C_{\gamma_i} (\|u_n\|_{W(R^N)}^{\gamma_i-1} + \|u\|_{W(R^N)}^{\gamma_i-1}) \|u_n - u\|_{\gamma_i, R^N}
\]

for any \( C_b, C_{\gamma_i} > 0, i = 1, 2, \ldots, m \). Since \( \{u_n\} \) converges strongly to \( u (u_n \rightharpoonup u) \) in \( L^{s_i}(R^N), i = 1, 2, \ldots, m \), that is, \( \|u_n - u\|_{s_i, R^N} \rightarrow 0 \) as \( n \rightarrow \infty \), we get

\[
\langle \Psi'(u_n) - \Psi'(u), u_n - u \rangle = \int_{R^N} |(f(x, u_n)) - f(x, u)(u_n - u)| \, dx \rightarrow 0.
\]

(2.20)

From Lemma 2.2, we have known that \( J' \) is of \((S_+)\) type. By (2.20), we have \( I' \) is of \((S_+)\) type. Now,

\[
\langle I'(u_n) - I'(u), u_n - u \rangle \rightarrow 0
\]

as \( n \rightarrow \infty \). Thus

\[
o(1) = \langle I'(u_n) - I'(u), u_n - u \rangle = \langle J'(u_n) - J'(u), u_n - u \rangle - \langle \Psi'(u_n) - \Psi'(u), u_n - u \rangle.
\]

Since \( I' \) is an operator of type \((S_+)\), we conclude that \( u_n \rightharpoonup u \) in \( W(R^N) \), therefore \( I \) satisfies the Palais–Smale (PS) condition. Thus, we have \( \|u_n - u\|_{W(R^N)} \rightarrow 0 \) as \( n \rightarrow \infty \). The proof of Lemma 2.3 is completed.  \( \square \)
Let $X$ be a separable and reflexive Banach space, then there exist $\{e_n\}_n \subset X$ and $\{e_n^*\}_n \subset X^*$ such that
\[
\langle e_n^*, e_m \rangle = \delta_{n,m} = \begin{cases} 
1 & \text{if } n = m, \\
0 & \text{if } n \neq m,
\end{cases}
\]
and
\[
X = \text{span}\{e_n; 1, 2, \ldots\}, \quad X^* = \text{span}\{e_n^*; 1, 2, \ldots\}.
\]

We use Krasnoselskii’s genus theory (see [5,12]) to get the proof of our main results. So, we recall some basic notations of Krasnoselskii’s genus.

Set
\[
\mathcal{R} = \{A \subset X \setminus \{0\} : A \text{ is compact and } A = -A\}.
\]

**DEFINITION 2.1**

Let $A \subset \mathcal{R}$ and $X = \mathbb{R}^k$. The genus $\eta(A)$ of $A$ is defined by
\[
\eta(A) = \min\{k \geq 1 : \text{there exists an odd continuous mapping } \phi : A \to \mathbb{R}^k \setminus \{0\}\}.
\]

If such a mapping does not exist for any $k > 0$, we set $\eta(A) = +\infty$. Moreover, from definition, $\eta(\emptyset) = 0$. A typical example of a set of genus $k$ is a set, which is homeomorphic to a $(k-1)$-dimensional sphere via an odd map.

**Proof of Theorem 2.2.** Set (see [5,8])
\[
\mathcal{R}_k = \{A \subset \mathcal{R} : \eta(A) \geq k, k \in \mathbb{N}\}
\]
and
\[
c_k = \inf_{A \subset \mathcal{R}_k} \sup_{u \in A} I(u), \quad k = 1, 2, \ldots,
\]
we have
\[
-\infty < c_1 \leq c_2 \leq \cdots \leq c_k \leq c_{k+1} \leq \cdots.
\]

Now, we will show that $c_k < 0$ for every $k \in \mathbb{N}$. For each $k$, we take $k$ disjoint open sets disjoint open sets $K_i$ such that $\bigcup_{i=1}^k K_i \subset G$ (see, for example, [6,19]). For $i = 1, 2, \ldots, k$, let $u_i \in (W(\mathbb{R}^N) \cap C_0^\infty(K_i)) \setminus \{0\}$, $\|u_i\|_{W(\mathbb{R}^N)} = 1$ and
\[
E_k = \text{span}\{u_1, u_2, \ldots, u_k\}, \quad S_k = \{u \in E_k : \|u\|_{W(\mathbb{R}^N)} = 1\}.
\]
For any $u \in E_k$, there exists $\mu_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$ such that
\[
u(x) = \sum_{i=1}^k \mu_i u_i(x) \text{ for } x \in \mathbb{R}^N.
\] (2.21)
Then

\[
\|u\|_{\gamma_0,\mathbb{R}^N} = \left(\int_{\mathbb{R}^N} |u(x)|^{\gamma_0} \, dx \right)^{1/\gamma_0} = \left(\sum_{i=1}^{k} |\mu_i|^{\gamma_0} \int_{\mathbb{R}^N} |u_i(x)|^{\gamma_0} \, dx \right)^{1/\gamma_0} \tag{2.22}
\]

and

\[
\|u\|_{W_p(\mathbb{R}^N)}^p = \sum_{i=1}^{k} |\mu_i|^p \left(\int_{\mathbb{R}^N} \left|u_i(x) - u_i(y)\right|^p K(x, y) \, dx \, dy + \int_{\mathbb{R}^N} V(x)|u_i| \, dx\right)
\]

\[
= \sum_{i=1}^{k} |\mu_i|^p \|u_i\|_{W_p(\mathbb{R}^N)}^p = \sum_{i=1}^{k} |\mu_i|^p. \tag{2.23}
\]

As all norms of a finite dimensional normed space are equivalent, there is a constant \(C_1 > 0\) such that

\[
C_1 \|u\|_{W_p(\mathbb{R}^N)} \leq \|u\|_{\gamma_0,\mathbb{R}^N} \text{ for all } u \in E_k. \tag{2.24}
\]

By \((f_2)\) \((2.22)\), \((2.23)\) and \((2.24)\), we have

\[
I(tu) = \frac{1}{p} \mathcal{M}(\|tu\|_{W_p(\mathbb{R}^N)}^p) - \int_{\mathbb{R}^N} F(x, tu) \, dx
\]

\[
\leq \frac{a_1}{p\alpha} t^{p\alpha} - \sum_{i=1}^{k} \int_{K_i} F(x, tu_i(u(x))) \, dx
\]

\[
\leq \frac{a_1}{p\alpha} t^{p\alpha} - \gamma_0 b_0 t^{\gamma_0} \sum_{i=1}^{k} |\mu_i|^{\gamma_0}
\]

\[
\int_{K_i} |u_i|^{\gamma_0} \, dx = \frac{a_1}{p\alpha} t^{p\alpha} - \gamma_0 b_0 t^{\gamma_0} \|u\|_{\gamma_0,\mathbb{R}^N}^{\gamma_0}
\]

\[
\leq \frac{a_1}{p\alpha} t^{p\alpha} - \gamma_0 b_0 C_1^{-\gamma_0} t^{\gamma_0},
\]

for all \(u \in S_k\) and \(0 < t \leq \delta, \delta\) be given in \((f_2)\). Since \(1 < \gamma_0 < p < p\alpha\), we can find \(t_0 = t(k) \in (0, 1)\) and \(\varepsilon_0 = \varepsilon(k) > 0\) such that

\[
I(t_0u) \leq -\varepsilon_0 < 0 \text{ for all } u \in S_k,
\]

that is,

\[
I(u) \leq -\varepsilon_0 < 0 \text{ for all } u \in S_0^k = \{t_0u : u \in S_k\}, k \in \mathbb{N}
\]
which, together with the fact that $I \in C^1(W(\mathbb{R}^N), \mathbb{R})$ and is even, implies that $S_k^{t_0} \in \mathcal{R}$. On the other hand, it follows from (2.21) and (2.23) that

$$S_k^{t_0} = \left\{ \sum_{i=1}^{k} \mu_i u_i : \sum_{i=1}^{k} |\mu_i|^p = t_0^p \right\}.$$

So we define a map $\Upsilon : S_k^{t_0} \to \partial \Sigma$ as follows:

$$\Upsilon (u) = (\mu_1, \mu_2, \ldots, \mu_k), \quad \forall u \in S_k^{t_0},$$

where

$$\Sigma = \left\{ (\mu_1, \mu_2, \ldots, \mu_k) \in \mathbb{R}^k : \sum_{i=1}^{k} |\mu_i|^p < t_0^p \right\}.$$

It is easy to verify that $\Upsilon : S_k^{t_0} \to \partial \Sigma$ is an odd homeomorphic map. By Proposition 7.7 in [24], $\eta(S_k^{t_0}) = k$ and so $-\infty < c_k \leq -\varepsilon_0 < 0$, that is, for any $k \in \mathbb{N}$, $c_k$ is a real negative number. Since $k$ is arbitrary, we obtain infinitely many critical points of $I$, and hence there is a sequence of solutions $\{\pm u_k : k = 1, 2, \ldots\}$ of problem (P) such that $I(\pm u_k) = c_k < 0$. The proof of Theorem 2.2 is completed. \hfill $\Box$

**Acknowledgement**

The author would like to thank the referees for their helpful suggestions.

**References**


COMMUNICATING EDITOR: Parameswaran Sankaran