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The convolution equation $\sigma * \mu = \mu$ on non-compact non-abelian semigroups

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Abstract. In probability theory, often in connection with problems on weak convergence, and also in other contexts, convolution equations of the form $\sigma * \mu = \mu$ come up. Many years ago, Choqet and Deny (*C. R. Acad. Sci. Paris* **250** (1960) 799–801) studied these equations in locally compact abelian groups. Later, Szekely and Zeng (*J. Theoret. Probab*. **3(2)** (1990) 361–365) studied these equations in abelian semigroups. Like in [2], the results in [7] are also complete. Thus, these equations are studied here for the first time on non-compact non-abelian semigroups. Our main results are Theorems 3.1 and 3.3 in section 3. They are new results as far as we know, and also the best possible under a minor condition. All semigroups in this paper are, unless otherwise mentioned, locally compact Hausdorff second countable topological semigroups. Theorems 3.1 and 3.3 hold for these semigroups. Local compactness may not be necessary when all measures appearing in this context are assumed to be regular.

Keywords. Convolution equation; abelian semigroups; completely simple semigroups; weak convergence; topological semigroups.

Mathematics Subject Classification. 60B99.

1. Introduction

In this paper, we study the classical Choquet–Deny convolution equation that occurs in probability theory in many different contexts, see [2,4,7]. This area is already quite old. Thus we must address the natural question: how does this paper enrich this area? The author answers the question as follows.

We study in this paper, for the first time, the one-sided Choquet–Deny convolution equation given by

$$\sigma * \mu = \mu, \tag{1.1}$$

where σ is a given probability measure on a locally compact Hausdorff second countable topological semigroup T. Let S be the closed semigroup generated by $S(\sigma)$, the support of σ, i.e.,

$$S = \text{cl}\left[\bigcup\{S(\sigma^n) : 1 \leq n < \infty\}\right].$$

Here σ^n is the n-th convolution power of σ. Also, for α and β, any two probability measures in $\mathcal{P}(S)$, the set of all regular probability measures on the Borel subsets of S, $\alpha * \beta(B)$ is given by

$$\alpha * \beta(B) = \int \alpha(Bx^{-1})\beta(\mathrm{d}x), \tag{1.2}$$

where $Bx^{-1} = \{y \in S : yx \in B\}$, and (1.2) holds for all Borel sets $B \subset S$. It is well-known that

$$S(\alpha * \beta) = \mathrm{cl}\,[S(\alpha)S(\beta)]\,,$$

where the set product $AB = \{ab \mid a \in A, b \in B\}$. Thus, $S(\sigma^n) = \mathrm{cl}\,[S(\sigma)]^n$.

Going back to (1.1), we see that σ is the given probability measure on the semigroup T or S, and μ is the unknown measure in $\mathcal{P}(S)$ satisfying equation (1.1). The problem is: whether such a μ exists, and when it does, what can we say about it?
The main reason for considering this problem in general (meaning not necessarily abelian or compact) semigroups is that the problem was solved in (discrete abelian) semigroups in the following way.

When S is abelian, μ satisfies (1.1) iff $\mu_x = \mu$ for each x in $S(\sigma)$, where the probability measure μ_x is defined by $\mu_x(B) = \mu(Bx^{-1})$, $B \subset S$, B Borel.

This nice result was proven by Szekely and Zeng in [7]. They used effectively probability limit laws and needed abelian property to do this. Thus, the Szekely–Zeng paper raised the obvious question: What happens in the non-abelian case? After some consideration, it became clear that methods as well as results will need to be new.

Finally, let us mention that our main results in this paper are Theorems 3.1 and 3.3 in section 3. Sufficient background information is given in section 2 for the ease of following our proofs in section 3.

Our main results roughly say the following: When the sequence (σ^n) is tight (that is, given $\varepsilon > 0$, there is a compact subset $K \subset S$ such that for each $n \geq 1$, $\sigma^n(K) > 1 - \varepsilon$), every solution μ of equation (1.1) is idempotent, and one of these solutions is the weak limit of

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n} \sigma^k,$$

as $n \to \infty$. For details, see section 3. The tightness condition above is also necessary when there is a positive integer n such that

$$\sigma^n(S(\mu)) > 0$$

for some positive integer n. When S is a semigroup of $d \times d$ $(d > 1)$ nonnegative matrices with no zero rows or zero columns, then we do not need the tightness condition for (σ^n).

2. Background information for section 3

Let T be a locally compact Hausdorff second countable topological semigroup. Let $\mathcal{P}(T)$ be the set of all regular probability measures defined on the class of all Borel subsets of T. Let σ be a given probability measure in $\mathcal{P}(T)$. Let S be the closed subsemigroup of T generated by $S(\sigma)$, the support of σ, so that

$$S = \mathrm{cl}\left\{\bigcup_{n=1}^{\infty}[S(\sigma)]^n\right\}.$$

Let μ be in $\mathcal{P}(T)$ satisfying the convolution equation

$$\sigma * \mu = \mu \tag{2.1}$$

where $\sigma * \mu$ is defined by

$$\sigma * \mu(B) = \int \sigma(Bx^{-1})\mu(\mathrm{d}x) \tag{2.2}$$

$$= \int \mu(x^{-1}B)\sigma(\mathrm{d}x), \tag{2.3}$$

where $Bx^{-1} = \{y \in T : yx \in B\}$ and $x^{-1}B = \{y \in T : xy \in B\}$; and B is any Borel subset of T.

The one-sided convolution equation (2.1) is the main subject of study in this paper and section 3. The problem is to find μ, given σ. The corresponding abelian problem (that is, when the semigroup T is abelian) was studied in [7] by Szekely and Zeng. They showed that when T is abelian, μ satisfies (2.1) iff

$$\mu(Bx^{-1}) = \mu(B)$$

for each x in $S(\sigma)$ and any Borel set $B \subset T$. Their method does not work in the non-abelian case. In this paper, we use the theory of idempotent probability measures on semigroups to tackle equation (2.1). This is discussed in detail in Chapter 2 of [4] and [5].

Let $\beta \in \mathcal{P}(S)$. If $\beta = \beta * \beta$, β is called an idempotent probability measure. Support of such a probability measure is known to be a completely simple semigroup. That is, it is a semigroup that is simple (no proper ideals) and has a primitive idempotent. Also, the semigroup S is completely simple iff it is topologically isomorphic to a product semigroup $X \times G \times Y$, where $X = \mathscr{I}(Se)$, $G = eSe$, $Y = \mathscr{I}(eS)$, e is any fixed idempotent element of S, $YX \subset G$, and the multiplication is given by

$$(x_1, g_1, y_1)(x_2, g_2, y_2) = (x_1, g_1(y_1x_2)g_2, y_2).$$

In the above, $\mathscr{I}(A)$ represents the set of all idempotents in A. Let us remark that when S is completely simple, then X is a left zero semigroup and Y is a right zero semigroup.

Finally, $\beta \in \mathcal{P}(S)$ and $\beta = \beta * \beta$ iff we can write β in the form

$$\beta = \beta_1 * \beta_2 * \beta_3,$$

where $\beta_1 \in \mathcal{P}(X)$, β_2 is the Haar probability measure on G, which is always a compact subgroup and $\beta_3 \in \mathcal{P}(Y)$.

Finally, it might be useful to mention that for $\beta \in \mathcal{P}(S)$ and $\eta \in \mathcal{P}(S)$,

$$S(\beta * \eta) = \mathrm{cl}\big(S(\beta) \cdot S(\eta)\big),$$

so that $S(\sigma^n) = \mathrm{cl}\big([S(\sigma)]^n\big)$. Also, $\beta \in \mathcal{P}(S)$ is called r^*-invariant when $\beta(Bx^{-1}) = \beta(B)$ for all Borel subsets B of S and each x in $S(\beta)$. If β is r^*-invariant, then β is idempotent, since

$$\beta * \beta(B) = \int \beta(Bx^{-1})\beta(\mathrm{d}x)$$

$$= \beta(B), \quad B \subset S.$$

Thus, if we define $\beta_x(B) = \beta(Bx^{-1})$ for all $B \subset S$, then for an r^*-invariant measure in $\mathcal{P}(S)$, β is idempotent and $S(\beta)$ is a completely simple semigroup of the form $X \times G \times Y$ (as mentioned earlier) and also, for each x in $S(\beta)$

$$S(\beta) = S(\beta_x) = \mathrm{cl}\big(S(\beta)x\big).$$

It follows that $\beta \in \mathcal{P}(S)$ iff $S(\beta) = H_1 \times H_2$, where H_1 is a left zero semigroup and H_2 is a compact group, and the multiplication is given by

$$(a, b)(c, d) = (a, bd),$$

and also, $\beta = \beta_1 \times \beta_2$, $\beta_1 \in \mathcal{P}(H_1)$ and $\beta_2 \in \mathcal{P}(H_2)$, β_2 is the Haar measure on H_2.

3. Main results

Let T be a locally compact Hausdorff topological semigroup. Let us also assume that T is second countable. Though often the topological assumptions above will not be necessary, we assume them here for technical simplicity and ease of writing.

Let $\mathcal{P}(T)$ represent all (regular) probability measures on the Borel subsets of T. Let $\sigma \in \mathcal{P}(T)$. Let S be the closed semigroup generated by $S(\sigma)$, the support of σ. In other words,

$$S = \mathrm{cl}\left\{\bigcup_{n=1}^{\infty}[S(\sigma)]^n\right\}. \tag{3.1}$$

Assume that $\mu \in \mathcal{P}(S)$ and μ satisfies the one-sided convolution equation

$$\mu = \sigma * \mu. \tag{3.2}$$

As mentioned earlier, this equation is well-known and studied by different authors specially in the two-sided case given by

$$\mu = \sigma * \mu = \mu * \sigma \tag{3.3}$$

for groups; and, for abelian semigroups by Szekely and Zeng [7]. Our main theorem below considers the non-abelian semigroup case and equation (3.1).

Theorem 3.1. *Let μ, σ and S be as in equations* (3.1) *and* (3.2). *Let us assume that the sequence (σ^n) is tight. Then the following results hold*:

(*a*) *The weak limit σ_0 given by*

$$\sigma_0 = (w) \lim_{n\to\infty} \sigma_n, \quad \sigma_n = \frac{1}{n}\sum_{k=1}^{n}\sigma^k \tag{3.4}$$

exists; *moreover*, $\sigma_0 * \sigma = \sigma * \sigma_0 = \sigma_0$, $\sigma_0 \in \mathcal{P}(S)$, $\sigma_0 * \mu = \mu$, $S(\mu) \subset S(\sigma_0)$, *and* $\sigma_0 = (\sigma_0)^2$.

(*b*) *Like σ_0, μ is also idempotent. Consequently, their supports are both completely simple subsemigroups of S. Their standard product representations are given by*

$$S(\sigma_0) = X \times G \times Y, \quad S(\mu) = X \times G \times B, \tag{3.5}$$

where $B \subset Y$. Finally, the product representations of σ_0 and μ are given by

$$\sigma_0 = \alpha * \beta * \gamma, \quad \mu = \alpha * \beta * \gamma_0, \tag{3.6}$$

where $\alpha \in \mathcal{P}(X)$, $\beta \in \mathcal{P}(G)$, $\gamma \in \mathcal{P}(Y)$, $\gamma_0 \in \mathcal{P}(B)$. See section 2 for details.

(c) *For any x in $S(\mu)$, any s in S, and any Borel subset $B \subset S$, we have*

$$\mu(Bx^{-1}) = \mu(Bx^{-1}s^{-1}). \tag{3.7}$$

(d) *When S is right cancellative, then for each s in S, $\mu = \mu_s$ and both are right invariant (that is, $\mu(B) = \mu(Bx^{-1})$) for each x in $S(\mu)$ and each Borel $B \subset S$, and the same is true for μ_s; also, their common support is the left group $X \times G \times \{b\}$, where b is a single point in B; in this case, $Y = B = \{b\}$, and $\sigma_0 = \mu = \mu_s$ for any s in S. Thus, when S is right cancellative,*

$$\mu = \sigma * \mu \text{ implies } \mu = \mu * \sigma.$$

Finally, when S is abelian or cancellative, $\mu = \mu_s = \sigma_0$ for each $s \in S$, and each is the Haar probability measure on their common support G, a compact subgroup of S.

Proof.

(a) For this proof, we need the tightness of the convolution sequence (σ^n). This means that given $\varepsilon > 0$, there is a compact subset $K \subset S$ such that for each $n \geq 1$,

$$\sigma^n(K) > 1 - \varepsilon.$$

In other words, every weak*-limit point of (σ^n) is a probability measure. This also means that the sequence (σ_n) is tight (see (3.4)). Let β_1 and β_2 be any two weak*-limit points of (σ_n). Notice that for $n \geq 1$, $\sigma_n * \sigma = \sigma * \sigma_n = \frac{1}{2}[\sigma^2 + \sigma^3 + \cdots + \sigma^{n+1}]$, and thus

$$\lim_{n\to\infty} \|\sigma_n * \sigma - \sigma_n\| = 0. \tag{3.8}$$

This means that for $n \geq 1$, $\sigma^n * \beta_1 = \beta_1 * \sigma^n = \beta_1$; consequently, for $n \geq 1$, $\sigma_n * \beta_1 = \beta_1 * \sigma_n = \beta_1$ and similarly, $\sigma_n * \beta_2 = \beta_2 * \sigma_n = \beta_2$. Using (3.8), it follows easily that

$$\beta_1 * \beta_2 = \beta_2 * \beta_1 = \beta_1 = \beta_2.$$

It is also clear from above and (3.8) that the weak limit of (σ_n) exists and if we call this limit σ_0, then

$$\sigma_0 = \sigma_0 * \sigma_0 = \underbrace{\sigma_0 * \sigma}_{3.8} = \sigma * \sigma_0.$$

Using equation (3.1), we have, for $n \geq 1$,

$$\sigma_n * \mu = \mu.$$

Consequently,

$$\sigma_0 * \mu = \mu. \tag{3.9}$$

From (3.8), we have $S(\sigma_0)S(\sigma) \subset S(\sigma_0)$ and also, $S(\sigma)S(\sigma_0) \subset S(\sigma_0)$. Thus, $S(\sigma_0)$ is an ideal of S, and since $\sigma_0 = \sigma_0^2$, $S(\sigma_0)$ is the closed minimal ideal of S. It follows from (3.9) that

$$S(\mu) = \mathrm{cl}\big(S(\sigma_0)S(\mu)\big) \subset \mathrm{cl}\big(S(\sigma_0)S\big) \subset S(\sigma_0)$$

This completes the proof of (a).
(b) From (a), we know that $S(\mu) \subset S(\sigma_0)$, $\sigma_0 * \mu = \mu$. Since $\sigma_0 = \sigma_0^2$, $S(\sigma_0)$ can be represented as $S(\sigma_0) = X \times G \times Y$ (see section 2 for details about X, a left zero semigroup, G a compact group, and Y a right zero semigroup, and $YX \subset G$). From (a), we know that

$$S(\sigma_0)S(\mu) \subset S(\mu) \subset S(\sigma_0);$$

thus, since $S(\sigma_0) = X \times G \times Y$, we must have $S(\mu) = X \times G \times B$ for some subset $B \subset Y$.

Let $z \in S(\mu)$. We claim that

$$\mu * \delta_z = \sigma_0 * \delta_z \tag{3.10}$$

Let us write: $z = xgb$, where $x \in X$, $g \in G$, $b \in B$. Then we have

$$\begin{aligned}\mu * \delta_z &= \sigma_0 * \mu * (\delta_x * \delta_g * \delta_b)\\ &= \alpha * \beta * \gamma * \mu * \delta_{xg} * \delta_b\\ &= \alpha * [\beta * (\gamma * \mu * \delta_{xg})] * \delta_b,\end{aligned}$$

where $\sigma_0 = \sigma_0^2 = \alpha * \beta * \gamma$, $\alpha \in \mathcal{P}(X)$, β is the Haar probability on the compact group G, and $\gamma \in \mathcal{P}(Y)$ (see section 2). Notice that $S\big(\beta * \gamma * \mu * \delta_{xg}\big]$ is equal to the closure of

$$S(\beta) \cdot \big[S(\gamma)S(\mu)\big]xg = G\big[Y(XGB)xg\big] \subset G,$$

since $YX \subset G$ and G is a compact group. Since $\beta = \beta * \pi$ for any $\pi \in \mathcal{P}(G)$, it is clear that $\mu * \delta_z = \alpha * \beta * \delta_b = \sigma_0 * \delta_z$. Thus, for any Borel set $B \subset S$,

$$\begin{aligned}\mu(B) &= \sigma_0 * \mu(B)\\ &= \int \sigma_0(Bz^{-1})\mu(\mathrm{d}z)\\ &= \int \mu(Bz^{-1})\mu(\mathrm{d}z)\\ &= \mu^2(B).\end{aligned}$$

Thus, μ is also idempotent, and we can represent μ as a product measure on its support $X \times G \times B$. To obtain it, we write again

$$\mu = \sigma_0 * \mu = (\alpha * \beta * \gamma) * \mu. \tag{3.11}$$

Since $\mu = \mu^2$, μ has a product representation of the form

$$\mu = \alpha_0 * \beta * \gamma_0, \tag{3.12}$$

where $\alpha_0 \in \mathcal{P}(X)$, $\gamma_0 \in \mathcal{P}(B)$, and β is the Haar probability measure on G. It follows from (3.11) and (3.12) that

$$\begin{aligned}\mu &= \alpha * \beta * \gamma * \alpha_0 * \beta * \gamma_0\\ &= \alpha * \big[\beta * (\gamma * \alpha_0) * \beta\big] * \gamma_0\\ \text{or } \mu &= \alpha * \beta * \gamma_0 \quad \text{and} \quad \sigma_0 = \alpha * \beta * \gamma.\end{aligned}$$

This completes the proof of (b).

(c) We know that $S(\mu) = X \times G \times B$, where X is a left zero semigroup, $B \subset Y$, and Y is a right zero semigroup, and G is a compact group, and $YX \subset G$. Let $x \in X$, $g \in G$, and $b \in B$; so, the element $z = (x, g, b) \in S(\mu)$ and the element $(x, (bx)^{-1}, b)$ $(= u$, say) also belongs to $S(\mu)$. Note that $uz = z$. Let $s \in S$. Then $sz = (su)z$. Remember that $\sigma * \mu = \mu$, which implies that $S\big(S(\mu)\big) \subset S(\mu)$; this means that $sz \in S(\mu)$. From (b), we know that $\mu = \alpha * \beta * \gamma_0$. Notice that $\gamma_0 \in \mathcal{P}(B)$ and $B \subset Y$. Thus

$$\mu * \delta_z = \alpha * \beta * (\gamma * \delta_z) = \alpha * \beta * \delta_{\{b\}},$$

since $\alpha * \big[\beta * (\gamma * \delta_x) * \delta_g\big] * \delta_b = \alpha * \beta * \delta_b$. Also, $su \in S(\mu)$ and $uz = z = (x, g, b) \in S(\mu)$. Thus, for any $s \in S$, $sz = (su)z \in S(\mu)z \subset X \times G \times \{b\}$. Thus, we can and do assume that $sz = (t, h, b)$, where $t \in X$, $h \in G$ and b is the B co-ordinate of z, so $b \in B$ as before. We now have

$$\begin{aligned}\mu * \delta_{sz} &= \alpha * \big[\beta * (\gamma_0 * \delta_t) * \delta_h\big] * \delta_b \\ &= \alpha * \beta * \delta_b = \mu * \delta_z.\end{aligned}$$

This proves that $\mu(Bz^{-1}) = \mu(Bz^{-1}s^{-1})$ for any $z \in S(\mu)$, $s \in S$ and any Borel set $B \subset S$. The proof of (c) is complete.

(d) Let us assume that S is right cancellative. Then it is easy to verify that the right factor Y of the completely simple semigroup $X \times G \times Y$, which is actually the support of σ_0, must be a single point b, that is, $Y = \{b\}$ for some point b in S. In this case, $S(\mu) = X \times G \times B = X \times G \times \{b\} = S(\sigma_0)$, since $B \subset Y$. Thus, when S is right cancellative, we have

$$\sigma_0 = \mu = \alpha * \beta * \delta_b.$$

Let $a \in X$, then $ba \in G$, and the point $u = (a, (ba)^{-1}, b) \in S(\mu)$. Notice that

$$\begin{aligned}\mu * \delta_u &= \alpha * \big[\beta * \delta_{ba} * \delta_{(ba)^{-1}}\big] * \delta_b \\ &= \alpha * \beta * \delta_b = \mu.\end{aligned}$$

By the result in (c), for any $s \in S$, $\mu = \mu * \delta_u$ implies that

$$\begin{aligned}\mu_s &= \mu * \delta_s = (\mu * \delta_u) * \delta_s \\ \text{or} \quad \mu_s &= \mu * \delta_{us}.\end{aligned} \tag{3.13}$$

Notice that in the present case, when S is assumed right cancellative, we have

$$\mu = \sigma_0 = \alpha * \beta * \delta_b.$$

Thus, since $\sigma_0 * \sigma = \sigma * \sigma_0 = \sigma_0$, $S(\sigma_0)$ is an ideal of S. Now that $\mu = \sigma_0$, $S(\mu)$ is also an ideal of S. Thus, since $u \in S(\mu)$, $us \in S(\mu)$. Thus, we can assume that $us = (x, g, b)$ for some $x \in X$ and $g \in G$. Now (3.13) above gives us

$$\mu_s = \mu * \delta_{us} = \alpha * \beta * (\delta_{bx} * \delta_g) * \delta_b = \alpha * \beta * \delta_b = \mu.$$

Thus, we have proven that for any $s \in S$, $\mu_s = \mu$. This means that

$$\mu * \sigma = \mu \tag{3.14}$$

even though we started with the one-sided convolution equation $\sigma * \mu = \mu$.
Lastly, when S is abelian or bicancellative, it is easy to verify that the completely simple semigroups $S(\sigma_0) = X \times G \times Y$ and $S(\mu) = X \times G \times B$ must both be equal to G, the compact subgroup of S. Then as a result, $\sigma_0 = \mu = \beta$ $(= \mu_s$, for any $s \in S)$, where B is the Haar probability measure on G. This completes the proof of (d) and also the proof of Theorem 3.1. □

Let us now introduce a simple condition on σ, where σ and μ are as before in Theorem 3.1 (that is, $\sigma * \mu = \mu$, S is as before, and $\mu \in \mathcal{P}(S)$).

Condition A. There exists a positive integer N such that

$$\sigma^N\big(S(\mu)\big) > 0.$$

Then the following lemma holds.

Lemma 3.2. *Let $\sigma * \mu = \mu$, where S is as in Theorem* 3.1 *and $\mu \in \mathcal{P}(S)$. Then $S(\mu)$ is a semigroup, $S \cdot S(\mu) \subset S(\mu)$, and*

$$\lim_{n\to\infty} \sigma^n\big(S(\mu)\big) = 1,$$

whenever condition (*A*) *holds.*

Proof. It is clear that $S \cdot S(\mu) \subset S(\mu)$, since $\sigma * \mu = \mu$ implies that

$$S(\sigma) \cdot S(\mu) \subset S(\mu).$$

Also, notice that for any $n \geq 1$,

$$\sigma^{n+1}\big(S(\mu)\big) \geq \sigma^n\big(S(\mu)\big)$$

since for any x in $S(\sigma)$,

$$S \cap x^{-1}\big(S(\mu)\big) = \{y \in S : xy \in S(\mu)\} \supset S(\mu),$$

and thus,

$$\sigma^{n+1}\big(S(\mu)\big) = \int \sigma^n\big(x^{-1}S(\mu)\big)\sigma(\mathrm{d}x) \geq \sigma^n\big(S(\mu)\big),$$

and thus, $\sigma^n\big(S(\mu)\big)$ is an increasing sequence.

Now to prove that $\sigma^n\big(S(\mu)\big) \to 1$ as $n \to \infty$, under condition (A), we consider a sequence of i.i.d. random variables with values in $S(\mu)$. Let us call them (X_n) so that for each $n \geq 1$, $\mathcal{P}\big(X_n \in S(\mu)\big) = \sigma^N\big(S(\mu)\big)$ and

$$\sum_{n=1}^{\infty} \mathcal{P}\big(X_n \in S(\mu)\big) = \infty.$$

By the Borel–Cantelli lemma, $\mathcal{P}\big(X_n \text{ i.o. in } S(\mu)\big) = 1$. This means that for a given $\varepsilon > 0$, there exists a positive integer m_0 such that for $m > m_0$,

$$\mathcal{P}\left(\bigcup_{n=m}^{\infty} \{X_n \in S(\mu)\}\right) > 1 - \varepsilon$$

In other words, for some $m \geq m_0$, we have

$$\sigma^{mN}\big(S(\mu)\big) > 1 - \varepsilon.$$

In the above, we used the fact that $S(\mu)$ is a semigroup and that $S\big(S(\mu)\big) \subset S(\mu)$. Thus, it is clear that for some sequence (m_k) of positive integers,

$$\lim_{k\to\infty} \sigma^{m_k}\big(S(\mu)\big) = 1.$$

But we also know from above that $\sigma^n\big(S(\mu)\big)$ is an increasing sequence in n and this finally shows that

$$\lim_{n\to\infty} \sigma^{n}\big(S(\mu)\big) = 1.$$

This completes the proof. □

The next result is the last theorem of this paper. Recall that the results (a), (b), (c) and (d) in Theorem 3.1 were obtained for the convolution equation

$$\sigma * \mu = \mu,$$

assuming that the sequence (σ^n) was tight. Theorem 3.3 below shows that under condition (A), the above tightness condition is necessary for each of the results (a), (b), (c) and (d) in Theorem 3.1.

Theorem 3.3. *Let σ and S be as given in Theorem* 3.1, $\mu \in \mathcal{P}(S)$, *and $\sigma * \mu = \mu$. Assume that condition* (A) *holds. Then the tightness condition for* (σ^n) *is necessary for each of the results in* (*a*), (*b*), (*c*) *and* (*d*) *in Theorem* 3.1.

Proof. Assume that condition (A) holds. Now we assume that (a) in Theorem 3.1 holds. This means that the weak limit $\sigma_0 = \sigma_0^2$, $\sigma_0 * \sigma = \sigma * \sigma_0 = \sigma_0$, $\sigma_0 * \mu = \mu$, and $S(\sigma_0) \supset S(\mu)$ all hold. Using the same notations as in Theorem 3.1, $S(\sigma_0)$ is a completely simple semigroup and $S(\sigma_0) = X \times G \times Y$. By Lemma 3.2, since $S(\mu) \subset S(\sigma_0)$,

$$\lim_{n\to\infty} \sigma^{n}\big(X \times G \times Y\big) = 1.$$

Let $\varepsilon > 0$. Then there exists $p > 1$ such that for $n \geq p$, $\sigma^n\big(X \times G \times Y\big) > 1 - \varepsilon$. Let E and F be compact subsets $E \subset X$, $F \subset Y$, and

$$\sigma^{p}\big(E \times G \times F\big) > 1 - 2\varepsilon.$$

Then it is clear that for $n \geq 3p$, we have

$$\begin{aligned}\sigma^n\big(E \times G \times F\big) &\geq \sigma^p\big(E \times G \times F\big) \cdot \sigma^{n-2p}(X \times G \times Y) \cdot \sigma^p\big(E \times G \times F\big)\\ &\geq (1-2\varepsilon)^2 \cdot (1-\varepsilon)\\ &> 1 - 3\varepsilon.\end{aligned}$$

Note that in the above, we have used the fact that

$$E \times G \times F = (E \times G \times F)(X \times G \times Y)(E \times G \times F) = E \times G \times F.$$

This proves that the sequence (σ^n) is tight.

Similarly as above, we can show that (b) in Theorem 3.1 implies the tightness of (σ^n). We *omit* this proof.

Let us now assume condition (A) and part (c) in Theorem 3.1. By part (c), we have $\sigma * \mu = \mu$, S as before, $\mu \in \mathcal{P}(S)$, and

$$\mu(Bx^{-1}) = \mu(Bx^{-1}s^{-1}) \tag{3.15}$$

for $x \in S(\mu)$ and $s \in S$. By (3.15), we have, for any $x \in S(\mu)$ and any Borel set $B \subset S$,

$$\begin{aligned}\mu(Bx^{-1}) &= \int \mu(Bx^{-1}s^{-1})\mu(\mathrm{d}s)\\ &= \mu^2(Bx^{-1}),\end{aligned}$$

since $S(\mu) \subset S$ and therefore, we can integrate with respect to μ. Thus, integrating again with respect to μ, this time on both sides, we have $\mu^2 = \mu^3$ or $\mu^2 = \mu^3 = \mu^4$. Thus, we get μ^2 idempotent. By Lemma 3.2, using condition (A), we have again

$$\lim_{n\to\infty} \sigma^n\big(S(\mu^2)\big) = 1,$$

since $S(\mu^2) \supset S(\mu)S(\mu)$. Thus, we can use the same proof as used above, and then it is clear that (3.15) implies that (σ^n) is tight.

Finally, when part (d) in Theorem 3.1 holds, then in the case when S is right cancellative, $S(\mu) = X \times G \times \{b\}$; and when S is abelian, $S(\mu) = G$, a compact group. By Lemma 3.2, $\lim_{n\to\infty} \sigma^n\big(S(\mu)\big) = 1$. Thus, the same proof that was used in the above shows that we can find a compact set $K \subset S(\mu)$ for a given $\varepsilon > 0$, such that $\sigma^n(K) > 1 - \varepsilon$, for n sufficiently large. In the abelian case, $K = S(\mu)$. Thus, (σ^n) is tight again when part (d) occurs.

The proof of Theorem 3.3 is complete. □

Remark 3.1. In the theorems proven above in this section, the convolution equation we studied was $\sigma * \mu = \mu$ (equation (3.2)), where $\mu \in \mathcal{P}(S)$, S being the closed semigroup generated by $S(\sigma)$ (see equation (3.1). The condition that $S(\mu) \subset S$ was necessary to conclude in Theorem 3.1(b) was that $\sigma_0 * \mu = \mu$ implies $S(\mu) \subset S(\sigma_0)$. This allowed us to conclude that $S(\mu) = X \times G \times B$, $B \subset Y$, when $S(\sigma_0) = X \times G \times Y$. This need not be true when $S(\mu)$ is not contained in S.

Here is a simple example:

Let S_1 and S_2 be two disjoint finite semigroups. For $x \in S_1$ and $y \in S_2$, we define $xy = yx = y$. Let σ and μ be such that $S(\sigma) = S_1$ and $S(\mu) = S_2$. Then in the semigroup $S_1 \cup S_2$, $\sigma * \mu = \mu * \sigma = \mu$, and $S(\sigma_0)$ is contained in S_1, while $S(\mu) \subset S_2$.

Remark 3.2. Let σ be the unit mass at $\{\frac{1}{2}\}$ and μ be the unit mass at $\{0\}$. Then the multiplicative semigroup $S = \{0\} \cup \{\frac{1}{2^n} : n \geq 1\}$. Notice that σ_0 in Theorem 3.1 is equal to μ. Clearly, $\sigma^n\big(S(\mu)\big) = 0$ for $n \geq 1$. Thus, condition (A) in Lemma 3.2 does not hold, and yet $\mu = \sigma_0$, and (σ^n) is tight.

If instead, we chose

$$\sigma = \frac{1}{2}\delta_{\{0\}} + \frac{1}{2}\delta_{\left\{\frac{1}{2}\right\}} \quad \text{and} \quad \mu\{0\} = 1,$$

then $\sigma * \mu = \mu$, $S = \{0\} \cup \{\frac{1}{2^n} : n \geq 1\}$. Also, condition (A) of Lemma 3.2 clearly holds since $\sigma\big(S(\mu)\big) = \frac{1}{2}$ and therefore, $\sigma^n\big(S(\mu)\big) \to 1$ as $n \to \infty$.

Remark 3.3. The result in part (c) of Theorem 3.1 appeared in [5]. However, the proof in [5] is not adequate (the compactness of S was assumed there, but somehow omitted in the final print there). Here, we have given a complete and different proof when (σ^n) is tight, a condition less restrictive than in [5].

Remark 3.4. Consider the equation $\sigma * \mu = \mu$, the equation (3.1), where σ, μ and S are all as before. However, we now assume that S is a semigroup of $d \times d$ nonnegative matrices, $1 < d < \infty$, containing only matrices with no zero rows or zero columns. Then the sequence (σ^n) must be tight. To prove this, let us first show that for any two compact subsets $A \subset S$ and $B \subset S$, the set $A^{-1}B$, where $A^{-1}B = \{y \in S : xy \in B \text{ for some } x \in A\}$ is compact. Let us first observe that $A^{-1}B$ is a closed set with the usual matrix (or Euclidean) topology in S. Let (y_n) be a sequence in $A^{-1}B$. If we can show that (y_n) is bounded above, that will confirm that $A^{-1}B$ is compact.

Now there exists a sequence (x_n) in A such that $x_n y_n \in B$ for each n.

Let $1 \leq i < d$, $1 \leq j < d$. Then there exists a sequence (n_k) of positive integers such that

$$x_{n_k} y_{n_k} \to z, \quad x_{n_k} \to x,$$

where $x \in A$ and $z \in B$. Since x has no zero column, there exists s, $1 \leq s < d$, such that $(x)_{si} > 0$. Thus, it is clear that there exists $\delta > 0$ such that

$$(x)_{si} > \delta, \quad (x_{n_k})_{si} > \delta,$$

for all sufficiently large k. It follows that for all sufficiently large k,

$$(x_{n_k})_{si}(y_{n_k})_{ij} \leq z_{sj}.$$

It follows that (y_n) is bounded and $A^{-1}B$ is compact. A similar proof also shows that the set $AB^{-1} = \{y \in S : yx \in A \text{ for some } x \in B\}$ is also compact whenever A and B are compact subsets.

Now we go back to equation (3.1) and we have

$$\sigma^n * \mu = \mu, \quad n \geq 1.$$

Thus, for any compact set $K \subset S$,

$$\begin{aligned}\mu(K) &= \sigma^n * \mu(K) \\ &= \int \sigma^n(Kx^{-1})\mu(\mathrm{d}x) \\ &\leq \sigma^n(KK^{-1}) + \varepsilon,\end{aligned}$$

if we choose K such that

$$\mu(K) > 1 - \varepsilon,$$

for all $n \geq 1$, ε being any given positive number. Since KK^{-1} is compact, it follows that (σ^n) is a tight sequence. The remark is justified.

4. An application

Consider the three functions $\{f_1, f_2, f_3\}$ each from the plane $\mathbb{R}^2$ into $\mathbb{R}^2$ given by $f_1(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right)$, $f_2(x, y) = \left(\frac{x+1}{2}, \frac{y}{2}\right)$, and $f_3(x, y) = \left(\frac{x}{2}, \frac{y+1}{2}\right)$. We can also consider them as three 3×3 nonnegative matrices given by

$$F_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, F_2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } F_3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

Let σ be a probability measure on the set $\{f_1, f_2, f_3\}$. Let S be the compact semigroup generated by the sub-stochastic matrices F_1, F_2 and F_3 with matrix multiplication as semigroup operation and usual matrix topology as the topology that converts S into a compact Hausdorff topological semigroup. Thus, S is the closed semigroup generated by $S(\sigma) = \{F_1, F_2, F_3\}$. By our Theorem 3.1, the unique solution μ of the equation $\sigma * \mu = \mu$ is $\mu = \sigma_0$. In fact, σ^n converges weakly to μ. To see this, notice that $\mu = \sigma_0$ and thus, $S(\mu)$ is compact and the smallest closed ideal (or the kernel) of S. Then we can easily show (like in Lemma 3.2) that $\sigma^n(C) \to 1$ as $n \to \infty$ for any open set C containing $S(\mu)$. This means that if λ is a weak*-limit point of the sequence (σ^n), then $S(\lambda) \subset S(\mu)$. Now $S(\mu)$, being the kernel of a semigroup of matrices, must consist of all matrices in S with minimal rank (rank one in this case). Such matrices are of the form

$$\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

This means that $S(\mu)$ must be a left zero semigroup (that is, $AB = A$ for A, B in $S(\mu)$). If λ_1 and λ_2 are any two weak*-limit points of (σ^n), then $S(\lambda_1) \subset S(\mu)$ and $S(\lambda_2) \subset S(\mu)$. Also, $\lambda_1 * \lambda_2 = \lambda_2 * \lambda_1$. Since $S(\mu)$ is a left zero semigroup, it follows that

$$\lambda_1 * \lambda_2 = \lambda_1, \ \lambda_2 * \lambda_1 = \lambda_2.$$

Thus, $\lambda_1 = \lambda_2$. This means that σ^n must converge weakly to μ, since μ must be equal to the idempotent cluster (or limit) point of (σ^n).

Thus, in this example, the equation $\sigma * \mu = \mu$ has a unique solution which is the weak limit of (σ^n).

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