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On the gaps in multiplicatively closed sets generated by at most two elements

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Abstract. We prove in the the main theorem, Theorem 3.2, that the multiplicatively closed subset of natural numbers, generated by two elements $1 < p_1 < p_2$ with $\alpha = \frac{\log\ p_1}{\log\ p_2}$ irrational, has arbitrarily large gaps by explicitly constructing large integer intervals, with known factorization for the endpoints in terms of generators p_1, p_2 obtained from the stabilization sequence of the irrational α (Definition 3.1). Example 5.6 is also illustrated. In the Appendix, for a finitely generated multiplicatively closed subset of natural numbers, we mention another constructive proof (refer to Theorem A.1) for arbitrarily large gap intervals, where the factorization of the right endpoint is not known in terms of generators unlike in the constructive proof of the main result. The suggested general Question 1.1 remains still open.

Keywords. Multiplicatively closed sets; log-rationality; gaps; Weyl-equidistributive criterion.

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1. Introduction

This article concerns about producing arbitrarily large gaps explicitly in a multiplicatively closed subset of natural numbers generated by at most two generators which is a particular case of the following open question.

1.1 *An open question*

Question 1.1. Let $S = \{1 < a_1 < a_2 < \cdots <\} \subset \mathbb{N}$ be a finitely generated multiplicatively closed infinite set generated by positive integers $d_1, d_2, \ldots, d_n$. How do we construct explicitly arbitrarily large integer intervals which do not contain any elements from the set S, with known factorization of the endpoints with explicit formulae, in terms of the positive integers $d_1, d_2, \ldots, d_n$?

Section 4 mentions some existence proofs of arbitrarily large gaps in certain subsets of integers that are present in the literature. However, constructive proofs, in particular, those which give the formulae for the endpoints of the arbitrarily large gap intervals have not been there. Here in this article, we are interested in one such constructive proof in section 5 and the consequences of the method of proof for generalization to more than two generators with regard to Question 1.1 in section 6. Section 5 answers Question 1.1 for two generators. For more than two generators, the general Question 1.1, about an explicit generation of large gap intervals with endpoints in the multiplicatively closed set is still open. In section 6, we mention a similar open question, Question 6.2, in the same context for three generators.

The Appendix is added to give another constructive proof of Theorem A.1, where the factorization of the right-end point of the gap interval is not known in terms of the generators.

Before we get to the main result, Theorem 3.2, with the required definitions, it is necessary to go through section 2 as a motivation to Definition 3.1 required for stating the main Theorem 3.2.

Finally, at the beginning of each section, we have motivated and summarized briefly the results in the section.

2. Irrationals and behaviour of rational approximations, arithmetic progressions, stabilization

We start this section by proving a theorem below on increasing gaps for the successive approximate inverses (defined later in Definition 3.1) as stated in the theorem below. The results in this section motivate the required definitions for the main result.

Theorem 2.1. *Let p, q be two positive integers with $gcd(p, q) = 1$, $p < q$. Consider the arithmetic progressions $p\mathbb{Z}^+$ and $q\mathbb{Z}^+$. Consider the sequence $(p\mathbb{Z}^+ \cup q\mathbb{Z}^+ \cup \{0\}) \cap \{0, 1, 2, \ldots, qp\}$ in the set $\{0, 1, 2, \ldots, qp\}$.*

$$\begin{aligned}
&l_0 = 0, p, 2p, 3p, \ldots, l_1 p, q, \\
&(l_1 + 1)p, (l_1 + 2)p, \ldots, l_2 p, 2q, \\
&(l_2 + 1)p, \ldots, l_i p, iq, \\
&(l_i + 1)p, \ldots, (q - 1)p, qp.
\end{aligned}$$

Now consider the sequence of numbers

$$\begin{aligned}
&\{l_0 = 0\} \cup \{l_j \mid q \geq j \geq 1, (l_j + 1)p - jq < \min_{0 \leq i < j}\{(l_i + 1)p - iq\}\} \\
&\quad = \{l_{j_1}, l_{j_2}, \ldots, l_{j_r}\} \\
&\quad \begin{cases} = \{0 = l_0 = l_{j_1} = p^{-1} - 1 \mod q\} \text{ if } p = 1 \\ = \{0 = l_{j_1} = l_0 < l_{j_2} = l_1 < l_{j_3} < \ldots < (l_{j_r} = p^{-1} - 1 \mod q)\} \\ \text{ if } p \neq 1. \end{cases}
\end{aligned}$$

Then the gaps $l_{j_{i+1}} - l_{j_i}$ in the above sequence is increasing.

Proof. If $p = 1$, then there is nothing to prove. So assume that $p > 1$. First, we observe that p is a unit in $\mathbb{Z}/q\mathbb{Z} = \{0, 1, 2, \ldots, q - 1\}$. The values $(l_i + 1)$ decrease to the inverse of p because the least possible value for $(l_i + 1)p - iq$ is one. If we consider the sequence of multiples $\{(l_{j_1}+1)p \mod q, (l_{j_2}+1)p \mod q, \ldots, (l_{j_r}+1)p \mod q\}$, then the values are distinct and decrease to 1 as multiples of p given by $0, p, 2p, \ldots, (q-1)p$ give rise to all residue classes modulo q. Now suppose we consider three consecutive elements in the sequence $l_{j_i}, l_{j_{i+1}}, l_{j_{i+2}}$. Then we have

$$\begin{aligned}(l_{j_i}+1)p &= k_{j_i}q + x_{j_i},\\ (l_{j_{i+1}}+1)p &= k_{j_{i+1}}q + x_{j_{i+1}},\\ (l_{j_{i+2}}+1)p &= k_{j_{i+2}}q + x_{j_{i+2}},\end{aligned}$$

and the residue classes satisfy $x_{j_i} > x_{j_{i+1}} > x_{j_{i+2}}$, and moreover, for any $t < l_{j_{i+1}} - l_{j_i}$, we have that if $(l_{j_i}+1+t)p = kq + x$, then $x > x_{j_i}$, because of the minimality condition. So we have

$$\begin{aligned}(l_{j_{i+1}}+1+t)p &= (l_{j_{i+1}} - l_{j_i})p + (l_{j_i}+1+t)p\\ &= (k_{j_{i+1}} - k_{j_i} + k)q + x_{j_{i+1}} - x_{j_i} + x.\end{aligned}$$

Now in the right-hand side, we have the following inequalities for the residue classes mod q.

$$\begin{aligned}&0 < x_{j_i} < q\\ &0 < x_{j_{i+1}} < q\\ &0 < x_{j_i} - x_{j_{i+1}} < q\\ &0 < x_{j_{i+1}} < x_{j_{i+1}} - x_{j_i} + x < x < q.\end{aligned}$$

This is a subtle argument about the residue classes. Hence we have $l_{j_{i+2}} > l_{j_{i+1}} + t$ for all $t < l_{j_{i+1}} - l_{j_i}$ and for $t = l_{j_{i+1}} - l_{j_i}$, we have $x = x_{j_{i+1}}$, so a candidate for the residue class is $(2x_{j_{i+1}} - x_{j_i})$ and

$$(l_{j_{i+1}}+1+t)p = (2k_{j_{i+1}} - k_{j_i})q + (2x_{j_{i+1}} - x_{j_i}).$$

Hence if $0 < (2x_{j_{i+1}} - x_{j_i})$, then the residue class is $(2x_{j_{i+1}} - x_{j_i})$ and

$$0 < (2x_{j_{i+1}} - x_{j_i}) = x_{j_{i+1}} + x_{j_{i+1}} - x_{j_i} < x_{j_{i+1}} < q.$$

So $l_{j_{i+2}} = 2l_{j_{i+1}} - l_{j_i}$ or $l_{j_{i+2}} - l_{j_{i+1}} = l_{j_{i+1}} - l_{j_i}$. Otherwise, if $0 < x_{j_{i+1}} < 2x_{j_{i+1}} < x_{j_i} < q$, then the residue class is given by $q + 2x_{j_{i+1}} - x_{j_i}$, and we observe that

$$q > q + 2x_{j_{i+1}} - x_{j_i} > x_{j_{i+1}} \text{ because } q > q + x_{j_{i+1}} - x_{j_i} > 0,$$

and we conclude that $l_{j_{i+2}} > 2l_{j_{i+1}} - l_{j_i}$ or $l_{j_{i+2}} - l_{j_{i+1}} > l_{j_{i+1}} - l_{j_i}$. It is also clear that the residue classes decrease to one. Now Theorem 2.1 follows. □

With the notation in the proof of Theorem 2.1, we prove a stabilization theorem for the sequence

$$\{1 = l_0 + 1 = l_{j_1} + 1, l_{j_2} + 1, \ldots, l_{j_r} + 1\}.$$

The theorem is stated as follows.

Theorem 2.2. *Let p_n, q_n be a sequence of positive integers with $gcd(p_n, q_n) = 1$ and suppose $\frac{p_n}{q_n}$ is a Cauchy sequence converging to an irrational number $0 < \alpha < 1$. Define as in the previous lemma, the sequence $l_i(n)$ and consider the set*

$$\{0 = l_{j_1(n)}(n) < l_{j_2(n)}(n) = l_1(n) < l_{j_3(n)}(n) < \cdots < l_{j_{r_n}(n)}(n) \\ = p_n^{-1} - 1 \bmod q_n\}.$$

The values $j_i(n)$ stabilize and also $l_{j_i(n)}(n)$ is eventually a constant as $n \longrightarrow \infty$ for a stabilized j_i.

Proof. We can assume that $p_n < q_n$ and $p_n \neq 1$. If $p_n = 1$ for infinitely many positive integer $n > 0$, then $\frac{p_n}{q_n} \longrightarrow 0$, which is a contradiction. We observe that $l_i(n) = \lfloor \frac{iq_n}{p_n} \rfloor$ and for fixed i, $l_i(n)$ is eventually $\lfloor \frac{i}{\alpha} \rfloor$ as $n \longrightarrow \infty$. Also, we have the sequence $j_i(n)$ which stabilizes as $n \longrightarrow \infty$ because in the inductive definition, we have $j_i(n)$ which satisfies the property that

$$(l_{j_i(n)}(n) + 1)p_n - j_i(n)q_n < \min_{0 \leq i < j_i(n)} \{(l_i(n) + 1)p_n - iq_n\},$$

or equivalently that

$$(l_{j_i(n)}(n) + 1)\frac{p_n}{q_n} - j_i(n) < \min_{0 \leq i < j_i(n)} \left\{(l_i(n) + 1)\frac{p_n}{q_n} - i\right\}.$$

Now if $n \longrightarrow \infty$, then we get that $(l_i(n) + 1)\frac{p_n}{q_n} - i \longrightarrow \left(\lfloor \frac{i}{\alpha} \rfloor + 1\right)\alpha - i$, which is independent of n. Now the independence of n here implies the stabilization of $j_i(n)$ follows as $n \longrightarrow \infty$. This completes the proof of Theorem 2.2. □

The theorem below along with Weyl equidistributive criterion, Theorem 2.4 establishes the increasing nature of gaps in the stabilized approximate inverses for a converging sequence of rationals to an irrational number.

Theorem 2.3. *Let p_n, q_n be a sequence of positive integers with $gcd(p_n, q_n) = 1$ with $p_n < q_n$ and suppose $\frac{p_n}{q_n}$ is a Cauchy sequence converging to an irrational number $0 < \alpha < 1$. Define as in the previous lemma, the sequence $l_i(n)$ and consider the set*

$$\{0 = l_{j_1(n)}(n) < l_{j_2(n)}(n) = l_1(n) < l_{j_3(n)}(n) < \cdots < l_{j_{r_n}(n)}(n) \\ = p_n^{-1} - 1 \bmod q_n\}.$$

Using the previous lemma, let $j_i = \lim_{n \longrightarrow \infty} j_i(n)$, $l_i = \lim_{n \longrightarrow \infty} l_i(n)$. Then we have

$$\lim_{i \longrightarrow \infty} l_{j_{i+1}} - l_{j_i} = \infty.$$

Proof. We can assume that $p_n \neq 1$ eventually. We observe that by using Theorem 2.2, we have for every $i \in \mathbb{N}$, $l_{j_{i+2}} - l_{j_{i+1}} \geq l_{j_{i+1}} - l_{j_i}$. If the above limit is not infinity (say equal to d), then eventually l_{j_i} forms an arithmetic progression with common difference d. Then

$(l_{j_i}+1)=\lceil \frac{j_i}{\alpha}\rceil$ is in arithmetic progression with common difference d. On the one hand, the sequence

$$\left\lceil \frac{j_i}{\alpha} \right\rceil \alpha - j_i \searrow 0,$$

and on the other hand, the sequence has a distribution if l_{j_i} are in arithemetic progression. Because, if $l_{j_i} = l_{j_{i_0}} + kd$ with $k \in \mathbb{N}$, then the fractional parts z_{j_i} are such that $\frac{j_i}{\alpha} + z_{j_i} = \lceil \frac{j_i}{\alpha} \rceil = l_{j_i} + 1$. Then we get $(l_{j_{i_0}} + kd + 1)\alpha - j_i = z_{j_i}\alpha \searrow 0$. However, the fractional parts $\{(l_{j_{i_0}} + kd + 1)\alpha - j_i\} = \{(l_{j_{i_0}} + kd + 1)\alpha\}$ are distributed in the unit interval uniformly as $k \in \mathbb{N}$ by Weyl's criterion, Theorem 2.4. So this is a contradiction and Theorem 2.3 follows. □

We mention Weyl's equidistributive criterion here (see also [7]).

Theorem 2.4. *Let α be positive irrational. Let $0 \le a \le b \le 1$. For $x \in \mathbb{R}^+$, let $\{x\}$ denote the fractional part of x. Then we have*

$$\frac{\#\{n \mid a \le \{n\alpha\} \le b, 1 \le n \le N\}}{N} \longrightarrow (b-a) \text{ as } N \longrightarrow \infty.$$

3. Statement of the main theorem

Let $\mathbb{N}$ denote the set of natural numbers. Let $\mathbb{P} = \{2, 3, 5, \dots, \}$ denote the set of primes. Here we give using techniques from number theory, geometry and finite fields, a constructive proof of the main result, Theorem 3.2, where the factorization of the endpoints of the gap intervals in terms of generators are known.

Before we state the main result, we need a definition.

DEFINITION 3.1 (Stabilization sequence of an irrational using sequences of approximate inverses)

Let $0 < \alpha < 1$ be irrational. Let $\frac{p_n}{q_n}, gcd(p_n, q_n) = 1, p_n < q_n$ be any sequence of positive rationals converging to α. Now consider the arithmetic progressions $p_n\mathbb{Z}^+$ and $q_n\mathbb{Z}^+$. Consider the sequence

$$\left((p_n\mathbb{Z}^+ \cup q_n\mathbb{Z}^+ \cup \{0\}) \cap \{0, 1, 2, \dots, q_n p_n\}\right) \subset \{0, 1, 2, \dots, q_n p_n\}$$

given as follows:

$$\begin{aligned}
&l_0(n) = 0,\ p_n, 2p_n, 3p_n, \dots, l_1(n)p_n, q_n,\\
&(l_1(n)+1)p_n, (l_1(n)+2)p_n, \dots, l_2(n)p_n, 2q_n,\\
&(l_2(n)+1)p_n, \dots, l_i(n)p_n, iq_n,\\
&(l_i(n)+1)p_n, \dots, (q_n-1)p_n,\ p_nq_n.
\end{aligned}$$

For every $n \in \mathbb{N}$, define the sequence of numbers

$$\{l_{j_1(n)}(n), l_{j_2(n)}(n), \dots, l_{j_{r_n}(n)}(n)\}$$

given as follows. We define $j_1(n) = 0, l_0(n) = 0$. Now let $j(n) \in \{j_1(n), j_2(n), \ldots, j_{r_n}(n)\}$.
The defining/characterizing property for $l_{j(n)}(n)$ is given by

$$\boxed{q_n \geq j(n) \geq 1, (l_{j(n)}(n) + 1)p_n - j(n)q_n < \min\{(l_i(n) + 1)p_n - iq_n \mid 0 \leq i < j(n)\}.}$$

We have

$$\{l_{j_1(n)}(n), l_{j_2(n)}(n), \ldots, l_{j_{r_n}(n)}(n)\}$$
$$\begin{cases} = \{0 = l_0(n) = l_{j_1(n)}(n) = p_n^{-1} - 1 \mod q_n\} \text{ if } p_n = 1 \\ = \{0 = l_{j_1(n)}(n) = l_0(n) < l_{j_2(n)}(n) = l_1(n) < l_{j_3(n)}(n) \\ \quad < \ldots < (l_{j_{r_n}(n)}(n) = p_n^{-1} - 1 \mod q_n)\} \text{ if } p_n \neq 1. \end{cases}$$

The sequence

$$\{1 = l_0 + 1 = l_{j_1(n)}(n) + 1, l_{j_2(n)}(n) + 1, \ldots, l_{j_{r_n}(n)}(n) + 1\}$$

is the sequence of approximate inverses of $p_n \mod q_n$. By using Theorems 2.1, 2.2, 2.3, we conclude that the gaps

- $l_{j_{i+1}(n)}(n) - l_{j_i(n)}(n)$ in the above sequence is increasing.
- The values $j_i(n)$ stabilize and also $l_{j_i(n)}(n)$ is eventually a constant as $n \longrightarrow \infty$ for a stabilized j_i. Let the stabilized constant be denoted by l_{j_i}.
- We have $\lim_{i \longrightarrow \infty}(l_{j_{i+1}} - l_{j_i}) \nearrow \infty$.

This stabilized approximate inverse sequence $\{l_{j_i} + 1 : i \in \mathbb{N}\}$ is called the stabilization sequence of the irrational α.

Now we state the main result.

Theorem 3.2. *Let p_1, p_2 be two positive integers, which are not log-rational to each other with $p_1 < p_2$. Let $\alpha = \frac{\log(p_1)}{\log(p_2)}$ be the associated irrational less than one. Let $\{s_i : i \in \mathbb{N}\}$ be the stabilization sequence of α. Let $t_i = \lfloor s_i \alpha \rfloor$. Then*

(1) *The set of integers*

$$\left\{p_2^{t_{i+1}-1}, p_2^{t_{i+1}-1} + 1, \ldots, p_1^{s_i} p_2^{t_{i+1}-t_i-1}\right\}$$

contains no element in the multiplicatively closed set generated by the positive integers p_1, p_2 apart from the end values.

(2) *The limit of the gaps $(p_1^{s_i} p_2^{t_{i+1}-t_i-1} - p_2^{t_{i+1}-1})$ tends to infinity as $i \longrightarrow \infty$.*

This theorem is illustrated with Example 5.6. As an application of Theorem 3.2, we have the following corollary which is proved after the proof of Theorem 3.2.

COROLLARY 3.3

Let A be a finite set of positive natural numbers. Let $\{1\} \neq T \subset \mathbb{N}$ be a singleton set or $T = \{p_1 < p_2\} \subset \mathbb{N}\backslash\{1\}$ with $\frac{\log(p_1)}{\log(p_2)}$ irrational. Let $S = \{1 < a_1 < a_2 < \cdots <$

$a_n < \cdots\} \subset \mathbb{N}$ be the infinite multiplicatively closed set generated by A. Suppose the multiplicatively closed set $S \subset \langle T\rangle$ is the multiplicatively closed set generated by the set T. Then we have

- $\limsup_{n\longrightarrow\infty}(a_{n+1} - a_n) = \infty$.
- *We have explicit expressions for the endpoints of certain arbitrarily large gap intervals in the set S using the generators of T.*

4. Short survey

In this section, we mention a short survey and summarize the proof of the main Theorem 3.2. The distribution of integers with exactly k-distinct prime factors has been studied by many authors. It was first shown by Landau [3] that for a fixed $k \geq 1$, the function defined by

$$\pi(x,k) = \sum_{n\leq x} f_k(n),$$

where $f_k(n) = 1$ if n has exactly k-prime factors and 0 otherwise satisfies

$$\pi(x,k) = \left(\frac{x}{\log\ x}\right)\frac{(\log\ \log\ x)^{k-1}}{(k-1)!}(1+o(1)). \tag{4.1}$$

The other authors who have obtained a similar or better asymptotic expressions are Sathe [4,5], Selberg [6], Hensely [1] and, Hildebrand and Tenenbaum [2].

Let $\{p_1, p_2, \ldots, p_k\}$ be any set of k-distinct primes. Let $S_{\{p_1,p_2,\ldots,p_k\}}$ be the multiplicatively closed set generated by 1, and numbers which have exactly and all the factors from $\{p_1, p_2, \ldots, p_k\}$. Let $\mathcal{C}$ be the collection of all k-subsets of prime numbers. Consider the set

$$S_k = \bigcup_{c\in\mathcal{C}} S_c.$$

Using any of the results, say the result by Landau [3] about asymptotics of $\pi(x,k)$, we conclude that there are arbitrarily large gaps in S. We observe here that by using equation (4.1), we have

$$\lim_{x\longrightarrow\infty} \frac{\pi(x,k)}{x} = 0.$$

If the gaps were bounded, then we have

$$\liminf_{x\longrightarrow\infty} \frac{\pi(x,k)}{x} > 0$$

which would be a non-zero constant. Hence the gaps must be arbitrarily large in the set S_k.

With an additional bit of effort on the result of Landau [3], we can extend and conclude arbitrarily large gaps for the set

$$\bigcup_{i=1}^{k} S_i.$$

Now choose a base, say $b = 2$. If we use asymptotics for a multiplicatively closed set T generated by primes $\{p_1, p_2, \ldots, p_k\}$, then we get for large x, the following inequality:

$$\left\lceil \frac{\log_b\ x}{\sum_{i=1}^{k}\log_b\ p_i}\right\rceil \leq \#(T\cap[1,x]) \leq \prod_{i=1}^{k}\left\lceil\frac{\log_b\ x}{\log_b\ p_i}\right\rceil. \tag{4.2}$$

Again we have

$$\lim_{x \longrightarrow \infty} \frac{\#(T \cap [1, x])}{x} = 0$$

from which we will be able to conclude that there are arbitrarily large gaps in T.

However, in this article, we give a constructive proof for the multiplicatively closed sets, which are contained in doubly generated multipicatively closed sets with known generators. First, we consider multiplicatively closed sets generated by two primes, or more generally, two positive integers (>1) which are not log-rational to each other. We note here that the multiplicatively closed set can contain numbers with single prime factors unlike the set, which is considered in the result by Landau [3]. Using the technique of rational approximation and stabilization of the sequences of approximate inverses and increasing gaps between two such successive ones, we explicitly construct by locating large intervals of natural numbers, which do not contain any element in the given multiplicatively closed set, thereby proving the main result, Theorem 3.2, as given below.

4.1 *Summary of the proof of the main result*

In section 5, we prove our main theorem, Theorem 3.2. We consider a multiplicatively closed set S generated by two positive numbers $p_1 < p_2$, which are log irrational to each other, i.e., $\frac{\log(p_1)}{\log(p_2)}$ is irrational. We apply Theorems 2.2 and 2.3 to $\frac{\log(p_1)}{\log(p_2)}$ for a suitable sequence of positive rationals obtained in Lemma 5.3 and conclude increasing gaps for the stabilized sequence. Then we locate integer intervals in Lemma 5.4 of arbitrarily large size, which has no elements from the multiplicatively closed set S. This finally proves our main Theorem 3.2 and Corollary 3.3.

5. Proof of the main theorem and construction of arbitrarily large gaps

Before we prove the main Theorem 3.2, we prove Lemmas 5.1, 5.3 and 5.4.

Lemma 5.1. *Let $p_1 < p_2$ be two natural numbers such that $gcd(p_1, p_2) = 1$. Then*

- *either $p_1 = 1$ or*
- *$\frac{\log(p_1)}{\log(p_2)}$ is irrational.*

Proof. If $p_1 = 1$, then there is nothing to prove. Suppose $\frac{\log(p_2)}{\log(p_1)} = \frac{m}{n}$ for some positive integers $m, n > 0$. Then we have $p_2^n = p_1^m$, a contradiction to unique factorization into primes. So $\frac{\log(p_1)}{\log(p_2)}$ is irrational. □

DEFINITION 5.2

We say a pair $(p_1, p_2) \in \mathbb{N}^2$ is an irrational pair if $p_1 \neq 1 \neq p_2 \neq p_1$ and $\frac{\log(p_1)}{\log(p_2)}$ is an irrational. For example, a GCD-one pair $(p_1, p_2) \in \mathbb{N}^2$, where $p_1 \neq 1 \neq p_2$ is an irrational pair.

Lemma 5.3. *Let* $(p_1, p_2) \in \mathbb{N}^2$ *be such that* $p_1 < p_2$ *and it is an irrational pair. Let* $\alpha = \frac{\log(p_1)}{\log(p_2)} < 1$ *and* $x_2(i) = \lceil \frac{i}{\alpha} \rceil$. *For every positive integer* i, *let*

$$z_i = -i + x_2(i)\alpha.$$

Define a subsequence with the property that

$$z_{k_j} < z_{k_{j-1}} = \min\{z_1, z_2, \dots, z_{k_j - 1}\}.$$

Then

(1) $z_{k_j} \searrow 0$,
(2) $k_j - k_{j-1}$ *is increasing.*
(3) $\lim_{j \longrightarrow \infty} (k_j - k_{j-1}) = \infty$.

Proof. First, we define a sequence of number parts $0 < y_i < 1$ defined by the equation

$$y_i + \frac{i}{\alpha} = \left\lceil \frac{i}{\alpha} \right\rceil = x_2(i).$$

Define a subsequence with the property that

$$y_{k_j} < y_{k_{j-1}} = \min\{y_1, y_2, \dots, y_{k_j - 1}\}.$$

Since the number parts of $\{\frac{i}{\alpha} \mid i \in \mathbb{N}\}$ is also dense in [0, 1], we have that $y_{k_j} \searrow 0$.

We also have for every i, $z_i = y_i\alpha$. So, z_{k_j} also satisfies the property that

$$z_{k_j} < z_{k_{j-1}} = \min\{z_1, z_2, \dots, z_{k_j - 1}\}.$$

Now we have $x_2(k_j) = \frac{k_j}{\alpha} + y_{k_j}$ and $y_{k_j} \searrow 0$. Since $y_{k_j}\alpha < 1$, then

$$\lfloor x_2(k_j)\alpha \rfloor = k_j.$$

Now we apply Theorems 2.2 and 2.3 as follows. The sequence

$$\frac{k_j}{x_2(k_j)} = \alpha - \frac{z_{k_j}}{x_2(k_j)} \longrightarrow \alpha \text{ as } j \longrightarrow \infty.$$

In Theorems 2.2 and 2.3, we choose α which is an irrational satisfying the property that $0 < \alpha < 1$ and the sequence of rationals $\frac{k_j}{x_2(k_j)} = \frac{p_j}{q_j} \longrightarrow \alpha$ as $j \longrightarrow \infty$, where $gcd(p_j, q_j) = 1$. Now by the very definition of z_{k_j} and using the properties of stabilization and eventual invariance, we have

- $x_2(k_j) - x_2(k_{j-1})$ is increasing,
- $\lim_{j \longrightarrow \infty} (x_2(k_j) - x_2(k_{j-1})) = \infty$.

This implies that we also have

- $k_j - k_{j-1}$ is increasing,
- $\lim_{j \longrightarrow \infty} (k_j - k_{j-1}) = \infty$.

This proves Lemma 5.3. □

Next we prove Lemma 5.4.

Lemma 5.4. *Let $p_1 < p_2$ be two positive integers such that (p_1, p_2) is an irrational pair. Using the notations of Lemma 5.3, we have, for any integer $0 \leq t < k_{j+1} - k_j$, there are no numbers of the form $p_1^b p_2^a$ in the set of consecutive integers*

$$\left\{p_2^{k_j+t} + 1, \ldots, p_1^{x_2(k_j)} p_2^t - 1\right\}.$$

Proof. Let $\alpha = \frac{\log(p_1)}{\log(p_2)} < 1$. Here we use the following fact. We have $\lfloor x_2(k_j)\alpha \rfloor = k_j$. Suppose, if there exists a number $p_2^{k_j+t} < p_1^b p_2^a < p_1^{x_2(k_j)} p_2^t$, then we have

$$\begin{aligned} &k_j + t < a + b\alpha < t + x_2(k_j)\alpha < t + k_j + 1 \\ &\quad \to k_j < -t + a + b\alpha < x_2(k_j)\alpha < k_j + 1 \\ &\quad \to k_j + t - a < b\alpha < k_j + t - a + 1. \end{aligned}$$

So we have $b \neq 0$. Similarly $b \neq x_2(k_j)$. If $b = x_2(k_j)$, then we get that $k_j = k_j + t - a$, which implies that $t = a$. Hence $p_1^b p_2^a$ is an endpoint, which is not considered.
Let $b\alpha = k_j + t - a + z$. Consider the case $k_j + t - a < k_{j+1}$. Then, by the definition of $z_{k_j}, z_{k_{j+1}}$ and since $b \neq x_2(k_j)$, we have $z \geq z_{k_j} > z_{k_{j+1}}$. Hence

$$k_j < k_j + z = -t + a + b\alpha < x_2(k_j)\alpha = k_j + z_{k_j} < k_j + 1.$$

We get $z < z_{k_j}$, which is a contradiction. Hence we must have $k_j + t - a \geq k_{j+1}$, which implies that $t \geq k_{j+1} - k_j + a \geq k_{j+1} - k_j$, which is again a contradiction to the hypothesis $0 \leq t < k_{j+1} - k_j$. This proves Lemma 5.4. □

Using Lemmas 5.1, 5.3 and 5.4, we prove our main theorem, Theorem 3.2.

Proof. Suppose $S = \{p_1^i p_2^j \mid i, j \geq 0\}$ and $\frac{\log(p_1)}{\log(p_2)}$ is irrational. Then in Lemma 5.4, we substitute $t = k_{j+1} - k_j - 1$ and we obtain a gap of size

$$0 < p_1^{x_2(k_j)} p_2^t - p_2^{k_j+t} = p_2^t (p_1^{x_2(k_j)} - p_2^{k_j}) \geq p_2^t = p_2^{k_{j+1}-k_j-1}.$$

Hence the limit superior of the gaps tend to infinity in the multiplicatively closed set S, by using Lemma 5.3. Now Theorem 3.2 follows. □

Note 5.5. Via the sequence k_j we know the factorization in terms of generators p_1, p_2 of the endpoints $p_2^{k_j+t}, p_1^{x_2(k_j)} p_2^t$ for $0 \leq t < k_{j+1} - k_j$. These give rise to all gap intervals whose initial point is a power of p_2.

Now we prove Corollary 3.3.

Proof. Suppose $S = \{1, f, f^2, \ldots\}$ is a singly generated multiplicatively closed set. Then we immediately have $\lim_{j \longrightarrow \infty} (f^{j+1} - f^j) = \infty$.

Now suppose that $S = \{g_1^i g_2^j \mid i, j \geq 0\}$, and $\frac{\log(g_1)}{\log(g_2)} = \frac{m}{n}$ is rational. Then S is contained in a singly generated multiplicatively closed set $T = \langle f \rangle$, where f is given as follows. If $g_1 \prod_{i=1}^{k} p_i^{a_i}$ and $g_2 = \prod_{i=1}^{k} p_i^{b_i}$, then we have

$$\begin{aligned} f &= \prod_{i=1}^{k} p_i^{\frac{a_i}{gcd(a_1,a_2,\ldots,a_k)}} = \prod_{i=1}^{k} p_i^{\frac{a_i n}{gcd(a_1 n,a_2 n,\ldots,a_k n)}} \\ &= \prod_{i=1}^{k} p_i^{\frac{b_i m}{gcd(b_1 m,b_2 m,\ldots,b_k m)}} = \prod_{i=1}^{k} p_i^{\frac{b_i}{gcd(b_1,b_2,\ldots,b_k)}} \\ g_1 &= f^{gcd(a_1,a_2,\ldots,a_k)}, \; g_2 = f^{gcd(b_1,b_2,\ldots,b_k)}. \end{aligned}$$

So there exist arbitrarily large gaps in S as well which can be expressed using the generator of T.

Finally, we can use Theorem 3.2 whenever $S \subset T =< p_1, p_2 >$ and $\frac{\log(p_1)}{\log(p_2)}$ is irrational. □

Here, we give an example illustrating the ideas used to prove Theorem 3.2.

Example 5.6 (*Main example*). Consider the irrational $\frac{\log(2)}{\log(3)}$. The first few terms of the sequence k_j, which is defined by the fractional parts

$$z_{k_j} < z_{k_{j-1}} = \min\{z_1, z_2, \ldots, z_{k_j - 1}\}$$

is given by

$$\{1, 3, 5, 17, 29, 41, 94, 147, 200, 253, 306, 971, 1636, 2301, 2966, 3631, 4296, 4961, 5626, 6291, 6956, 7621, 8286, 8951, 9616, 10281, 10946, 11611, 12276, 12941, 13606, 14271, 14936, 15601, 47468, 79335, 190537\}.$$

The corresponding first few terms of the sequence $x_2(k_j)$ is given by

$$\{2, 5, 8, 27, 46, 65, 149, 233, 317, 401, 485, 1539, 2593, 3647, 4701, 5755, 6809, 7863, 8917, 9971, 11025, 12079, 13133, 14187, 15241, 16295, 17349, 18403, 19457, 20511, 21565, 22619, 23673, 24727, 75235, 125743, 301994\}.$$

The first few terms of the rational approximation seqence to α is given by

$$\left\{\frac{1}{2},\frac{3}{5},\frac{5}{8},\frac{17}{27},\frac{29}{46},\frac{41}{65},\frac{94}{149},\frac{147}{233},\frac{200}{317},\frac{253}{401},\frac{306}{485},\frac{971}{1539},\frac{1636}{2593},\frac{2301}{3647},\frac{2966}{4701},\right.$$
$$\frac{3631}{5755},\frac{4296}{6809},\frac{4961}{7863},\frac{5626}{8917},\frac{6291}{9971},\frac{6956}{11025},\frac{7621}{12079},\frac{8286}{13133},\frac{8951}{14187},\frac{9616}{15241},$$
$$\frac{10281}{16295},\frac{10946}{17349},\frac{11611}{18403},\frac{12276}{19457},\frac{12941}{20511},\frac{13606}{21565},\frac{14271}{22619},\frac{14936}{23673},\frac{15601}{24727},$$
$$\left.\frac{47468}{75235},\frac{79335}{125743},\frac{190537}{301994}\right\}.$$

This seqence for approximate inverses for the fraction $\frac{190537}{301994}$ is given by

$$\{1, 2, 5, 8, 27, 46, 65, 149, 233, 317, 401, 485, 1539, 2593, 3647, 4701, 5755, 6809, 7863, 8917, 9971, 11025, 12079, 13133, 14187, 15241, 16295, 17349, 18403, 19457, 20511, 21565, 22619, 23673, 24727, 75235, 125743, 301994\}.$$

We note that it matches with $x_2(k_j)$. Actually, this can be obtained for any suitable rational approximation sequence for α. The first few gaps of intervals with the prime factorization of endpoints of the gap intervals are of the form

$$\{p_2^{k_{j+1}-1}, p_2^{k_{j+1}-1}+1,\ldots,p_1^{x_2(k_j)}p_2^{k_{j+1}-k_j-1}\}$$

which using this method is given by

$$\{3^2,3^2+1,\ldots,2^2 3^1\},\{3^4,3^4+1,\ldots,2^5 3^1\},\{3^{16},3^{16}+1,\ldots,2^8 3^{11}\},$$
$$\{3^{28},3^{28}+1,\ldots,2^{27}3^{11}\},\{3^{40},3^{40}+1,\ldots,2^{46}3^{11}\},$$
$$\{3^{93},3^{93}+1,\ldots,2^{65}3^{52}\},$$
$$\{3^{146},3^{146}+1,\ldots,2^{149}3^{52}\},\{3^{199},3^{199}+1,\ldots,2^{233}3^{52}\},$$
$$\{3^{252},3^{252}+1,\ldots,2^{317}3^{52}\},$$
$$\{3^{305},3^{305}+1,\ldots,2^{401}3^{52}\},\{3^{970},3^{970}+1,\ldots,2^{485}3^{664}\},$$
$$\{3^{1635},3^{1635}+1,\ldots,2^{1539}3^{664}\},$$
$$\{3^{2300},3^{2300}+1,\ldots,2^{2593}3^{664}\},\{3^{2965},3^{2965}+1,\ldots,2^{3647}3^{664}\},$$
$$\{3^{3630},3^{3630}+1,\ldots,2^{4701}3^{664}\},\{3^{4295},3^{4295}+1,\ldots,2^{5755}3^{664}\},$$
$$\{3^{4960},3^{4960}+1,\ldots,2^{6809}3^{664}\},\{3^{5625},3^{5625}+1,\ldots,2^{7863}3^{664}\},$$
$$\{3^{6290},3^{6290}+1,\ldots,2^{8917}3^{664}\},\{3^{6955},3^{6955}+1,\ldots,2^{9971}3^{664}\},$$
$$\{3^{7620},3^{7620}+1,\ldots,2^{11025}3^{664}\},\{3^{8285},3^{8285}+1,\ldots,2^{12079}3^{664}\},$$
$$\{3^{8950},3^{8950}+1,\ldots,2^{13133}3^{664}\},\{3^{9615},3^{9615}+1,\ldots,2^{14187}3^{664}\},$$
$$\{3^{10280},3^{10280}+1,\ldots,2^{15241}3^{664}\},\{3^{10945},3^{10945}+1,\ldots,2^{16295}3^{664}\},$$
$$\{3^{11610},3^{11610}+1,\ldots,2^{17349}3^{664}\},\{3^{12275},3^{12275}+1,\ldots,2^{18403}3^{664}\},$$
$$\{3^{12940},3^{12940}+1,\ldots,2^{19457}3^{664}\},\{3^{13605},3^{13605}+1,\ldots,2^{20511}3^{664}\},$$
$$\{3^{14270},3^{14270}+1,\ldots,2^{21565}3^{664}\},\{3^{14935},3^{14935}+1,\ldots,2^{22619}3^{664}\},$$
$$\{3^{15600},3^{15600}+1,\ldots,2^{23673}3^{664}\},\{3^{47467},3^{47467}+1,\ldots,2^{24727}3^{31866}\},$$
$$\{3^{79334},3^{79334}+1,\ldots,2^{75235}3^{31866}\},$$
$$\{3^{190536},3^{190536}+1,\ldots,2^{125743}3^{111201}\}.$$

6. On a generalization of this method to more than two generators

In the proof of the main Theorem 3.2, we know factorizations in terms of generators of both the endpoints of the gap interval via the stabilization sequence. In an attempt to answer this open Question 1.1, in section 6, we mention that a generalization of the proof of Theorem 3.2 is not directly feasible by proving Lemma 6.1 via an example. In particular, we prove a lemma which says that the same technique may or may not be extendable for explicit generation of gap intervals. We mention the open Question 6.2 below.

Lemma 6.1. *Let* $G = \{p_1 < p_2 < \cdots < p_l\}$ *be a finite set of primes. Let k be any positive integer. Consider the monoid* $T = \{\sum_{i=1}^{l-1} x_i \frac{\log(p_i)}{\log(p_l)} \mid x_i \in \mathbb{N} \cup \{0\}\}$. *Consider the set* $T_k = T \cap (k, k+1)$. *Let* $z_k = \min(T_k - k)$ *and let* z_{k_j} *be a monotone decreasing sequence converging to zero constructed from* z_k *defined by the property that*

$$z_{k_j} < z_{k_{j-1}} = \min\{z_1, z_2, \ldots, z_{k_j - 1}\}.$$

Then the sequence of integers $\{k_{j+1} - k_j : j \in \mathbb{N}\}$ *need not be increasing.*

Proof. Consider the following example. Let $\{p_1 = 2 < p_2 = 3 < p_3 = 5\}$. By calculating the logarithm of numbers to the base 5 in the sequence $\{2^i 3^j \mid 0 \le i, j \le 50\}$ or by actually showing inequalities we obtain

- $k_0 = 0, z_{k_0} = z_0 = \log_5(2) - 0$.
- $k_1 = 1, z_{k_1} = z_1 = \log_5(2.3) - 1$.
- $k_2 = 2, z_{k_2} = z_2 = \log_5(3^3) - 2$.
- $k_3 = 3, z_{k_3} = z_3 = \log_5(2^7) - 3$.
- $k_4 = 7, z_{k_4} = z_7 = \log_5(2^2.3^9) - 7$.
- $k_5 = 8, z_{k_5} = z_8 = \log_5(2^{17}.3) - 8$.
- $k_6 = 13, z_{k_6} = z_{13} = \log_5(2^8.3^{14}) - 13$.
- $k_7 = 14, z_{k_7} = z_{14} = \log_5(2^{23}.3^6) - 14$.

We can show the inequalities $z_{k_0} > z_{k_1} > z_{k_2} > z_{k_3} > z_{k_4} > z_{k_5} > z_{k_6} > z_{k_7}$ and

$$\begin{aligned} k_1 - k_0 = k_2 - k_1 = k_3 - k_2 = 1 < k_4 - k_3 = 4 > k_5 - k_4 = 1 \\ < k_6 - k_5 = 5 > k_7 - k_6 = 1, \end{aligned}$$

which is not increasing. This proves the lemma. However we mention that it is possible that $\limsup_{j \longrightarrow \infty}(k_{j+1} - k_j) = \infty$, which additionally requires a proof. □

The open question is stated as follows.

Question 6.2. Let p_1, p_2, p_3 be three primes.

(1) Can we express the endpoints of arbitrarily large gap intervals in the multiplicatively closed set $M = \langle p_1, p_2, p_3 \rangle$ using $\frac{\log(p_i)}{\log(p_j)}$, $1 \le i \ne j \le 3$?

(2) Let $n = p_1^i p_2^j p_3^k$ for some fixed $i, j, k \in \mathbb{N} \cup \{0\}$. Is there a formula for expressing the next number in M using p_1, p_2, p_3, i, j, k?

For two generators $1 < p_1 < p_2$ (which need not be prime) such that $\frac{\log(p_1)}{\log(p_2)}$ is irrational we refer to Lemma 5.4 and Note 5.5 for a formula for the next number of p_2^j in the doubly generated multiplicatively closed set.

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Appendix

This section is added to give another constructive proof in Theorem A.1 and discuss advantages and disadvantages with respect to the above given constructive proof in Note A.2.

Theorem A.1 (Another constructive proof). *The multiplicatively closed subset of* $\mathbb{N}$ *generated by finitely many positive integers S has arbitrarily large gaps.*

Proof. We give here another constructive proof in this theorem. Let K be an arbitrary positive integer. Let $n_1, n_2, \ldots, n_k$ be the generators of the multiplicatively closed set. Define $\lceil \frac{\log(K)}{\log(n_i)} \rceil = a_i$. Then we have for all $t_i \geq a_i$,

$$t_i \in \mathbb{N}, n_i^{t_i+1} - n_i^{t_i} = n_i^{t_i}(n_i - 1) \geq n_i^{t_i} \geq K.$$

The gap between $n_1^{t_1} n_2^{t_2} \ldots n_k^{t_k}$ and the next number l in the set $S_1 S_2 \ldots S_k$ is at least K. Let $l = n_1^{s_1} n_2^{s_2} \ldots n_k^{s_k}$ be the next number. Then there is at least one $i = i_0$ such that $s_i > t_i$. Let $a = \frac{n_1^{s_1} n_2^{s_2} \ldots n_k^{s_k}}{n_i^{s_i}}$ and $b = \frac{n_1^{t_1} n_2^{t_2} \ldots n_k^{t_k}}{n_i^{t_i}}$. So, we get that $n_1^{s_1} n_2^{s_2} \ldots n_k^{s_k} - n_1^{t_1} n_2^{t_2} \ldots n_k^{t_k} = n_i^{s_i} a - n_i^{t_i} b = n_i^{t_i}(n_i^{s_i - t_i} a - b) \geq n_i^{t_i} \geq K$. □

Note A.2. The difference between this constructive proof and the other constructive proof is that we do not exactly know the factorization of the right endpoint l of this gap interval in terms of the generators. However we were able to locate a point $n_1^{a_1} n_2^{a_2} \ldots n_k^{a_k}$ and a gap interval of size at least K with this integer as the left endpoint for every positive integer $K > 0$.

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