Heptavalent symmetric graphs of order 24p

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MS received 17 July 2018; revised 24 August 2018; accepted 13 September 2018; published online 24 June 2019

Abstract. A graph is symmetric if its automorphism group acts transitively on the set of arcs of the graph. In this paper, we classify connected heptavalent symmetric graphs of order 24p for each prime p. As a result, there are twelve sporadic such graphs: one for p = 2, four for p = 3, one for p = 5 and six for p = 13.

Keywords. Symmetric graph; s-transitive graph; coset graph; orbital graph.

2000 Mathematics Subject Classification. 05C25, 20B25.

1. Introduction

Throughout this paper, graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here, we refer the reader to [23, 26] or [1, 2], respectively. Let G be a permutation group on a set Ω and v ∈ Ω. Denote by Gv the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is semiregular on Ω if Gv = 1 for every v ∈ Ω and regular if G is transitive and semiregular.

For a graph X, denote by V(X), E(X) and Aut(X) its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be G-vertex-transitive if G ≤ Aut(X) acts transitively on V(X). X is simply called vertex-transitive if it is Aut(X)-vertex-transitive. An s-arc in a graph is an ordered (s + 1)-tuple (v0, v1, . . . , vs−1, vs) of vertices of the graph X such that vi−1 is adjacent to vi for 1 ≤ i ≤ s, and vi−1 ≠ vi+1 for 1 ≤ i ≤ s − 1. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup G ≤ Aut(X), a graph X is said to be (G, s)-arc-transitive or (G, s)-regular if G is transitive or regular on the set of s-arcs in X, respectively. A (G, s)-arc-transitive graph is said to be (G, s)-transitive if it is not (G, s + 1)-arc-transitive. In particular, a (G, 1)-arc-transitive graph is called G-symmetric. A graph X is simply called s-arc-transitive, s-regular or s-transitive if it is (Aut(X), s)-arc-transitive, (Aut(X), s)-regular or (Aut(X), s)-transitive, respectively.

As we all know that the structure of the vertex stabilizers of symmetric graphs is very useful to classify such graphs, and this structure of the cubic or tetravalent case was given by Miller [20] and Potočnik [22]. Thus, classifying symmetric graphs with valency 3 or 4...
Table 1. Heptavalent symmetric graphs of order $24p$.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$s$-transitivity</th>
<th>$\text{Aut}(X)$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L^G_{48}$</td>
<td>1-transitive</td>
<td>$\text{PGL}(2, 7) \times S_3$</td>
<td>Construction 3.1, $p = 2$</td>
</tr>
<tr>
<td>$L^G_{72}^i$</td>
<td>1-transitive</td>
<td>$\text{PSL}(2, 8) \times Z_2$</td>
<td>Construction 3.2, $p = 3$, $i = 1, 2, 3$</td>
</tr>
<tr>
<td>$U^G_{72}$</td>
<td>2-transitive</td>
<td>$\text{PSU}(3, 3) \times Z_2$</td>
<td>Construction 3.4, $p = 3$</td>
</tr>
<tr>
<td>$S^G_{120}$</td>
<td>2-transitive</td>
<td>$S_7$</td>
<td>Construction 3.5, $p = 5$</td>
</tr>
<tr>
<td>$L^G_{312}^1$</td>
<td>2-transitive</td>
<td>$\text{PGL}(2, 13) \times Z_2$</td>
<td>Construction 3.6, $p = 13$, $i = 1, 2, 3, 4$</td>
</tr>
<tr>
<td>$L^G_{312}^5$</td>
<td>2-transitive</td>
<td>$(\text{PSL}(2, 13) \times Z_2) \times Z_2$</td>
<td>Construction 3.6, $p = 13$</td>
</tr>
<tr>
<td>$L^G_{312}^6$</td>
<td>2-transitive</td>
<td>$\text{PSL}(2, 13) \times D_8$</td>
<td>Construction 3.6, $p = 13$</td>
</tr>
</tbody>
</table>

has received considerable attention and a lot of results have been achieved, see [7, 29, 30]. Guo [9] determined the exact structure of pentavalent case. Following this structure, a series of pentavalent symmetric graphs were classified in [11, 16–18, 27, 28]. In particular, Ling classified pentavalent symmetric graphs of order $24p$ in [17], and Guo [10] gave the exact structure of heptavalent case. Thus, as an application, we classify connected heptavalent symmetric graphs of order $24p$ for each prime $p$ in this paper.

**Theorem 1.1.** Let $X$ be a connected heptavalent symmetric graph of order $24p$ with $p$ a prime. Then $X$ is isomorphic to one of the graphs in Table 1.

2. Preliminary results

Let $X$ be a connected $G$-symmetric graph with $G \leq \text{Aut}(X)$, and let $N$ be a normal subgroup of $G$. The quotient graph $X_N$ of $X$ relative to $N$ is defined as the graph with vertices the orbits of $N$ on $V(X)$ and with two orbits adjacent if there is an edge in $X$ between those two orbits. In view of [19, Theorem 9], we have the following.

**Proposition 2.1**

Let $X$ be a connected heptavalent $G$-symmetric graph with $G \leq \text{Aut}(X)$, and let $N$ be a normal subgroup of $G$. Then one of the following holds:

1. $N$ is transitive on $V(X)$;
2. $X$ is bipartite and $N$ is transitive on each part of the bipartition;
3. $N$ has $r \geq 3$ orbits on $V(X)$, $N$ acts semiregularly on $V(X)$, the quotient graph $X_N$ is a connected heptavalent $G/N$-symmetric graph.

The following proposition characterizes the vertex stabilizers of connected heptavalent $s$-transitive graphs (see [10, Theorem 1.1]).

**Proposition 2.2**

Let $X$ be a connected heptavalent $(G, s)$-transitive graph for some $G \leq \text{Aut}(X)$ and $s \geq 1$. Let $v \in V(X)$. Then $s \leq 3$ and one of the following holds:
PROPOSITION 2.3

obtain the following proposition by checking the orders of non-abelian simple groups:

Thus, \( p \) is a prime. Let \( G_v \cong F_{42}, F_{42} \times \mathbb{Z}_2, F_{42} \times \mathbb{Z}_3, \text{PSL}(3, 2), \text{A}_7, \text{S}_7, \mathbb{Z}_2^3 \rtimes \text{SL}(3, 2) \) or \( \mathbb{Z}_4^2 \rtimes \text{SL}(3, 2) \);

(3) For \( s = 3 \), \( G_v \cong F_{42} \times \mathbb{Z}_6, \text{PSL}(3, 2) \times \text{S}_4, \text{A}_7 \times \text{A}_6, \text{S}_7 \times \text{S}_6, (\text{A}_7 \times \text{A}_6) \times \mathbb{Z}_2, \mathbb{Z}_2^2 \rtimes (\text{SL}(2, 2) \times \text{SL}(3, 2)) \) or \( [2^{20}] \times (\text{SL}(2, 2) \times \text{SL}(3, 2)) \).

From [8, pp. 12–14], [25, Theorem 2], [4, Theorem 1] and [14, Theorem A], we may obtain the following proposition by checking the orders of non-abelian simple groups:

PROPOSITION 2.3

Let \( p \) be a prime, and let \( G \) be a non-abelian simple group of order \(|G|\) \((2^{27} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot p)\). Then \( G \) has 3-prime factor, 4-prime factor or 5-prime factor, and is one of the groups given in Table 2.

Remark. Let \( p \) and \( q \) be two primes. Then by [25, Theorem 2], \( \text{PSL}(2, p) \) satisfying the relation \( p^2 - 1 = 2^a 3^b r^c \) is an infinite family of simple \( K_4 \)-group, where \( a, b, c \) are positive integers. Note that \(|\text{PSL}(2, p)| = p(p^2 - 1)/2 \) for \( p \) an odd prime. In Proposition 2.3, \(|G|\) \((2^{27} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot p)\). Let \( G = \text{PSL}(2, p) \) be a simple \( K_4 \)-group. Clearly, \( p \neq 5 \) and 7. Thus, \( p \geq 11 \) and \( r = 5 \) or 7. The upper bound of \(|G|\) forcing that \( 1 \leq a \leq 28, 1 \leq b \leq 5, \) and if \( r = 5 \), then \( 1 \leq c \leq 2 \); if \( r = 7 \), then \( c = 1 \). With these relations about \( a, b, c \), we can use MAGMA [3] to search all the possibilities of \( a, b, c \) such that \((2^a 3^b r^c + 1)\) is a prime square \( p^2 \). Thus, we can have \( p = 11, 13, 19, 31 \) or 127. If \( G \) is other infinite family of simple \( K_4 \)-group or \( K_5 \)-group listed in [25, Theorem 2] and [14, Theorem A], then we can also use this approach and achieve the other groups listed in the Table 2.

In view of [15, Theorem 1.1] and [21, Theorem 1.1], the classifications of connected heptavalent symmetric graphs of order \( kp \) are given, where \( k = 6, 8 \) or 12, and \( p \) is a prime. Thus, we have that following characterization of these graphs.

PROPOSITION 2.4

Let \( X \) be connected heptavalent symmetric graph of order \( kp \) with \( k = 6, 8 \) or 12 and \( p \) a prime. Let \( v \in V(X) \) and \( s \) a positive integer. Then the descriptions of \(|V(X)|, \text{Aut}(X), s\)-transitivity and \( \text{Aut}(X)_v \) are as given in Table 3.

3. Graph constructions

In this section, we construct some heptavalent symmetric graphs of order \( 24p \) with \( p \) a prime. To do this, we need to introduce the so called coset graph (see [20,24]) constructed from a finite group \( G \) relative to a subgroup \( H \) of \( G \) and a union \( D \) of some double cosets of \( H \) in \( G \) such that \( D^{-1} = D \). The coset graph \( \text{Cos}(G, H, D) \) of \( G \) with respect to \( H \) and \( D \) is defined to have vertex set \([G : H]\), the set of right cosets of \( H \) in \( G \), and edge set \( \{[H g, H d g] \mid g \in G, d \in D\} \). The graph \( \text{Cos}(G, H, D) \) has valency \(|D|/|H|\) and is connected if and only if \( D \) generates the group \( G \). The action of \( G \) on \( V(\text{Cos}(G, H, D)) \) by right multiplication induces a vertex-transitive automorphism group, which is arc-transitive if and only if \( D \) is a single double coset. Moreover, this action is faithful if
and only if \( H_G = 1 \), where \( H_G \) is the largest normal subgroup of \( G \) in \( H \). Clearly, \( \text{Cos}(G, H, D) \cong \text{Cos}(G, H^\alpha, D^\alpha) \) for every \( \alpha \in \text{Aut}(G) \). For more details regarding coset graphs, see, for example [6,19,24].
From [12, Theorem 1.1], we know that there is only one heptavalent symmetric graph of order 48.

**Construction 3.1.** Let \( G = \text{PGL}(2, 7) = \langle (1, 2, 6)(3, 4, 8), (3, 8, 7, 6, 5, 4) \rangle \). Then \( G \) has a Sylow 7-subgroup \( H = \langle (2, 5, 7, 6, 3, 4, 8) \rangle \). Let \( g = (1, 2)(3, 5)(6, 8) \). Define the coset graph

\[
\mathcal{L}G_{48} = \text{Cos}(G, H, HgH).
\]

Then any connected heptavalent symmetric graph of order 48 is isomorphic to \( \mathcal{L}G_{48} \) and \( \text{Aut}(\mathcal{L}G_{48}) = \text{PGL}(2, 7) \times S_3 \).

Next, we discuss the connected heptavalent symmetric coset graphs on the groups \( \text{PSL}(2, 8) \) and \( \text{PSL}(2, 8) \times \mathbb{Z}_2 \).

**Construction 3.2.** Let \( a = (3, 5, 6, 7, 9, 8, 4), b = (1, 3, 2)(4, 7, 8)(5, 6, 9) \) and \( c = (10, 11) \). Then \( T = \langle a, b \rangle \cong \text{PSL}(2, 8) \) and \( G = \langle a, b, c \rangle \cong \text{PSL}(2, 8) \times \mathbb{Z}_2 \). Take two elements \( d = (1, 8, 2, 7, 4, 9, 3) \) and \( e = (1, 8)(2, 3)(5, 6)(7, 9) \). It is easy to see that \( H = \langle d \rangle \cong \mathbb{Z}_7 \) and \( K = \langle d, e \rangle \cong D_{14} \). Take \( x = (2, 4)(3, 6)(5, 8)(7, 9) \) and \( y = (2, 6)(3, 4)(5, 9)(7, 8) \) in \( T \), and \( z = (2, 4)(3, 6)(5, 8)(7, 9)(10, 11) \) in \( G \). We can define the following three coset graphs:

\[
\mathcal{L}G_{72}^1 = \text{Cos}(T, H, HxH), \quad \mathcal{L}G_{72}^2 = \text{Cos}(T, H, HyH), \quad \mathcal{L}G_{72}^3 = \text{Cos}(G, K, KzK).
\]

Then by MAGMA [3], \( \text{Aut}(\mathcal{L}G_{72}^i) \cong \text{PSL}(2, 8) \times \mathbb{Z}_2 \) with \( i = 1, 2, 3 \).

By using the isomorphisms of coset graph and calculation of MAGMA [3], we have that there are only three non-isomorphic heptavalent symmetric graphs of order 72 admitting \( \text{PSL}(2, 8) \times \mathbb{Z}_2 \) as an automorphism group.

**Lemma 3.3.** Let \( X \) be a connected heptavalent symmetric graph of order 72 and \( A = \text{Aut}(X) \). Then

1. If \( A \) has an arc-transitive subgroup isomorphic to \( \text{PSL}(2, 8) \), then \( X \cong \mathcal{L}G_{72}^i \) for \( i = 1 \) or 2;
2. If \( A \) has an arc-transitive subgroup isomorphic to \( \text{PSL}(2, 8) \times \mathbb{Z}_2 \), then \( X \cong \mathcal{L}G_{72}^i \) for \( i = 1, 2 \) or 3;

**Proof.** We use the same notations as Construction 3.2 to prove results. Assume that \( A \) has an arc-transitive subgroup \( T \cong \text{PSL}(2, 8) \). Then \( X \cong \text{Cos}(T, T_v, T_vtT_v) \) with \( T_v \cong \mathbb{Z}_7 \), \( \langle T_v, t \rangle = T \), \( |T G_v : T G_v \cap T_v| = 7 \) and \( t \) a 2-element. Clearly, \( T \) has one conjugacy class of subgroup isomorphic to \( \mathbb{Z}_7 \). Without loss of generality, we take \( T_v = H \). By MAGMA [3], \( t \) has 42 choices, denoted this set by \( U \), and \( N_T(H) \cong D_{14} \) acting on \( U \) has 6 orbits. With the isomorphisms of coset graph, we know that the coset graph formed by the elements in the same orbit are isomorphic each other. Thus, we obtain six coset graphs. With the calculation of MAGMA [3], these six representatives of 6 orbits form two graphs, that is, \( \mathcal{L}G_{72}^1 \) and \( \mathcal{L}G_{72}^2 \).
Assume that $A$ has an arc-transitive subgroup $G$. Then $X \cong \text{Cos}(G, G_v, G_v g G_v)$ with $G_{v T} \cong D_{14}, \langle G_v, g \rangle = G$ and $|G_v : G_v^g \cap G_v| = 7$. In particular, $g$ can be chosen as a 2-element. By MAGMA [3], $G$ has two conjugacy classes of subgroups isomorphic to $D_{14}$ with $K$ and $L = \langle d, (1, 8)(2, 3)(5, 6)(7, 9)(10, 11) \rangle$ as their representatives.

Suppose that $G_v = K$. Then by MAGMA [3], $g$ has 42 choices, and $N_G(K) \cong D_{28}$ acting on these elements has six orbits. These six orbits form six coset graphs, which are isomorphic each other, that is, $L_C G_{72}^3$.

Suppose that $G_v = L$. Then by MAGMA [3], $g$ has 78 choices, and $N_G(L) \cong D_{28}$ acting on these elements has 12 orbits. These orbits form 12 coset graphs and by MAGMA [3], there are 2 non-isomorphic graphs, that is, $X \cong L_G G_{72}^i$ with $i = 1, 2$. □

As we all know that if a symmetric graph has an arc-transitive subgroup $G$, then this graph can be viewed as an orbital graph of the group $G$. Thus, we use orbital graph to construct another heptavalent symmetric graph of order 72.

Construction 3.4. Let $G \cong \text{Aut}(\text{PSU}(3, 3)) \cong \text{PSU}(3, 3) \rtimes \mathbb{Z}_2$. Then by Atlas [5], $G$ has one conjugacy class of subgroup $H \cong \text{PSL}(2, 7)$ and $|G : H| = 72$. Thus, $G$ has only one permutation representation on 72 points. By MAGMA [3], this representation has two self-paired suborbit of length 7, and the corresponding orbital graphs are isomorphic each other, denoted by $U_G G_{72}$. Moreover, Aut($U_G G_{72}$) $\cong G$. Conversely, any connected heptavalent symmetric graph of order 72 admitting $G$ as an arc-transitive automorphism group is isomorphic to $U_G G_{72}$.

The following graph is a vertex-primitive 2-transitive graph of order 120, defined on the group $S_7$.

Construction 3.5. Let $G = S_7$. Then $G$ has a maximal subgroup $H \cong F_{42}$ and hence $G$ has a primitive permutation representation on 120 points. By MAGMA [3], this representation has one self-paired suborbit of length 7, and denoted the corresponding orbital graph by $S_G G_{120}$. Moreover, Aut($S_G G_{120}$) $\cong G$. Conversely, any connected heptavalent symmetric graph of order 120 admitting $S_7$ or $A_7$ as an arc-transitive automorphism group is isomorphic to $S_G G_{120}$.

Now with the calculation of MAGMA [3], we consider the heptavalent symmetric orbital graphs on the group $\text{PGL}(2, 13) \times \mathbb{Z}_2$.

Construction 3.6. Take the following three elements:

\[ a = (3, 9, 10, 12, 6, 13, 4, 11, 5, 7, 8, 14), \]
\[ b = (1, 14, 2)(3, 8, 13)(4, 10, 11)(5, 6, 12), \]
\[ c = (15, 16), d = (1, 12, 10, 13, 3, 6, 4)(2, 11, 7, 9, 5, 14, 8). \]

Then $G = \langle a, b \rangle \cong \text{PGL}(2, 13)$ and $H = \langle a^2, b, c \rangle \cong \text{PSL}(2, 13) \times \mathbb{Z}_2$. By Atlas [5], $G$ and $H$ both have one conjugacy class of subgroups isomorphic to $K = \langle d \rangle \cong \mathbb{Z}_7$. It follows that $G$ and $H$ both have a permutation representation on 312 points. By MAGMA [3], for these two representation, the lengths of suborbits both are $1^4, 74^4$.

For the representation of $G$, there are 10 suborbits such that their orbital graphs are connected heptavalent symmetric graphs. By MAGMA [3], up to isomorphism, there
are four such graphs, denoted by $LG^i_{312}$ and $\text{Aut}(LG^i_{312}) \cong \text{PGL}(2, 13) \times \mathbb{Z}_2$ with $i = 1, 2, 3, 4$. For the representation of $H$, there are 12 suborbits such that their orbital graphs are connected heptavalent symmetric graphs. By MAGMA [3], up to isomorphism, there are five such graphs. However, there are three graphs isomorphic to that of $LG$ with $i = 2, 3, 4$. Thus, we denote the other two by $LG^5_{312}$ and $LG^6_{312}$. By MAGMA [3], $\text{Aut}(LG^i_{312}) \cong (\text{PSL}(2, 13) \times \mathbb{Z}_2) \times \mathbb{Z}_2$ and $\text{Aut}(LG^i_{312}) \cong \text{PGL}(2, 13) \times D_8$.

Conversely, let $X$ be a connected heptavalent symmetric graph of order 312. If $X$ is PGL(2, 13)-symmetric, then $X \cong LG^i_{312}$ with $i = 1, 2, 3, 4$. If $X$ is PSL(2, 13) × $\mathbb{Z}_2$-symmetric, then $X \cong LG^i_{312}$ with $i = 2, 3, 4, 5, 6$. If $X$ is PGL(2, 13) × $\mathbb{Z}_2$-symmetric, then $X \cong LG^i_{312}$ with $i = 1, 2, 3, 4$.

4. Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. Let $X$ be a connected heptavalent symmetric graph of order $24p$ and $A = \text{Aut}(X)$. If $p = 2$, then $|V(X)| = 48$. By [12, Theorem 1.1], $X \cong G_{48}$ and $\text{Aut}(X) \cong \text{PGL}(2, 7) \times S_3$. Thus, in what follows, we may assume that $p \geq 3$. Take $v \in V(X)$. Then by Proposition 2.2, $|A_v| \geq 2^{24} \cdot 3^4 \cdot 5^2 \cdot 7$ and hence $|A| \geq 2^{27} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot p$. We separate the proof into two cases: $A$ has a solvable minimal subgroup; $A$ has no solvable minimal normal subgroup.

Case 1. $A$ has a solvable minimal normal subgroup. Let $N$ be a solvable minimal normal subgroup of $A$. Then $|N| \geq 2^{27} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot p$, and $N$ is elementary abelian. Thus, $N \cong \mathbb{Z}_q^k$ with $q = 2, 3, 5, 7$ or $p$ and $k$ a positive integer. By Proposition 2.1, $N$ is semiregular and $X_N$ is also a connected heptavalent $A/N$-symmetric graph. It follows that $|N| \geq 24p$ and $N \cong \mathbb{Z}_2, \mathbb{Z}_2^2, \mathbb{Z}_2^2, \mathbb{Z}_3$ or $\mathbb{Z}_p$. Note that there is no connected heptavalent regular graph of odd order. Thus, $N \equiv \mathbb{Z}_3$.

Assume that $N \cong \mathbb{Z}_p$. Then $X_N$ is a heptavalent symmetric graph of order 24. By Proposition 2.4, $A/N \leq \text{PGL}(2, 7)$. The symmetry of $A/N$ forces that $24 \cdot 7 | |A/N|$. By Atlas [5], $\text{PSL}(2, 7) \cong A/N \leq \text{PGL}(2, 7)$, and hence $A/N$ has a normal arc-transitive subgroup $M/N \cong \text{PSL}(2, 7)$. Clearly, $N \leq C_M(N)$ and by ‘$N/C$ theorem’ (see [13, Chapter I, Theorem 4.5]), $M/C_M(N) \leq \text{Aut}(N) \cong \mathbb{Z}_p - 1$. If $N = C_M(N)$, then $\text{PSL}(2, 7) \cong M/C_M(N) \leq \mathbb{Z}_p - 1$, a contradiction. Thus, $N < C_M(N)$. On the other hand, since $C_M(N)/N \leq M/N \cong \text{PSL}(2, 7)$, we have that $C_M(N)/N = M/N$ and hence $C_M(N) = M$. By Atlas [5], $\text{Mult}(\text{PSL}(2, 7)) \cong \mathbb{Z}_2$. It follows that $M \cong \text{SL}(2, 7)$ with $p = 2$ or $\text{PSL}(2, 7) \times \mathbb{Z}_p$. Clearly, by our assumption, $M \equiv \text{SL}(2, 7)$. If $M \equiv \text{PSL}(2, 7) \times \mathbb{Z}_p$, then $M$ has a normal subgroup $H \equiv \text{PSL}(2, 7)$. The block graph $X_H$ has order $p$ and $M/H \equiv \mathbb{Z}_p$ is arc-transitive on $X_H$. Thus, the only possibility is that $p = 2$, which is contrary to our assumption.

Assume that $N \cong \mathbb{Z}_3$. Then $X_N$ is a heptavalent symmetric graph of order $8p$. By Proposition 2.4, $A/N \leq S_8 \times \mathbb{Z}_2$ for $p = 2$ or $\text{PGL}(2, 7)$ for $p = 3$. Note that $p \geq 3$. Thus, $|V(X_N)| = 24$ and $A/N \leq \text{PGL}(2, 7)$. By MAGMA [3], $\text{PGL}(2, 7)$ has a minimal arc-transitive subgroup $\text{PSL}(2, 7)$. It follows that $A/N$ has an arc-transitive subgroup $M/N \equiv \text{PSL}(2, 7)$. Since $\text{Mult}(\text{PSL}(2, 7)) \equiv \mathbb{Z}_2$, we have that $M \equiv \text{PSL}(2, 7) \times \mathbb{Z}_3$. Thus, $M$ has a normal subgroup $H \equiv \text{PSL}(2, 7)$, and $H$ has three orbits on $V(X)$. By Proposition 2.1, $H$ is semiregular and hence $|H| \geq 24 \cdot 3$, a contradiction.

Assume that $N \cong \mathbb{Z}_2^2$. Then $X_N$ is a heptavalent symmetric graph of order $6p$. By Proposition 2.4, $A/N \leq S_8$ for $p = 5$ or $\text{PGL}(2, 7)$ for $p = 13$. 
Let $A/N \leq S_8$ with $p = 5$. Then by MAGMA [3], $S_8$ has a minimal arc-transitive subgroup $S_7$. It follows that $A/N$ has an arc-transitive subgroup $M/N \cong S_7$, and $M/N$ has a normal subgroup $H/N \cong A_7$, which has two orbits. By ‘$N/C$ theorem’, $H/C_H(N) \leq \text{Aut}(N) \cong \text{GL}(2, 2)$. Since $\text{Aut}(N) \cong \text{GL}(2, 2)$ is solvable, we have that $C_H(N) = H$. This implies that $N \leq Z(H)$. By Atlas [5], $\text{Mult}(A_7) \cong Z_6$, and hence $H \cong A_7 \times Z_2$ or $Z_2 \times A_7 \times Z_2$. If $H \cong A_7 \times Z_2$, then $H$ has a characteristic subgroup $K \cong A_7$. The normality of $H$ in $M$ forces that $K \leq M$. The block graph $X_K$ has order 8, and by Proposition 2.1, $K$ is semiregular and $|K| = 24p$. This is impossible because $K \cong A_7$. If $H \cong Z_2 \times A_7 \times Z_2$, since $H$ has two orbits, we have that $|H_v| = 168$. By Proposition 2.2, $H_v \cong \text{PSL}(2, 7)$. However, by MAGMA [3], $Z_2 \cdot A_7 \times Z_2$ has no subgroup isomorphic to $\text{PSL}(2, 7)$, a contradiction.

Let $A/N \leq \text{PGL}(2, 13)$ with $p = 13$. Then $\text{PGL}(2, 13)$ has a minimal arc-transitive subgroup $M/N \cong \text{PSL}(2, 13)$. With a similar argument as in the above paragraph, we have that $M \cong \text{PSL}(2, 13) \times Z_2$ or $\text{SL}(2, 13) \times Z_2$. Since $M$ is arc-transitive, $|M_v| = 14$, and by Proposition 2.2, $M_v \cong D_{14}$. However, $\text{SL}(2, 13) \times Z_2$ has no subgroup isomorphic to $D_{14}$, a contradiction. Thus, $M \cong \text{PSL}(2, 13) \times Z_2$. In this case, $M$ has a normal subgroup $H \cong \text{PSL}(2, 13)$. Since $M_v \cong D_{14}$, we have that $H_v \cong D_{14}$. It follows that $H$ has 4 orbits. By Proposition 2.1, $H$ is semiregular and $|H| = 24p$. This is impossible because $H \cong \text{PSL}(2, 13)$.

Assume that $N \cong Z_2$. Then $X_N$ is a heptavalent symmetric graph of order $12p$. Note that $p \geq 3$. By Proposition 2.4, $A/N \leq \text{PSL}(2, 8)$ for $p = 3$ and $\text{PGL}(2, 13) \times Z_2$ for $p = 13$.

Let $A/N \leq \text{PSL}(2, 8)$ with $p = 3$. Then since $A/N$ is arc-transitive, $2^2 \cdot 3^2 \cdot 7 \mid |A/N|$ and hence $A/N \cong \text{PSL}(2, 8)$. Note that $\text{Mult}(\text{PSL}(2, 8)) = 1$ by Atlas [5]. It forces that $A \cong \text{PSL}(2, 8) \times Z_2$. By Construction 3.2 and Lemma 3.3, $X \cong LG_{72}^i$ with $i = 1, 2, 3$.

Let $A/N \leq \text{PGL}(2, 13) \times Z_2$ with $p = 13$. By Proposition 2.4 and with the calculation of MAGMA [3], the minimal arc-transitive subgroups are $\text{PSL}(2, 13)$, $\text{PGL}(2, 13) \times Z_2$ and $\text{PGL}(2, 13)$. It follows that $A/N$ has an arc-transitive subgroup $M/N \cong \text{PSL}(2, 13)$, $\text{PSL}(2, 13) \times Z_2$ or $\text{PGL}(2, 13)$. In each case, $M/N$ has a normal subgroup $H/N \cong \text{PSL}(2, 13)$. For the former case, $H$ is arc-transitive, and for the latter two cases, $H$ has two orbits on $V(X)$. Since $\text{Mult}(\text{PGL}(2, 13)) \cong Z_2$, we have that $H \cong \text{SL}(2, 13)$ or $\text{PSL}(2, 13) \times Z_2$. Assume that $H \cong \text{SL}(2, 13)$. Then $H_v \cong Z_7$. By MAGMA [3], there is no heptavalent coset graph on $\text{SL}(2, 13)$ for $(\text{SL}(2, 13))_v \cong Z_7$. Thus, $H \cong \text{PSL}(2, 13) \times Z_2$ and by Construction 3.6, $X \cong LG_{312}^i$ with $i = 2, 3, 4, 5, 6$. Assume that $H$ has two orbits on $V(X)$. Then $H_v \cong D_{14}$. Note that $\text{SL}(2, 13)$ has only one element of order 2, which is the central element in $\text{SL}(2, 13)$. This implies that the subgroup of order 14 in $\text{SL}(2, 13)$ is isomorphic to $Z_{14}$, and $H \cong \text{PSL}(2, 13) \times Z_2$. Clearly, $H$ has a subgroup $K \cong \text{PSL}(2, 13) \leq M$ and $K_v \cong D_{14}$ or $Z_7$. If $K_v \cong D_{14}$, then $K$ has four orbits on $V(X)$. By Proposition 2.1, $K$ is semiregular, a contradiction. Thus, $K_v \cong Z_7$ and $K$ also has two orbits on $V(X)$. It forces that $M/N \cong \text{PSL}(2, 13) \times Z_2$ or $\text{PGL}(2, 13)$. For the former, $M \cong \text{PSL}(2, 13) \times Z_4$ or $\text{PSL}(2, 13) \times Z_2^2$. By MAGMA [3], there is no such graph on these two groups, a contradiction. For the latter, $M \cong \text{PSL}(2, 13) \times Z_4$ or $\text{PGL}(2, 13) \times Z_2$. By Construction 3.6 and the calculation of MAGMA [3], if $M \cong \text{PSL}(2, 13) \times Z_4$, then $X \cong LG_{312}^5$; if $M \cong \text{PGL}(2, 13) \times Z_2$, then $X \cong LG_{312}^i$ with $i = 1, 2, 3, 4$.

Case 2. A has no solvable normal subgroup. For convenience, we still use $N$ to denote a minimal normal subgroup of $A$. Then $N$ is non-solvable. Let $N = T^k$ with $T$ a
non-abelian simple group and $k$ a positive integer. Then $T$ has at least 3-prime factors. Since $|T| = 2^{27}.3^5.5^2.7.p$, we have that $T$ is one of the simple groups listed in Proposition 2.3. By Proposition 2.1, $N$ has at most two orbits on $V(X)$, and hence $|N| = 24p|N_v|$ or $12p|N_v|$.

Assume that $k \geq 2$. Since $T$ is a non-abelian simple group, we have that $2^2 | |T|$ and $T_v \neq 1$. If $p > 7$, then $p \not| |T|$ because $p^2 \not| |N| = |T^k|$. It follows that $p$ divides the order of $X_N$. By Proposition 2.1, $N = T^k$ is semiregular and hence $T_v = 1$, a contradiction. Thus, $p \leq 7$. Recall that $p \geq 3$, $|T^k| = 2^{27}.3^5.5^2.7.p$ and $12.p | |T^k|$.

Let $p = 3$ or 5. Then $7^2 | |T^k|$ and $T$ is a simple $\{2, 3, 5\}$-group. By Proposition 2.3, $T \cong A_5$ or $A_6$. The normality of $N$ in $A$ implies that $N_v \leq A_v$. If $k \geq 3$ or $p = 3$, then $5^2 | |N_v|$ and hence $N_v$ has a subgroup isomorphic to $A_7$ by Proposition 2.2. It follows that $7 | |T|$, a contradiction. Thus, $k = 2$ and $p = 5$. Suppose that $T \cong A_5$. Then $N \cong A_5^2$ and $|N_v| = 30$ or 60. By Atlas [5], $N_v \cong \mathbb{Z}_3 \times D_{10}$ or $S_3 \times S_5$ for $|N_v| = 30$, and $N_v \cong S_3 \times D_{10}$, $A_4 \times \mathbb{Z}_5$ or $A_5$ for $|N_v| = 60$. Since $N \leq A$, we have that $N_v \leq A_v$. However, $A_v$ has no such normal subgroups by Proposition 2.2, a contradiction. Suppose that $T \cong A_6$. Then $N \cong A_6^2$ and $|N_v| = 2^{4}.3^3.5$ or $2^3.3^3.5$. By Atlas [5], $N_v \cong A_6 \times S_3$ or $A_5 \times F_{36}$ for $|N_v| = 2^4.3^3.5$, and $N_v \cong A_6 \times \mathbb{Z}_3$ or $A_5 \times F_{18}$ for $|N_v| = 2^3.3^3.5$. However, $A_v$ has no such normal subgroups, a contradiction.

Let $p = 7$. Then $|T^k| = 2^{27}.3^5.5^2.7^2$ and $12.7 | |T^k|$. It follows that $7 | |T|$ and $k = 2$. By Proposition 2.3, $T$ is isomorphic to one of the following groups:

- $\text{PSL}(2, 7)$, $\text{PSL}(2, 8)$, $\text{A}_7$, $\text{A}_8$, $\text{PSL}(3, 4)$.

If $T \cong \text{PSL}(2, 7)$, then $|N_v| = 2^4.3^7$ or $2^3.3^7$. By Atlas [5], for $|N_v| = 2^4.3^7$, $N_v \cong \text{PSL}(2, 7) \times \mathbb{Z}_2^2$; for $|N_v| = 2^3.3^7$, $N_v \cong \text{PSL}(2, 7)$, $S_4 \times \mathbb{Z}_7$ or $F_{21} \times D_8$. The normality of $N_v$ in $A_v$ implies that $A_v \cong \text{PSL}(2, 7)$ or $\text{PSL}(2, 7) \times S_4$ and $N_v \cong \text{PSL}(2, 7)$ by Proposition 2.2 and MAGMA [3]. It follows that $N$ is transitive on $V(X)$ and hence $N \cong \text{PSL}(2, 7)^2$ is arc-transitive. However, by MAGMA [3], there is no heptavalent symmetric coset graph of order 24.7 = 168 on $\text{PSL}(2, 7)^2$, a contradiction. If $T \cong \text{PSL}(2, 8)$, then $|N_v| = 2^4.3^3.7$ or $2^3.3^3.7$. By MAGMA [3], $N_v \cong \text{PSL}(2, 8) \times S_4$ or $\text{PSL}(2, 8) \times \mathbb{Z}_3$. However, $A_v$ has no such normal subgroup by Proposition 2.2, a contradiction. If $T \cong \text{A}_8$, then by MAGMA [3], the only possibility is that $|N_v| = 2^9.3^3.5^2.7$ and $N_v \cong \text{A}_8 \times S_5$. Similarly, $A_v$ has no such normal subgroup, a contradiction. If $T \cong \text{A}_7$ or $\text{PSL}(3, 4)$, then $|N_v| = |N|/24p$ or $|N|/12p$. By MAGMA [3], $A_7^2$ or $\text{PSL}(3, 4)^2$ has no subgroups of such orders, a contradiction.

Thus, $k = 1$ and $N = T$ is a non-abelian simple group. It follows that $2^2 | |N|$ and $N_v \neq 1$. By Proposition 2.1, $N$ has at most two orbits on $V(X)$ and $|N_v| = |N|/24p$ or $|N|/12p$.

Subcase 2.1. Suppose that $p = 3$. Then $|N| = 2^{27}.3^6.5^2.7$ and $2^2.3^2 | |N|$. By Proposition 2.3, $N$ is isomorphic to one of the following simple groups:

- $A_6$, $\text{PSL}(2, 8)$, $\text{PSU}(4, 2)$, $\text{PSU}(3, 3)$, $\text{A}_7$, $\text{A}_8$
- $A_9$, $A_{10}$, $\text{PSL}(3, 4)$, $\text{PSp}(6, 2)$, $J_2$, $\text{PSO}^+(8, 2)$.

Note that $|N_v| = |N|/24.3$ or $|N|/12.3$.

Let $N \cong \text{A}_6$. Then $N_v \cong \mathbb{Z}_5$ or $D_{10}$. By Proposition 2.2, $A_v$ has no normal subgroup isomorphic to $\mathbb{Z}_5$ or $D_{10}$, a contradiction.

Let $N \cong \text{PSL}(2, 8)$. Then by Atlas [5], $N_v \cong D_{14}$ or $\mathbb{Z}_7$. If $N_v \cong D_{14}$, then $N$ has two orbits on $V(X)$. Since $N_v \leq A_v$, we have that $A_v$ is solvable by Proposition 2.2. It
forces that $A/N$ is solvable. Note that $N$ is simple. Thus, $N \cap C_A(N) = 1$. It follows that $C_A(N) \cong C_A(N)N/N \leq A/N$ and hence $C_A(N)$ is solvable. By our assumption, $A$ has no solvable normal subgroup. Thus, $C_A(N) = 1$ and by ‘$N/C$ theorem’, $A \cong A/C_A(N) \lesssim \text{Aut}(N) \cong \text{PSL}(2,8) \times \mathbb{Z}_3$. Clearly, $A$ cannot act transitively on the two orbits of $N$, a contradiction. Thus, $N_v \cong \mathbb{Z}_7$ and $N$ is arc-transitive. By Construction 3.2 and Lemma 3.3, $X \cong \mathcal{L}G_{72}^1$ and $A \cong \text{PSL}(2,8) \times \mathbb{Z}_2$. This is impossible because $A$ has a solvable normal subgroup.

Let $N \cong \text{PSU}(4,2)$. Then by Atlas [5], $N_v \cong S_6$ or $A_6$. The normality of $N_v$ in $A_v$ implies that $A_v \cong A_7 \times A_6$, $S_7 \times S_6$ or $(A_7 \times A_6) \times \mathbb{Z}_2$. By our assumption, $C_A(N)$ is non-solvable. Since $NC_A(N) = N \times C_A(N)$ and $|A : A_vN| \leq 2$, we have that $C_A(N)$ has a non-solvable normal subgroup $M \cong A_7$ and $M$ is also normal in $A$. By Proposition 2.1, $M$ has at most two orbits on $V(X)$. It follows that $|M_v| = 35$ or $70$. However, $A_7$ has no subgroups of such orders, a contradiction.

Let $N \cong \text{PSU}(3,2)$. Then by Atlas [5], $|N_v| = 2^3 \cdot 3 \cdot 7$ and $N_v \cong \text{PSL}(2,7)$. It follows that $N$ has two orbits on $V(X)$. Since $N_v \leq A_v$, we have that $A_v/N_v$ is solvable and $A/N$ is solvable. It forces that $C_A(N) = 1$ and $A \cong A/C_A(N) \lesssim \text{Aut}((\text{PSU}(3,2)) \cong \text{PSU}(3,2) \times \mathbb{Z}_2$. Thus, $A \cong \text{PSU}(3,2) \times \mathbb{Z}_2$. By Construction 3.4, $X \cong \mathcal{L}G_{72}$.

Let $N \cong A_7, A_9, A_{10}, \text{PSL}(3,4)$, $J_2$ or $\text{PSU}^+(8,2)$. Then $|N_v| = |N|/243$ or $|N|/123$. However, by Atlas [5], $A_7$ has no subgroups of such orders, a contradiction.

Let $N \cong A_9$. Then by Atlas [5], $N_v \cong S_7$ or $A_7$. If $N_v \cong S_7$, then $N$ has two orbits on $V(X)$. Since $N_v \leq A_v$, we have that $A_v \cong S_7 \times S_6$ or $S_7$ by Proposition 2.2. For the former, $C_A(N)$ has a normal subgroup $H \cong A_6$, which is also normal in $A$. It follows that $H_v \cong \mathbb{Z}_5$ or $D_{10}$ is normal in $A_v$, a contradiction. For the latter, $C_A(N) = 1$ and $A \cong \text{Aut}(A_9) \cong S_9$. However, by MAGMA [3], there is no heptavalent symmetric coset graph of order 72 on $S_9$, a contradiction. If $N_v \cong A_7$, then $N$ is transitive on $V(X)$ and hence arc-transitive. We can deduce a similar contradiction by MAGMA [3] as above.

Let $N \cong \text{PSp}(6,2)$. Then by Atlas [5], $N_v \cong S_8$ or $A_8$. However, $A_v$ has no such normal subgroups by Proposition 2.2, a contradiction.

**Subcase 2.2.** Suppose that $p = 5$. Then $|N| \left| 2^{27} \cdot 3^5 \cdot 5^3 \cdot 7$. By Proposition 2.3, $N$ is isomorphic to one of the following simple groups:

- $A_5, A_6, \text{PSU}(4,2), A_7, A_8, A_9, A_{10}$,
- $\text{PSL}(3,4), \text{PSp}(6,2), J_2, \text{PSU}^+(8,2)$.

Note that $|N_v| = |N|/65$ or $|N|/125$.

Let $N \cong A_5$. Then $N_v = 1$ and $N$ has two orbits on $V(X)$. It follows that $A = (A_vN) \cdot \mathbb{Z}_2$. If $C_A(N) = 1$, then $A \cong A/C_A(N) \lesssim \text{Aut}(N) \cong S_5$. This is impossible because $7 \not| |S_5|$. Thus, $C_A(N) \neq 1$ and by our assumption, $C_A(N)$ is non-solvable. Since $C_A(N) \cong C_A(N)N/N \leq A/N \cong (A_v, \mathbb{Z}_2)$, we have that $A_v$ is non-solvable and $C_A(N)$ has a normal subgroup $H \cong A_6, A_7$ or $\text{PSL}(2,7)$, which is also normal in $A$, by Proposition 2.2. By Proposition 2.1, $H$ has at most two orbits on $V(X)$. It forces that $5 \not| |H|$ and $H \cong A_6$ or $A_7$. The information of Atlas [5] implies that $H_v \cong \mathbb{Z}_3$ or $S_3$ for $H \cong A_6$ and $H_v \cong F_{21}$ or $F_{42}$ for $H \cong A_7$. On the other hand, the normality of $H$ in $A$ implies that $H_v \subseteq A_v$. Note that $A_v$ is non-solvable. By Proposition 2.2, there are no such non-solvable vertex stabilizers containing normal subgroup isomorphic to $\mathbb{Z}_3, S_3, F_{21}$ or $F_{42}$, a contradiction.

Let $N \cong A_6$. Then $N_v \cong \mathbb{Z}_3$ or $S_3$. Clearly, $A_v$ has no subgroup isomorphic to $S_3$. Thus, $N_v \cong \mathbb{Z}_3$ and $N$ is transitive on $V(X)$. Since $\mathbb{Z}_3 \cong N_v \leq A_v$, we have that $A_v$ is solvable by Proposition 2.2. Note that $A = A_vN$. It follows that $C_A(N) = 1$, otherwise $C_A(N)$
is solvable which is contrary to our assumption. This implies that $A \cong A/C_A(N) \leq \text{Aut}(N) \cong A_6 \cdot \mathbb{Z}_2^2$. However, $7 \mid |\text{Aut}(N)|$, a contradiction.

Let $N \cong \text{PSU}(4, 2)$. Then $|N_v| = 2^3 \cdot 3^3$ or $2^4 \cdot 3^3$. By MAGMA [3], $\text{PSU}(4, 2)$ has no subgroup of order $2^3 \cdot 3^3$ and $|N_v| = 2^3 \cdot 3^3$. The subgroups of order $2^3 \cdot 3^3$ in $\text{PSU}(4, 2)$ have normal Sylow 3-subgroups. The normality of $N_v$ in $A_v$ implies that $A_v$ has a normal subgroup of order $3^3$, this is impossible by Proposition 2.2.

Let $N \cong A_7$. Then $N_v \cong F_{21}$ or $F_{42}$. Since $N_v \not\leq A_v$, we have that $A_v$ is solvable by Proposition 2.2. This forces that $C_A(N) = 1$ and $A \cong A/C_A(N) \leq S_7$. For $N_v \cong F_{21}$, we have that $N$ is arc-transitive, and for $N_v \cong F_{42}$, we have that $A = S_7$. By Construction 3.5, $X \cong S_6 \cdot G_{120}$ and $A \cong S_7$.

Let $N \cong A_8$. Then by Atlas [5], $|N_v| = 2^3 \cdot 3^7$ and $N_v \cong \mathbb{Z}_2^3 \rtimes F_{21}$ or $\text{PSL}(2, 7)$. Clearly, $N$ is arc-transitive. However, by MAGMA [3], there is no heptavalent symmetric coset graph of order 120 on $A_8$, a contradiction.

Let $N \cong A_9$. Then by Atlas [5], $|N_v| = 2^3 \cdot 3^3 \cdot 7$ and $N_v \cong \text{PSL}(2, 8) \rtimes \mathbb{Z}_3$. However, $A_v$ has no normal subgroup isomorphic to $\text{PSL}(2, 8) \rtimes \mathbb{Z}_3$ by Proposition 2.2, a contradiction.

Let $N \cong A_{10}$. Then by Atlas [5], $|N_v| = 2^4 \cdot 3^3 \cdot 7$ and $N_v \cong (A_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. However, $A_v$ has no such normal subgroup, a contradiction.

Let $N \cong \text{PSL}(3, 4)$. Then by Atlas [5], $N_v \cong \text{PSL}(2, 7)$ and $N$ is arc-transitive. By MAGMA [3], there is no such coset graph on $\text{PSL}(3, 4)$, a contradiction.

Let $N \cong \text{PSp}(6, 2)$. Then by Atlas [5], $N_v \cong \text{PSU}(3, 3) \rtimes \mathbb{Z}_2$. However, $A_v$ has no such normal subgroup, a contradiction.

Let $N \cong J_2$. Then by MAGMA [3], $J_2$ has no subgroup of order $2^4 \cdot 3^2 \cdot 5^7$ or $2^5 \cdot 3^2 \cdot 5^7$, a contradiction.

Let $N \cong P\Omega^+(8, 2)$. Then by MAGMA [3], $|N_v| = 2^9 \cdot 3^4 \cdot 5^7$ and $N_v \cong \text{PSp}(6, 2)$. Note that $N_v \not\leq A_v$. However, by Proposition 2.2, $A_v$ has no normal subgroup isomorphic to $\text{PSp}(6, 2)$, a contradiction.

Subcase 2.3. Suppose that $p = 7$. Then $|N| = 2^7 \cdot 3^5 \cdot 5^2 \cdot 7^2$ and $12 \cdot 7 \mid |N|$ or $24 \cdot 7 \mid |N|$. In particular, $7^2 \mid |N|$. By Proposition 2.3, $N$ is isomorphic to one of the following simple groups:

\[
\text{PSL}(2, 7), \text{PSL}(2, 8), \text{PSU}(3, 3), A_7, A_8, A_9, A_{10}, \text{PSL}(2, 49), \text{PSL}(3, 4), \text{PSp}(6, 2), J_2, P\Omega^+(8, 2).
\]

Let $N \cong \text{PSL}(2, 7)$. Then $N_v \cong \mathbb{Z}_2$ or 1. If $C_A(N) = 1$, then $A \cong A/C_A(N) \leq \text{Aut}(N) \cong \text{PGL}(2, 7)$, this is impossible because $7^2 \mid |A|$. Thus, $C_A(N) \neq 1$ and by our assumption, $C_A(N)$ is non-solvable. Assume that $N_v \cong \mathbb{Z}_2$. Then $N$ has two orbits on $V(X)$. It follows that $A = A_v \cdot N \cdot \mathbb{Z}_2$. Since $C_A(N) \cong C_A(N)N/N \cong A_vN/\mathbb{Z}_2N \cong A_v \cdot \mathbb{Z}_2/N_v$, we have that $A_v$ is non-solvable. On the other hand, $A_v$ has a normal subgroup $N_v \cong \mathbb{Z}_2$. By Proposition 2.2, $A_v \cong \mathbb{Z}_2^3 \rtimes \text{SL}(3, 2) \cong \text{ASL}(3, 2) \rtimes \mathbb{Z}_2$ and $A_v/N_v \cong \text{ASL}(3, 2)$. Thus, $C_A(N)$ is a normal non-solvable subgroup of $\text{ASL}(3, 2) \rtimes \mathbb{Z}_2$ and $C_A(N) \cong \text{ASL}(3, 2)$ or $\text{ASL}(3, 2).\mathbb{Z}_2$. It implies that $C_A(N)$ has a subgroup isomorphic to $\mathbb{Z}_2^3$, which is normal in $A$. This is contrary to our assumption that $A$ has no solvable normal subgroup. Assume that $N_v = 1$. Then $N$ is regular on $V(X)$ and $C_A(N) \cong C_A(N)N/N \cong A_vN/N \cong A_v$. Since $C_A(N)$ is non-solvable, we have that $A_v$ is also non-solvable. By our assumption, $C_A(N)$ has no solvable characteristic subgroup, and by Proposition 2.2, $A_v \cong \text{PSL}(2, 7)$, $A_7 \rtimes A_6$, $(A_7 \times A_6) \times \mathbb{Z}_2$ or $S_7 \times S_6$. For $A_v \cong \text{PSL}(2, 7)$, we have $C_A(N) \cong \text{PSL}(2, 7)$ and $A = \text{PSL}(2, 7)^2$, this is impossible. For the latter three cases, $C_A(N)$ has a
subgroup $K \cong A_7$ or $A_6$ and $K \unlhd A$. If $K \cong A_7$, then $|K_v| = 15$ or 30. This is impossible because $A_7$ has no subgroup of order 15 or 30. If $K \cong A_6$, then $7
ot| |K|$, and by Proposition 2.1, $K$ is semiregular. It forces that $|K| \mid 24-7$, a contradiction.

Let $N \cong PSL(2, 8)$. Then $N_v \cong S_3$ or $Z_3$. The normality of $N_v$ in $A_v$ implies that $N_v \cong Z_3$ and $A_v \cong F_21 \times Z_3$, $F_42 \times Z_3$ or $F_42 \times Z_6$, by Proposition 2.2. Since $N_v \cong Z_3$, we have that $N$ is transitive on $V(X)$. It follows that $A = A_vN$ and $C_A(N) \cong C_A(N)/N/N \not\cong A/N \cong A_v/N_v$. If $C_A(N) \neq 1$, then $C_A(N)$ is solvable, and hence $A$ has a solvable normal subgroup, contrary to our assumption. Thus, $C_A(N) = 1$ and by ‘$N/C$ theorem’, $A \cong A/C_A(N) \cong \text{Aut}(N) \cong PSL(2, 8) \times Z_3$. This is impossible because $7^2 \mid |PSL(2, 8) \times Z_3|$.

Let $N \cong PSU(3, 3), A_7, A_{10}, PSL(2, 49), PSL(3, 4), PSp(6, 2), J_2$ or $PSL^+(2, 2)$. Then $|N_v| = |N|/12-7$ or $|N|/24-7$. By MAGMA [3], $N$ has no subgroups of such orders, a contradiction.

Let $N \cong A_8, A_9$. Then by Atlas [5], $N_v \cong S_5$ for $N \cong A_8$, $N_v \cong A_6 \times Z_3$ or $(A_6 \times Z_3) \times Z_2$ for $N \cong A_9$. However, $A_v$ has no such normal subgroup, a contradiction. $Subcase$ 2.4. Suppose that $p > 7$. Then $|N| \mid 2^{27} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 13$ and $24 \cdot p \mid |N|$. By Proposition 2.3, $N$ is isomorphic to one of the following simple groups:

$$
PSL(2, 17), PSL(3, 3), PSL(2, 11), PSL(2, 13), PSL(2, 16), PSL(2, 19), PSL(2, 25), PSL(2, 27), PSL(2, 31), PSL(2, 81), PSL(2, 127), PSU(3, 4), PSU(3, 8), PSU(5, 2), PSp(4, 4), M_{11}, M_{12},
$$
$$\begin{aligned}
2F_4(2)' & , A_{11}, A_{12}, PSL(2, 29), PSL(2, 41), PSL(2, 71), PSL(2, 449), \\
PSL(2, 2^6) & , PSL(4, 4), PSL(5, 2), PSp(8, 2), M_{22}, PSp^-(8, 2), G_2(4).
\end{aligned}
$$

Let $N \cong PSL(2, 17)$. Then by Atlas [5], $N_v \cong A_4$ or $S_3$. Since $N_v \unlhd A_v$, we have that $A_v \cong PSL(2, 7) \times S_4$ by Proposition 2.2. As above, $C_A(N)$ has a subgroup $H \cong PSL(2, 7)$ and $H \unlhd A$. Since $p \not| |H|$, we have that $H$ has at least $p$ orbits on $V(X)$, and by Proposition 2.1, $H$ is semiregular and hence solvable, a contradiction.

Let $N \cong PSL(3, 3)$. Then $|N_v| = 2^2 \cdot 3^2$ or $2 \cdot 3^2$. By Atlas [5], $N_v$ has a characteristic subgroup isomorphic to $Z_3^2$. The normality of $N_v$ in $A_v$ implies that $A_v$ has a normal subgroup isomorphic to $Z_3^2$. This is impossible by Proposition 2.2.

Let $N \cong PSL(2, 11), PSL(2, 16), PSL(2, 25), M_{11}, M_{12}$ or $A_{12}$. Then by Atlas [5], $N_v \cong Z_5$ for $N \cong PSL(2, 11); N_v \cong D_{10}$ for $N \cong PSL(2, 16); N_v \cong Z_5^2 \times Z_2$ or $Z_3^2$ for $N \cong PSL(2, 15); N_v \cong A_5$ for $N \cong M_{11}; N_v \cong A_6$ or $A_6 \times Z_2$ for $N \cong M_{12}; N_v \cong A_{10}$ for $N \cong A_{12}$. However, by Proposition 2.2, $A_v$ has no such normal subgroups, a contradiction.

Let $N \cong PSL(2, 13)$. Then by Atlas [5], $N_v \cong Z_7$ and $N$ has two orbits on $V(X)$. Since $N_v \unlhd A_v$, we have that $A_v$ is solvable by Proposition 2.2. Note that $C_A(N) \cong C_A(N)/N/N \not\cong A/N$ and $|A/N : A_vN/N| \leq 2$. Thus, $C_A(N)$ is solvable. By our assumption, $C_A(N) = 1$ and by ‘$N/C$ theorem’, $A \cong A/C_A(N) \cong \text{Aut}(N) \cong PSL(2, 13)$. Since $A$ is arc-transitive, $A \cong PGL(2, 13)$. By Construction 3.6, $X \cong L_3^{11}$ with $i = 1, 2, 3, 4$ and $A \cong PGL(2, 7) \times Z_2$. This is impossible because $A$ has a solvable normal subgroup.

For the remaining simple groups listed above, with the calculation of MAGMA [3] or by checking the information of maximal subgroups in Atlas [5], we can deduce that all the groups do not have subgroups of order $|N_v| = |N|/12-7$ or $|N|/24\cdot p$, a contradiction.
Acknowledgements

This work was supported by the National Natural Science Foundation of China (11301154), and the Innovation Team Funding of Henan University of Science and Technology (2015XTD010).

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