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Symplectic reduction of Sasakian manifolds

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Abstract. When a complex semisimple group G acts holomorphically on a Kähler manifold (X, ω) such that a maximal compact subgroup $K \subset G$ preserves the symplectic form ω, a basic result of symplectic geometry says that the corresponding categorical quotient X/G can be identified with the quotient of the zero-set of the moment map by the action of K. We extend this to the context of a semisimple group acting on a Sasakian manifold.

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1. Introduction

Contact manifolds can be thought of as odd-dimensional analogs of symplectic manifolds. In the same way, Sasakian manifolds can be regarded as odd-dimensional analog of Kähler manifolds. These manifolds were introduced by Sasaki [11–13]. The topic remained dormant for more than thirty years until the following things happened:

(1) In the AdS/CFT correspondence discovered by Maldacena [7], it was realized that Sasakian manifolds play a key role in string theory. Over time, many works in this direction emerged (see [2,9,10] and references therein).
(2) Boyer and Galicki [1] worked systematically and published a series of papers investigating various differential geometric aspects of Sasakian manifolds (see [1] and references therein).

Let (X, ω) be a compact Kähler manifold, and let G be a complex semisimple affine algebraic group acting holomorphically on X such that the action of a maximal compact subgroup $K \subset G$ preserves the Kähler form ω. Let $\mu : X \longrightarrow \text{Lie}(K)^*$ be the moment map for this action. It is known that the categorical quotient X/G is identified with the quotient $\mu^{-1}(0)/K$; the reader is referred to [6] (see also [5,14]).

Here we take a Sasakian manifold (X, g, ξ); let (M, ω_M) be the associated Kähler manifold whose underlying manifold is $X \times \mathbb{R}_+$. Let r denote the standard coordinate on

$\mathbb{R}_+$. Let G be a complex semisimple affine algebraic group acting holomorphically on M such that the action of a maximal compact subgroup $K \subset G$ preserves X. The action of K preserves the contact one-form. We also assume the following:

- ξ is orthogonal to the orbits of the action of K,
- $\partial/\partial r$ is orthogonal, with respect to ω_M, to the distribution on M given by Lie(K), and
- $[\mathfrak{k}, \partial/\partial r] = 0$ (we denote by $\mathfrak{k}$ the distribution on M given by Lie(K).

Using the relationship between the Sasakian manifolds and Kähler manifolds, we prove that $\mu^{-1}(0)/K$ is a Sasakian manifold, where μ as before is the moment map, such that the categorical quotient M/G is the Kähler manifold associated to it (Theorem 12).

A similar result was proved earlier in [3]. But the methods employed here differ from that of [3].

2. Sasakian manifolds

We denote by (X, g) a connected, oriented Riemannian manifold equipped with the corresponding Levi–Civita connection ∇. It is called *Sasakian* if the metric cone

$$(X \times \mathbb{R}_+, dr^2 \oplus r^2 g)$$

is *Kähler*. We denote by J the complex structure, and

$$\xi = J\left(\frac{\partial}{\partial r}\right)\Big|_{X\times\{1\}} \tag{1}$$

is called the *Reeb* vector field, where $X \times \{1\}$ is identified with X.

The computation of the Nijenhuis torsion tensor in terms of the Reeb vector field leads to the following well-known characterization of a Sasakian manifold.

Theorem 1 ([1, Definition and Theorem 10]). *The following conditions for a Riemannian manifold (X, g) are equivalent*:

(*i*) *There is a Killing vector field ξ on X of unit length such that the section*

$$\Phi \in C^\infty(X, TX \otimes (TX)^*) \tag{2}$$

defined by $v \longmapsto -\nabla_v \xi$, $v \in TX$, satisfies the following identity for the Lie derivative of Φ, which is defined by $(\nabla_v \Phi)(w) = \nabla_v(\Phi(w)) - \Phi(\nabla_v(w))$:

$$(\nabla_v \Phi)(w) = g(v, w)\xi - g(\xi, w)v \tag{3}$$

for all $v, w \in T_x X$ and all $x \in X$.

(*ii*) *There is a Killing vector field ξ on X of unit length such that the Riemann curvature tensor R of (X, g) satisfies the identity*

$$R(v, \xi)w = g(\xi, w)v - g(v, w)\xi$$

for all v and w as above.

(*iii*) *The metric cone $(X \times \mathbb{R}_+, dr^2 \oplus r^2 g)$ is Kähler.*

We will point out some facts regarding the above equivalent conditions.

Given a Killing vector field ξ of unit length satisfying condition (i), the Kähler structure on $\mathbb{R}_+ \times X$ asserted in statement (iii) is constructed as follows. Let F be the distribution of X of rank $2n$ given by the orthogonal complement of ξ. The homomorphism Φ (defined in (2)) preserves the above defined distribution F on X, since $g(\Phi(v), \xi) = \frac{1}{2} v(g(\xi, \xi)) = 0$, and furthermore,

$$(\Phi|_F)^2 = -\mathrm{Id}_F\,. \tag{4}$$

Then an almost complex structure J on $\mathbb{R}_+ \times X$ defined by the following conditions:

$$J|_F = \Phi|_F \tag{5}$$

satisfying (1), and the corresponding equation

$$J(\xi) = -\frac{d}{dr}\,. \tag{6}$$

The almost complex structure J is in fact the complex structure stated in (iii). Condition (3) is equivalent to the vanishing of the Nijenhuis tensor, and the Riemannian metric $dr^2 \oplus r^2 g$ on $\mathbb{R}_+ \times X$ is Kähler with respect to J.

Conversely, if the metric cone $(X \times \mathbb{R}_+,\ dr^2 \oplus r^2 g)$ is Kähler, then consider the vector field ξ given by (1), where J is the almost complex structure on $X \times \mathbb{R}_+$. The vector field ξ defined this way satisfies condition (i), which is known to be equivalent to (ii) in Definition 1.

In this sense, the vector field ξ (or equivalently, the Kähler structure on $X \times \mathbb{R}_+$) can and will be considered as part of the definition of a Sasakian manifold to be denoted by (X, g, ξ).

Let X be a smooth oriented Riemannian manifold of dimension $2n + 1$ and $F \subset TX$ an oriented smooth distribution of rank $2n$. The quotient map

$$TX \longrightarrow TX/F =: N$$

defines a smooth one-form on X,

$$\omega \in C^\infty(X,\ T^*X \otimes N) \tag{7}$$

with values in the line bundle N. Since X is oriented, the orientation of F induces an orientation of the normal bundle N. Therefore, N has a canonical smooth section given by the positively oriented vectors of unit length in the fibers of N. Consequently, the form ω in (7) gives a nowhere vanishing smooth one-form on X. This one-form will also be denoted by ω. The distribution F is said to be a *contact structure* on X if the $(2n+1)$-form $(d\omega)^n \wedge \omega$ is nowhere vanishing (see [1] and references therein.)

Remark 2. The distribution F is integrable, if it satisfies the Frobenius condition which says that the one-forms ω satisfy the condition $(d\omega)\wedge\omega = 0$. Therefore, a contact structure F is not integrable.

Now let $(X,\ g,\ \xi)$ be a Sasakian manifold. The distribution F on X of rank $2n$ that is given by the orthogonal complement of the Killing vector field ξ defines a contact structure on X. We note that the corresponding one-form ω is the dual of ξ with respect to the metric g, i.e. $\omega(u) = g(\xi, u)$. From the condition that $(d\omega)^n \wedge \omega$ is nowhere vanishing, it follows that the restriction of $d\omega$ to F is fiberwise nondegenerate.

Lemma 3. *For all $x \in X$ and all $v,\ w \in F_x$,*

$$d\omega(v,\ w) = -g(\Phi(v),\ w)\,, \tag{8}$$

where Φ is defined in (2).

Proof. From the definition of Φ,

$$-g(\Phi(v),\ w) = g(\nabla_v \xi,\ w)\,.$$

Since ξ is a Killing vector field,

$$g(\nabla_v \xi,\ w) + g(\nabla_w \xi,\ v) = 0\,. \tag{9}$$

Extend v and w to smooth sections $\tilde{v}$ and $\tilde{w}$ of F. Since $\tilde{w}$ is orthogonal to ξ,

$$g(\nabla_v \xi,\ w) = -g(\xi,\ \nabla_v \tilde{w})\,.$$

Using (9),

$$g(\nabla_v \xi,\ w) = -g(\nabla_w \xi,\ v) = g(\xi,\ \nabla_w \tilde{v})$$

because $\tilde{v}$ is also orthogonal to ξ. Therefore,

$$-g(\Phi(v),\ w) = \frac{1}{2}(-g(\xi,\ \nabla_v \tilde{w}) + g(\xi,\ \nabla_w \tilde{v})) = -\frac{1}{2} g(\xi,\ [\tilde{v},\ \tilde{w}])\,.$$

But $-\frac{1}{2} g(\xi,\ [\tilde{v},\ \tilde{w}]) = d\omega(v,\ w)$ because both $\tilde{v}$ and $\tilde{w}$ are orthogonal to ξ. □

We now consider N as the subbundle of TX generated by ξ.

3. Symplectic quotients of Kähler manifolds

We will summarize some basic facts, which can be found in Kirwan's work [6].

Let $(M,\ \omega_M)$ be a compact Kähler manifold acted on holomorphically by a complex Lie group G, which is the complexification of a maximal compact subgroup K. Assume that the Kähler form is preserved by the action of K, meaning $k^* \omega_M = \omega_M$ for all $k \in K$.

Furthermore, we assume the existence of a *moment map*

$$\mu\ :\ M \longrightarrow \mathfrak{k}^*$$

for the underlying symplectic manifold, where $\mathfrak{k} = \mathrm{Lie}(K)$.

By definition, a moment map for the action of K on (M, ω_M) is K-equivariant with respect to the action of K on M and the co-adjoint action Ad^* of K on $\mathfrak{k}^*$ satisfies the following condition. Note first that for any $a \in \mathfrak{k}$, the composition $d\mu : TM \longrightarrow \mathfrak{k}^*$ with the evaluation at a defines a 1-form on M. This form is required to correspond under the duality defined by ω_M to the vector field on M that is induced by a: For all $x \in M$ and for all $\xi \in T_x M$,

$$d\mu(x)(\xi) \cdot a = \omega_M(\xi, a), \tag{10}$$

where $\cdot$ denotes the natural pairing of $\mathfrak{k}$ and $\mathfrak{k}^*$.

The Marsden–Weinstein theorem states that a moment map exists, and is uniquely determined, if the group K is semisimple [8]. Furthermore, it is known to exist, if $H^1(\mathfrak{g}) = 0$ and $H^2(\mathfrak{g}) = 0$ (cf. [4, section 26]). A moment map is explicitly given for the action of $U(N+1)$ on the complex projective space $\mathbb{P}_N$ equipped with the Fubini–Study form. In particular, if a compact group with a $U(N+1)$-representation acts on a projective variety $M \subset \mathbb{P}_N$, a moment map exists.

If the action of a connected reductive group G on a projective manifold M lifts to a (very) ample line bundle L, like in the above explicit case, the notion of stable and semistable points from geometric invariant theory for the group action hold. The loci $M^s \subset M^{ss} \subset M$ of stable and semistable points are known to be Zariski open in M, and the geometric and categorical quotients $M^s/G \subset M^{ss}/\!\!/G$ exist as projective varieties.

Kirwan showed in [6] that Mumford's geometric invariant theoretic quotient $M^{ss}/\!\!/G$ coincides with the Marsden–Weinstein quotient $\mu^{-1}(0)/K$ in the projective case under the assumption that the group G is semisimple with finite stabilizers, and that K acts on $\mu^{-1}(0)$ so that all points are stable. As a result, the quotient has the structure of a complex orbifold. The set $\mu^{-1}(0)$ is a differentiable manifold under this assumption because of (10), which implies that $d\mu(x)$ is surjective for all $x \in M$. Hence the quotient $\mu^{-1}(0)/K$ carries a natural orbifold structure.

The more general case (for Kähler manifolds and semistable actions) was solved by Heinzner and Loose in [5].

We state the Marsden–Weinstein theorem now. We assume that (M, ω_M) is a symplectic manifold on which a compact group K acts with a moment map $\mu : M \longrightarrow \mathfrak{k}^*$. Assume that $\mu^{-1}(0)$ is nonempty.

Theorem 4 [8]. *The set $\mu^{-1}(0)$ is K-invariant, and the quotient $\mu^{-1}(0)/K$ possesses a natural symplectic structure, if the stabilizer of K with respect to all $x \in \mu^{-1}(0)$ is finite.*

We will always assume that a reductive group G acts on M with finite stabilizers and that all G-orbits intersect $\mu^{-1}(0)$ avoiding categorical quotients.

In our situation the following version holds.

Theorem 5 [6]. *Let (M, ω_M) be a Kähler manifold on which a reductive complex Lie group G with maximal compact subgroup K acts such that ω_M is K-invariant. Assume that all isotropy groups are finite, and suppose the existence of a moment map $\mu : M \longrightarrow \mathfrak{k}^*$. Furthermore, suppose that all G-orbits intersect $\mu^{-1}(0)$. Then the inclusion $\mu^{-1}(0) \subset M$ induces a diffeomorphism of geometric quotients*

$$\pi : \mu^{-1}(0)/K \longrightarrow M/G$$

such that M/G carries the structure of a complex orbifold and the symplectic form on the symplectic quotient amounts to a Kähler orbifold form on M/G.

4. Quotients of Sasakian manifolds

We fix the assumption for our main theorem. Let (X, g, ξ) be a Sasakian manifold, and (M, ω_M) be the associated Kähler manifold, i.e. $X \times \mathbb{R}_+$ with the induced Kähler structure. We assume that a connected reductive Lie group G acts on M holomorphically fixing the Kähler form ω_M. Let $K \subset G$ be a maximal compact subgroup. Assume that this subgroup K fixes the subspace X; note that this condition implies that the action of K on X preserves the Riemannian metric g.

In particular, $\mathfrak{k} = \mathrm{Lie}(K)$ consists of Killing vector fields on X. We assume the existence of a moment map for the above group action.

To begin with, we make the extra assumption that both K and G act freely. We saw in section 3 that $\mu^{-1}(0)$ is a differentiable manifold so that the quotient $\mu^{-1}(0)/K$ is also smooth.

Assume that the elements of the Lie algebra $\mathfrak{k}$ are perpendicular to the Reeb field ξ, and to $\partial/\partial r$,

$$\xi \perp \mathfrak{k} \quad \text{with respect to } g \text{ on } X \simeq X \times \{1\}, \tag{11}$$

$$\partial/\partial r \perp \mathfrak{k} \quad \text{with respect to } \omega_M \text{ and } [\mathfrak{k}, \partial/\partial r] = 0. \tag{12}$$

The above assumption (12) implies that the group K acts on $X \simeq X \times \{1\}$ and all spaces $X \times \{r\}$ for all $r \in \mathbb{R}_+$.

We already know that K fixes $\mu^{-1}(0)$ from the Marsden–Weinstein theorem. In fact this follows readily from (10).

Lemma 6. *The action of $\mathbb{R}_+$ on $X \times \mathbb{R}_+$, given by the multiplication of $\mathbb{R}_+$ and the trivial action of $\mathbb{R}_+$ on X, commutes with the action of K and fixes the subset $\mu^{-1}(0)$.*

Proof. The first statement follows from $[\mathfrak{k}, \partial/\partial r] = 0$. Furthermore, because of (12) we have $d\mu(\partial/\partial r) = \omega_M(\xi, \partial/\partial r) = 0$, which shows the second claim. □

We study the compatibility of the action of K on $X \times \mathbb{R}_+$ and the complex structure of the associated Kähler structure. We already know the following.

Lemma 7. *Concerning the action of K and the almost complex structure $J|_F$, the following holds on X: For any $u \in \mathfrak{k}$ and $v \in \xi^\perp$,*

$$[u, (J|F)](v) = -(\nabla_u \Phi)(v) = -g(u, v)\xi\,. \tag{13}$$

The claim follows immediately from (3).

We will call the elements of $\mathfrak{k}$ also *vertical* vector fields, and those perpendicular to $\mathfrak{k}$ *horizontal*.

Lemma 8. *On (X, g, ξ), we have*

$$[\xi, \mathfrak{k}] = 0\,. \tag{14}$$

Proof. Let $v \in \mathfrak{k}$, and w be an arbitrary vector field. Note that v is Killing. Then

$$\begin{aligned} g([\xi, v], w) &= g(\nabla_\xi v, w) - g(\nabla_v \xi, w) = -g(\nabla_w v, \xi)) - g(\nabla_v \xi, w) \\ &= -wg(v, \xi) + g(v, \nabla_w \xi) - g(\nabla_v \xi, w). \end{aligned}$$

Now the first term vanishes, because $\xi \perp \mathfrak{k}$, and the second and third term together give zero, because ξ is a Killing vector field. □

We also have the action of $\mathbb{R}_+$ by multiplication on the second factor on $M = X \times \mathbb{R}_+$.

Lemma 9. *The group K acts in a free way on the differentiable manifold $(\mu^{-1}(0) \cap (X \times \{1\})) \times \mathbb{R}_+$. The natural bijection*

$$(\mu^{-1}(0) \cap (X \times \{1\})) \times \mathbb{R}_+ \longrightarrow \mu^{-1}(0), \qquad ((x, 1), r) \longmapsto (x, r)$$

induces an isomorphism

$$(\mu^{-1}(0) \cap (X \times \{1\}))/K \times \mathbb{R}_+ \longrightarrow \mu^{-1}(0)/K.$$

Proof. We consider the surjection $\mu^{-1}(0) \cap (X \times \{1\}) \times \mathbb{R}_+ \longrightarrow \mu^{-1}(0)/K$. Recall that $X \times \{1\}$ is preserved by the action of $K \subset G$ on M. Let $(x, r) = \gamma \cdot (\tilde{x}, s)$ for $x, \tilde{x} \in X = X \times \{1\}$ and $\gamma \in K$. We use Lemma 6 and see that

$$(x, s^{-1}r) = s^{-1}\gamma(\tilde{x}, s) = \gamma s^{-1}(\tilde{x}, s) = (\gamma\tilde{x}, 1)$$

using the action of K on $X = X \times \{1\}$. Hence $x = \gamma\tilde{x}$ and $s = r$. □

We denote the differentiable manifold $\mu^{-1}(0) \cap (X \times \{1\})$ by $\mu_X^{-1}(0)$, where μ_X stands for the restriction of μ to $X \times \{1\}$. The quotient manifold $\mu_X^{-1}(0)/K$ is denoted by Y, with projection map $\pi : \mu_X^{-1}(0) \longrightarrow Y$.

Since K acts in an isometric way on X, the restriction of g to horizontal tangent vectors defines a Riemannian metric g_Y on Y. We denote by ∇^Y the corresponding covariant differentiation.

We consider the restriction of ξ to $\mu_X^{-1}(0)$ with values in TX, and denote it with the same letter. This vector field is K-invariant and orthogonal to K-orbits by (11). Hence it descends to a vector field ξ_Y on Y.

We introduce the following notation: Given a vector field v on Y, we denote the horizontal lift of v to $\mu_X^{-1}(0)$ by $\tilde{v}$. In this sense $\xi = \tilde{\xi}_Y$.

Lemma 10. *Let u, v, w be vector fields on Y with horizontal lifts $\tilde{u}, \tilde{v}, \tilde{w}$ to $\mu_X^{-1}(0)$, and let $f \in C^\infty(Y)$. Then*

(*i*) $\pi^*([u, v]f) = [\tilde{u}, \tilde{v}](\pi^* f)$, *in particular* $[\tilde{u}, \tilde{v}] - \widetilde{[u, v]}$ *is tangent to the fibers of* π, *and invariant under the action of* K, *hence an element of* $\mathfrak{k}$.
(*ii*) $\widetilde{\nabla_u^Y v} - \nabla_{\tilde{u}} \tilde{v} \in \mathfrak{k}$.
(*iii*) *Let* $\Phi^Y(u) = -\nabla_u^Y(\xi_Y)$. *Then* $\widetilde{\Phi^Y(u)} - \Phi(\tilde{u}) \in \mathfrak{k}$.

Proof. The first identity follows from the definition, the second from (i) and applying the Koszul formula twice shows that

$$g(\nabla_{\tilde{u}}\tilde{v},\ \tilde{w}) = \pi^* g_Y(\nabla_u v,\ w)\,.$$

Now (ii) implies that $\widetilde{\nabla^Y_w \xi_Y} - \nabla_{\tilde{w}}\xi$ is a vertical vector field. □

In the above, we defined $Y = \mu_X^{-1}(0)/K = (\mu^{-1}(0) \cap (X \times \{1\}))/K$.

PROPOSITION 11

The manifold (Y, g_Y, ξ_Y) is Sasakian.

Proof. We first note that the horizontal lift of the Reeb field ξ_Y is equal to the original Reeb field ξ on X. We will verify condition (ii) of Theorem 1 on Y, using the tilde notation for horizontal lifts. Let u and v be vector fields on Y. We will apply Lemma 10 repeatedly, and use $\equiv_{\mathfrak{k}}$ for equivalence modulo elements of $\mathfrak{k}$,

$$\begin{aligned}\widetilde{(\nabla^Y_u \Phi^Y)}(v) &= \widetilde{\nabla^Y_u(\Phi^Y(v))} - \widetilde{\Phi^Y(\nabla^Y_u)(v))} \equiv_{\mathfrak{k}} \nabla_{\tilde{u}}(\widetilde{\Phi^Y(v)}) + \nabla_{\widetilde{\nabla^Y_u(v)}}(\xi) \\ &\equiv_{\mathfrak{k}} -\nabla_{\tilde{u}}\nabla_{\tilde{v}}(\xi) + \nabla_{\nabla_{\tilde{u}}\tilde{v}}(\xi) = (\nabla_{\tilde{u}}\Phi)(\tilde{v}) = g(\tilde{u},\tilde{v})\xi - g(\xi,\tilde{v})\tilde{u} \\ &= g_Y\widetilde{(u,v)}\xi_Y - g_Y\widetilde{(\xi_Y,v)}u.\end{aligned}$$

Hence

$$(\nabla^Y_u \Phi^Y)(v) = g_Y(u,v)\xi_Y - g_Y(\xi_Y, v)u\,.$$

□

5. Symplectic reduction for Sasakian manifolds

Let (X, g, ξ) be a Sasakian manifold, and (M, ω_M) the associated Kähler manifold. Let G be a semisimple complex Lie group acting holomorphically on M with finite stabilizers, fixing ω_M. We know that in this case a moment map $\mu : M \longrightarrow \mathfrak{k}^*$ exists. In the somewhat more general case of a reductive group G, we make the existence of a moment map an assumption. By Kirwan's result, the Marsden–Weinstein symplectic quotient

$$\mu^{-1}(0)/K \xrightarrow{\sim} M/G$$

possesses a complex structure turning the symplectic form on the quotient into a Kähler form.

Our geometric assumptions are (11) and (12).

Proposition 11 states that $(Y,\ g_Y,\ \xi_Y)$ is a Sasakian manifold, and by Lemma 9, we have

$$Y \times \mathbb{R}_+ \xrightarrow{\sim} \mu^{-1}(0)/K \xrightarrow{\sim} M/G\,.$$

So far we assumed free group actions. In case of finite stabilizers, Y is a Sasakian orbifold and $Y \times \mathbb{R}_+ \simeq M/G$ is a Kähler orbifold. The differential geometric computation remains the same.

Theorem 12. *The geometric Kähler quotient M/G is induced by a natural structure of a Sasakian orbifold on the quotient $\mu_X^{-1}(0)/K = \mu^{-1}(0) \cap (X \times \{1\})/K$.*

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