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An identity on generalized derivations involving multilinear polynomials in prime rings

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Abstract. Let R be a prime ring of characteristic different from 2 with its Utumi ring of quotients U, extended centroid C, $f(x_1, \dots, x_n)$ a multilinear polynomial over C, which is not central-valued on R and d a nonzero derivation of R. By $f(R)$, we mean the set of all evaluations of the polynomial $f(x_1, \dots, x_n)$ in R. In the present paper, we study $b[d(u), u] + p[d(u), u]q + [d(u), u]c = 0$ for all $u \in f(R)$, which includes left-sided, right-sided as well as two-sided annihilating conditions of the set $\{[d(u), u] : u \in f(R)\}$. We also examine some consequences of this result related to generalized derivations and we prove that if F is a generalized derivation of R and d is a nonzero derivation of R such that

$$F^2([d(u), u]) = 0$$

for all $u \in f(R)$, then there exists $a \in U$ with $a^2 = 0$ such that $F(x) = xa$ for all $x \in R$ or $F(x) = ax$ for all $x \in R$.

Keywords. Derivation; generalized derivation; prime ring; extended centroid; Utumi quotient ring.

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1. Introduction

A ring R is said to be *prime* if for any $a, b \in R$, $aRb = \{0\}$ implies either $a = 0$ or $b = 0$ and is said to be *semiprime* if for any $a \in R$, $aRa = \{0\}$ implies $a = 0$. Let $Z(R)$ denote the center of R and U be the Utumi ring of quotients of R and $C = Z(U)$. The symbols $[x, y]$ denote the Lie commutator $xy - yx$ for any $x, y \in R$. By a *derivation*, we mean an additive mapping $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$.

Several authors found a number of results investigating the relationship between the behaviour of additive mappings defined on a prime (or semiprime) ring R and the structure of R. Posner [17] proved that if R is a prime ring and d a nonzero derivation on R such that

$[d(r), r] \in Z(R)$, then R is commutative. Several authors have generalized the Posner's result.

Lee and Lee in [13] proved that if $[d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)]_k = 0$ for all $x_1, \ldots, x_n$ in some nonzero ideal of R, then $f(x_1, \ldots, x_n)$ is central-valued on R, except when char(R) $= 2$ and R satisfies $s_4(x_1, x_2, x_3, x_4)$, the standard identity in four variables. Later on, De Filippis and Di Vincenzo [5] considered the situation $\delta([d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)]) = 0$ for all $x_1, \ldots, x_n \in R$, where d and δ are two derivations of R. The statement of De Filippis and Di Vincenzo's theorem is the following:

Let K be a noncommutative ring with unity, R a prime K-algebra of characteristic different from 2, d and δ nonzero derivations of R, and $f(x_1, \ldots, x_n)$ a multilinear polynomial over K. If $\delta([d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)]) = 0$ for all $x_1, \ldots, x_n \in R$, then $f(x_1, \ldots, x_n)$ is central-valued on R.

It is natural to consider the situation when derivation δ is replaced by δ^2, that is, $\delta^2([d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)]) = 0$ for all $x_1, \ldots, x_n \in R$. In the present paper, we investigate a more general case replacing δ^2 with F^2, where F is a generalized derivation of R.

On the other hand, Dhara [7] studied $[d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] = 0$ for all $x_1, \ldots, x_n \in \rho$ in prime ring R, where d is a derivation of R and ρ is a nonzero right ideal of R.

We will continue the study of analogue problems involving generalized derivations on the appropriate subsets of prime rings. An additive mapping $F : R \to R$ associated with a derivation d on R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, is said to be *generalized derivation*. For some fixed $a, b \in U$, an additive mapping $F : R \to R$ defined as $F(x) = ax + xb$ for all $x \in R$ is an example of generalized derivation. In [2], the following result was obtained:

Let R be a prime ring of characteristic different from 2 with extended centroid C, $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. If d is a derivation of R, and F is a generalized derivation of R such that $F([d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)]) = 0$ for all $x_1, \ldots, x_n \in R$, then either $F = 0$ or $d = 0$.

In this line of investigation, in [4], De Filippis and Di Vincenzo proved the following:

Let R be a prime algebra over a commutative ring K with unity, and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over K, not central valued on R. Suppose that d is a nonzero derivation of R, and F is a nonzero generalized derivation of R such that $d([F(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)]) = 0$ for all $r_1, \ldots, r_n \in R$. If the characteristic of R is different from 2, then one of the following holds:

(1) there exists $\lambda \in C$, the extended centroid of R, such that $F(x) = \lambda x$, for all $x \in R$;
(2) there exists $a \in U$, the Utumi quotient ring of R, and $\lambda \in C = Z(U)$ such that $F(x) = ax + xa + \lambda x$ for all $x \in R$, and $f(x_1, \ldots, x_n)^2$ is central-valued on R.

Furthermore, Tiwari *et al.* [18] investigated $d([F^2(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)]) = 0$ for all $r_1, \ldots, r_n \in R$, where d is a nonzero derivation of R, and F is a generalized derivation of R.

In the present paper, we prove the following:

Main Theorem. *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and $f(x_1,\ldots,x_n)$ a non-central multilinear polynomial over the extended centroid C. If d is a nonzero derivation of R and F is a generalized derivation of R such that*

$$F^2([d(f(x_1,\ldots,x_n)), f(x_1,\ldots,x_n)]) = 0$$

for all $x_1,\ldots,x_n \in R$, then there exists $a \in U$ with $a^2 = 0$ such that $F(x) = xa$ for all $x \in R$ or $F(x) = ax$ for all $x \in R$.

Here we give an example which shows that in our result, the primeness of the ring is essential.

Example. Define $R = \left\{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in \mathbb{Z}\right\}$ and a multilinear polynomial $f(r,s) = rs$. We see that R is a ring under usual operations and $f(r,s)$ is not central valued on R. Also, note that R is not a prime ring. Now we define maps $d, F, g : R \to R$ such that $d\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$, $F\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$ and $g\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -y \\ 0 & 0 \end{pmatrix}$. Notice that d is a nonzero derivation on R and F is a generalized derivation associated to the derivation g on R. It can easily be seen that $F^2([d(f(r,s)), f(r,s)]) = 0$ for all $r, s \in R$. Thus R satisfies the hypothesis of the main theorem. However, the conclusion of the main theorem does not hold as g is a nonzero derivation of R.

2. Preliminaries

In what follows, R always denotes a prime ring and U denotes the Utumi ring of quotients of R. $f(x_1,\ldots,x_n)$ denotes the multilinear polynomial over C which is in the form

$$f(x_1,\ldots,x_n) = x_1x_2\cdots x_n + \sum_{\sigma\in S_n,\sigma\neq id} \alpha_\sigma x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)},$$

for some $\alpha_\sigma \in C$ and S_n the symmetric group of degree n.

The definition and axiomatic formulation of Utumi quotient ring U can be found in [1] and [3].

We have the following properties which we need:

(1) $R \subseteq U$;
(2) U is a prime ring with identity;
(3) The center of U is denoted by C and is called the extended centroid of R. C is a field.

Moreover, we recall some known facts.

Fact 1. Let $\mathcal{K}$ be an algebra over a field $\mathbb{E}$. A generalized polynomial identity (GPI) of $\mathcal{K}$ is a polynomial expression g in non commutative indeterminates and fixed coefficients from $\mathcal{K}$ between the indeterminates such that g vanishes on all replacements by elements of $\mathcal{K}$. The generalized polynomial in the context of Utumi quotient ring U is defined as follows:

Suppose that V is a set of C-independent vectors of U and $Y = \{y_1, y_2, y_3, \ldots\}$ is a countable set, where y_i are non commuting indeterminates. Let $C\langle Y\rangle$ be the free algebra over C in the set Y. Consider $\mathcal{W} = U *_C C\langle Y\rangle$, the free product of U and $C\langle Y\rangle$ over C. The elements of $\mathcal{W}$ are called generalized polynomials. An element $h \in \mathcal{W}$ of the form $h = s_0 x_1 s_1 x_2 s_2 \ldots x_n s_n$, where $\{s_0, \ldots, s_n\} \subseteq U$ and $\{x_1, \ldots, x_n\} \subseteq Y$ is said to be a monomial. Therefore, each $g \in \mathcal{W}$ can be represented as a finite sum of monomials. A V-monomial is of the form $e = v_0 x_1 v_1 x_2 v_2 \ldots x_n v_n$, where $\{v_0, \ldots, v_n\} \subseteq V$ and $\{x_1, \ldots, x_n\} \subseteq Y$. Thus an element $g \in \mathcal{W}$ can be written as $g = \sum_i \beta_i e_i$, where $\beta_i \in C$ and e_i are V-monomials. An element $g \in \mathcal{W}$ is trivial if and only if $\beta_i = 0$ for each i. For more details, we refer the reader to [1], [3].

Fact 2. If I is a two-sided ideal of R, then R, I and U satisfy the same generalized polynomial identities (GPIs) with coefficients in U (see [3]).

Fact 3. Every derivation d of R can be uniquely extended to a derivation of U (see Proposition 2.5.1 in [1]).

Fact 4. If I is a two-sided ideal of R, then R, I and U satisfy the same differential identities (see [14]).

Fact 5. Let d be a derivation on R. By $f^d(x_1, \ldots, x_n)$, $f^{d^2}(x_1, \ldots, x_n)$ and $f^{d^3}(x_1, \ldots, x_n)$, we denote the polynomials obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma)$, $d^2(\alpha_\sigma)$ and $d^3(\alpha_\sigma)$, respectively. Then we have

$$\begin{aligned} d(f(x_1, \ldots, x_n)) =& f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d(x_i), \ldots, x_n), \\ d^2(f(x_1, \ldots, x_n)) =& f^{d^2}(x_1, \ldots, x_n) + 2\sum_i f^d(x_1, \ldots, d(x_i), \ldots, x_n) \\ &+ \sum_i f(x_1, \ldots, d^2(x_i), \ldots, x_n) \\ &+ \sum_{i \neq j} f(x_1, \ldots, d(x_i), \ldots, d(x_j), \ldots, x_n) \end{aligned}$$

and

$$\begin{aligned} d^3(f(x_1, \ldots, x_n)) = & f^{d^3}(x_1, \ldots, x_n) + 3\sum_i f^{d^2}(x_1, \ldots, d(x_i), \ldots, x_n) \\ &+ 3\sum_i f^d(x_1, \ldots, d(x_i), \ldots, d(x_j), \ldots, x_n) \\ &+ 3\sum_i f^d(x_1, \ldots, d^2(x_i), \ldots, x_n) \\ &+ \sum_{i \neq j \neq k} f(x_1, \ldots, d(x_i), \ldots, d(x_j), \ldots, d(x_k), \ldots, x_n) \\ &+ 2\sum_{i \neq j} f(x_1, \ldots, d^2(x_i), \ldots, d(x_j), \ldots, x_n) \end{aligned}$$

$$+ 2\sum_{i\neq j} f(x_1,\ldots,d(x_i),\ldots,d^2(x_j),\ldots,x_n)$$
$$+ \sum_{i} f(x_1,\ldots,d^3(x_i),\ldots,x_n).$$

3. The case when F is inner

In this section, we study all the possible situation of annihilating condition of the set $\{[d(x),x] | x \in f(R)\}$, where d is a derivation of R. For any subset S of R, denote by $r_R(S)$ the right annihilator of S in R, that is, $r_R(S) = \{x \in R | Sx = 0\}$ and $l_R(S)$ the left annihilator of S in R, that is, $l_R(S) = \{x \in R | xS = 0\}$. If $r_R(S) = l_R(S)$, then $r_R(S)$ is called an annihilator ideal of R and is written as $\mathrm{ann}_R(S)$.

In [6], De Filippis and Di Vincenzo studied the left annihilating condition of the set $\{[d(x),x] | x \in f(R)\}$. More precisely, they proved that if R is a prime ring of $\mathrm{char}(R) \neq 2$ and d is a nonzero derivation of R satisfying $a[d(x),x] = 0$ for all $x \in f(R)$, then $a = 0$.

Now we will study a more general situation, involving left sided, right sided as well as two-sided annihilating conditions. More specifically, we study the situation $b[d(x),x] + p[d(x),x]q + [d(x),x]c = 0$ for all $x \in f(R)$, where $b, c, p, q \in R$.

First we consider that d is an inner derivation of R, that is, $d(x) = [a,x]$ for all $x \in R$. Then

$$b[d(f(r)), f(r)] + p[d(f(r)), f(r)]q + [d(f(r)), f(r)]c = 0$$

gives

$$\begin{aligned}&b\big(af(r)^2 - 2f(r)af(r) + f(r)^2a\big) + p(af(r)^2\\&\quad - 2f(r)af(r) + f(r)^2a\big)q\\&\quad + \big(af(r)^2 - 2f(r)af(r) + f(r)^2a\big)c = 0,\end{aligned}$$

that is,

$$\begin{aligned}&baf(r)^2 - 2bf(r)af(r) + bf(r)^2a + paf(r)^2q\\&\quad - 2pf(r)af(r)q + pf(r)^2aq\\&\quad + af(r)^2c - 2f(r)af(r)c + f(r)^2ac = 0\end{aligned}$$

for any $r = (r_1,\ldots,r_n) \in R^n$. We rewrite it as

$$\begin{aligned}&a_1f(r)^2 - 2a_2f(r)a_3f(r) + a_2f(r)^2a_3 + a_4f(r)^2a_5\\&\quad - 2a_6f(r)a_3f(r)a_5 + a_6f(r)^2a_7\\&\quad + a_3f(r)^2a_8 - 2f(r)a_3f(r)a_8 + f(r)^2a_9 = 0\end{aligned}$$

for any $r = (r_1,\ldots,r_n) \in R^n$, where $a_1 = ba, a_2 = b, a_3 = a, a_4 = pa, a_5 = q, a_6 = p, a_7 = aq, a_8 = c, a_9 = ac$. Now we study this situation in a matrix ring.

We need the following:

Lemma 3.1 [4, *Lemma* 1]. *Let F be an infinite field and $k \geq 2$. If $A_1, \ldots, A_n$ are not scalar matrices in $M_k(F)$ then there exists some invertible matrix $P \in M_k(F)$ such that any matrices $PA_1P^{-1}, \ldots, PA_nP^{-1}$ have all non-zero entries.*

PROPOSITION 3.2

Let $R = M_k(F)$ be the ring of all $k \times k$ matrices over the infinite field F, $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over F and $a_1, a_2, \ldots, a_9 \in R$. If

$$\begin{aligned} &a_1 f(r)^2 - 2a_2 f(r)a_3 f(r) + a_2 f(r)^2 a_3 + a_4 f(r)^2 a_5 \\ &\quad - 2a_6 f(r)a_3 f(r)a_5 + a_6 f(r)^2 a_7 \\ &\quad + a_3 f(r)^2 a_8 - 2f(r)a_3 f(r)a_8 + f(r)^2 a_9 = 0 \end{aligned}$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then either a_3 or a_5 or a_6 is central.

Proof. By the hypothesis, we have

$$\begin{aligned} &a_1 f(r_1, \ldots, r_n)^2 - 2a_2 f(r_1, \ldots, r_n)a_3 f(r_1, \ldots, r_n) + a_2 f(r_1, \ldots, r_n)^2 a_3 \\ &\quad + a_4 f(r_1, \ldots, r_n)^2 a_5 - 2a_6 f(r_1, \ldots, r_n)a_3 f(r_1, \ldots, r_n)a_5 \\ &\quad + a_6 f(r_1, \ldots, r_n)^2 a_7 \\ &\quad + a_3 f(r_1, \ldots, r_n)^2 a_8 - 2f(r_1, \ldots, r_n)a_3 f(r_1, \ldots, r_n)a_8 \\ &\quad + f(r_1, \ldots, r_n)^2 a_9 = 0. \end{aligned}$$

Suppose that $a_3 \notin Z(R)$, $a_5 \notin Z(R)$ and $a_6 \notin Z(R)$. Then we shall prove that this case leads to a contradiction.

Since $a_3 \notin Z(R)$, $a_5 \notin Z(R)$ and $a_6 \notin Z(R)$, by Lemma 3.1, there exists a F-automorphism ϕ of $M_k(F)$ such that $\phi(a_3)$, $\phi(a_5)$ and $\phi(a_6)$ have all nonzero entries. Clearly, R satisfies the GPI,

$$\begin{aligned} &\phi(a_1) f(r_1, \ldots, r_n)^2 - 2\phi(a_2) f(r_1, \ldots, r_n)\phi(a_3) f(r_1, \ldots, r_n) \\ &\quad + \phi(a_2) f(r_1, \ldots, r_n)^2 \phi(a_3) + \phi(a_4) f(r_1, \ldots, r_n)^2 \phi(a_5) \\ &\quad - 2\phi(a_6) f(r_1, \ldots, r_n)\phi(a_3) f(r_1, \ldots, r_n)\phi(a_5) \\ &\quad + \phi(a_6) f(r_1, \ldots, r_n)^2 \phi(a_7) \\ &\quad + \phi(a_3) f(r_1, \ldots, r_n)^2 \phi(a_8) - 2f(r_1, \ldots, r_n)\phi(a_3) f(r_1, \ldots, r_n)\phi(a_8) \\ &\quad + f(r_1, \ldots, r_n)^2 \phi(a_9) = 0. \end{aligned} \tag{1}$$

As usual, by e_{ij}, $1 \leq i, j \leq k$, we denote the matrix unit whose (i, j)-entry is equal to 1 and all its other entries are equal to 0. Since $f(x_1, \ldots, x_n)$ is non-central, by [13] (see also [15]), there exist $s_1, \ldots, s_n \in M_k(F)$ and $\beta \in F \backslash \{0\}$ satisfying $f(s_1, \ldots, s_n) = \beta e_{\mathrm{st}}$ with $s \neq t$. Moreover, since the set $\{f(y_1, \ldots, y_n) : y_1, \ldots, y_n \in M_k(F)\}$ is invariant under the action of all F-automorphisms of $M_k(F)$, for any $i \neq j$, there exists $u_1, \ldots, u_n \in M_k(F)$ such that $f(u_1, \ldots, u_n) = e_{ij}$. Hence by (1) we have

$$\begin{aligned}
&\phi(a_1)e_{ij}^2 - 2\phi(a_2)e_{ij}\phi(a_3)e_{ij} + \phi(a_2)e_{ij}^2\phi(a_3) + \phi(a_4)e_{ij}^2\phi(a_5) \\
&\quad - 2\phi(a_6)e_{ij}\phi(a_3)e_{ij}\phi(a_5) + \phi(a_6)e_{ij}^2\phi(a_7) \\
&\quad + \phi(a_3)e_{ij}^2\phi(a_8) - 2e_{ij}\phi(a_3)e_{ij}\phi(a_8) + e_{ij}^2\phi(a_9) = 0.
\end{aligned}$$

Multiplying left side and right side by e_{ij}, we obtain $2e_{ij}\phi(a_6)e_{ij}\phi(a_3)e_{ij}\phi(a_5)e_{ij} = 0$. Since $\mathrm{char}(R) \neq 2$, we have $\phi(a_6)_{ji}\phi(a_3)_{ji}\phi(a_5)_{ji} = 0$. This is a contradiction as $\phi(a_3)$, $\phi(a_5)$ and $\phi(a_6)$ have all nonzero entries. Thus we conclude that either a_3 or a_5 or a_6 is central. □

PROPOSITION 3.3

Let $R = M_k(F)$ be the ring of all matrices over the field F with $\mathrm{char}(R) \neq 2$, *$f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over F and $a_1, a_2, \ldots, a_9 \in R$. If*

$$\begin{aligned}
&a_1 f(r)^2 - 2a_2 f(r)a_3 f(r) + a_2 f(r)^2 a_3 + a_4 f(r)^2 a_5 \\
&\quad - 2a_6 f(r)a_3 f(r)a_5 + a_6 f(r)^2 a_7 \\
&\quad + a_3 f(r)^2 a_8 - 2f(r)a_3 f(r)a_8 + f(r)^2 a_9 = 0
\end{aligned}$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then either a_3 or a_5 or a_6 is central.

Proof. If F is an infinite field, then by Proposition 3.2, we get the desired result. Next, we assume that F is finite.

Let E be an infinite field extension of the field F. Suppose that $\bar{R} = M_k(E) \cong R \otimes_F E$. Note that the multilinear polynomial $f(r_1, \ldots, r_n)$ is central-valued on R if and only if it is central-valued on $\bar{R}$. R satisfies the GPI,

$$\begin{aligned}
\Psi(r_1, \ldots, r_n) = {} & a_1 f(r_1, \ldots, r_n)^2 - 2a_2 f(r_1, \ldots, r_n)a_3 f(r_1, \ldots, r_n) \\
&+ a_2 f(r_1, \ldots, r_n)^2 a_3 + a_4 f(r_1, \ldots, r_n)^2 a_5 \\
&- 2a_6 f(r_1, \ldots, r_n)a_3 f(r_1, \ldots, r_n)a_5 + a_6 f(r_1, \ldots, r_n)^2 a_7 \\
&+ a_3 f(r_1, \ldots, r_n)^2 a_8 - 2f(r_1, \ldots, r_n)a_3 f(r_1, \ldots, r_n)a_8 \\
&+ f(r_1, \ldots, r_n)^2 a_9 = 0
\end{aligned}$$

which is multi-homogeneous of multi-degree $(2, \ldots, 2)$ in the indeterminates $r_1, \ldots, r_n$. Thus the complete linearization of $\Psi(r_1, \ldots, r_n)$ is a multilinear generalized polynomial $\Phi(r_1, \ldots, r_n, r_1, \ldots, r_n)$ in $2n$ indeterminates. Clearly, $\Phi(r_1, \ldots, r_n, r_1, \ldots, r_n) = 2^n \Psi(r_1, \ldots, r_n)$.

Note that the multilinear polynomial $\Phi(r_1, \ldots, r_n, r_1, \ldots, r_n)$ is a generalized polynomial identity for both R and $\bar{R}$. Since $\mathrm{char}(F) \neq 2$, we obtain $\Psi(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in \bar{R}$. Hence by Proposition 3.2, the proof of proposition follows. □

Lemma 3.4. *Let R be a prime ring of characteristic different from 2 with extended centroid C and $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over C. Suppose that for some $a_1, a_2, \ldots, a_9 \in R$,*

$$\begin{aligned} &a_1 f(r)^2 - 2a_2 f(r)a_3 f(r) + a_2 f(r)^2 a_3 + a_4 f(r)^2 a_5 \\ &\quad - 2a_6 f(r)a_3 f(r)a_5 + a_6 f(r)^2 a_7 \\ &\quad + a_3 f(r)^2 a_8 - 2f(r)a_3 f(r)a_8 + f(r)^2 a_9 = 0 \end{aligned}$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then either a_3 or a_5 or a_6 is central.

Proof. Since R satisfies the generalized polynomial identity (GPI),

$$\begin{aligned} g(x_1, \ldots, x_n) = {} & a_1 f(x_1, \ldots, x_n)^2 - 2a_2 f(x_1, \ldots, x_n)a_3 f(x_1, \ldots, x_n) \\ & + a_2 f(x_1, \ldots, x_n)^2 a_3 + a_4 f(x_1, \ldots, x_n)^2 a_5 \\ & - 2a_6 f(x_1, \ldots, x_n)a_3 f(x_1, \ldots, x_n)a_5 + a_6 f(x_1, \ldots, x_n)^2 a_7 \\ & + a_3 f(x_1, \ldots, x_n)^2 a_8 - 2f(x_1, \ldots, x_n)a_3 f(x_1, \ldots, x_n)a_8 \\ & + vf(x_1, \ldots, x_n)^2 a_9 = 0 \end{aligned} \tag{2}$$

for all $x_1, \ldots, x_n \in R$. Assume that $a_3 \notin C$, $a_5 \notin C$ and $a_6 \notin C$. By Fact 2, R and U satisfy the same GPI, U satisfies $g(x_1, \ldots, x_n) = 0$. Suppose that $g(x_1, \ldots, x_n)$ is a trivial GPI for U. Let $\mathcal{W} = U *_C C\{x_1, x_2, \ldots, x_n\}$, the free product of U and $C\{x_1, \ldots, x_n\}$, the free C-algebra in noncommuting indeterminates $x_1, x_2, \ldots, x_n$. So $g(x_1, \ldots, x_n)$ is a zero element in $\mathcal{W} = U *_C C\{x_1, \ldots, x_n\}$. In equation (2), the term $-2a_6 f(x_1, \ldots, x_n)a_3 f(x_1, \ldots, x_n)a_5$ appears nontrivially, implying that

$$-2a_6 f(x_1, \ldots, x_n)a_3 f(x_1, \ldots, x_n)a_5 = 0 \in \mathcal{W}.$$

This implies that either a_3 or a_5 or a_6 is central.

Now assume that $g(x_1, \ldots, x_n)$ is a non-trivial GPI for U. In case C is infinite, we have $g(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in U \otimes_C \bar{C}$, where $\bar{C}$ is the algebraic closure of C. Moreover, both U and $U \otimes_C \bar{C}$ are prime and centrally closed algebras [8]. Hence, we substitute U or $U \otimes_C \bar{C}$ in place of R according to C finite or infinite respectively. Without loss of generality, we may suppose that $C = Z(R)$ and R is a centrally closed C-algebra. Using Martindale's theorem [16], R is then a primitive ring having nonzero Socle soc(R) with C as the associated division ring. Hence by Jacobson's theorem [10, p. 75], R is isomorphic to a dense ring of linear transformations of some vector space V over C.

First, suppose that V is finite dimensional over C, that is, $\dim_C V = k$. By density of R, we have $R \cong M_k(C)$. Since $f(r_1, \ldots, r_n)$ is not central-valued on R, R must be noncommutative and so $k \geq 2$. In this case, by Proposition 3.3, we get that either a_3 or a_5 or a_6 is in C, a contradiction.

If V is infinite dimensional over C, then for any $e^2 = e \in \text{soc}(R)$, we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. Since a_3, a_5 and a_6 are not in C, there exist $h_1, h_2, h_3 \in \text{soc}(R)$ such that $[a_3, h_1] \neq 0$, $[a_5, h_2] \neq 0$ and $[a_6, h_3] \neq 0$. By Litoff's theorem [9], there exists idempotent $e \in \text{soc}(R)$ such that $a_3h_1, h_1a_3, a_5h_2, h_2a_5, a_6h_3, h_3a_6, h_1, h_2,$

$h_3 \in eRe$. Since R satisfies GPI, it follows that

$$\begin{aligned}
&e\{a_1 f(ex_1e,\ldots,ex_ne)^2 - 2a_2 f(ex_1e,\ldots,ex_ne)a_3 f(ex_1e,\ldots,ex_ne)\\
&+a_2 f(ex_1e,\ldots,ex_ne)^2a_3 + a_4 f(ex_1e,\ldots,ex_ne)^2a_5\\
&-2a_6 f(ex_1e,\ldots,ex_ne)a_3 f(ex_1e,\ldots,ex_ne)a_5 + a_6 f(ex_1e,\ldots,ex_ne)^2a_7\\
&+va_3 f(ex_1e,\ldots,ex_ne)^2a_8 - 2f(ex_1e,\ldots,ex_ne)a_3 f(ex_1e,\ldots,ex_ne)a_8\\
&+f(ex_1e,\ldots,ex_ne)^2a_9\}e = 0,
\end{aligned}$$

where the subring eRe satisfies

$$\begin{aligned}
&ea_1ef(x_1,\ldots,x_n)^2 - 2ea_2ef(x_1,\ldots,x_n)ea_3ef(x_1,\ldots,x_n)\\
&+ea_2ef(x_1,\ldots,x_n)^2ea_3e + ea_4ef(x_1,\ldots,x_n)^2ea_5e\\
&-2ea_6ef(x_1,\ldots,x_n)ea_3ef(x_1,\ldots,x_n)ea_5e + ea_6ef(x_1,\ldots,x_n)^2ea_7e\\
&+ea_3ef(x_1,\ldots,x_n)^2ea_8e - 2f(x_1,\ldots,x_n)ea_3ef(x_1,\ldots,x_n)ea_8e\\
&+f(x_1,\ldots,x_n)^2ea_9e = 0.
\end{aligned}$$

Then by the above finite dimensional case, either ea_3e or ea_5e or ea_6e is the central element of eRe. This leads to a contradiction, since $a_3h_1 = (ea_3e)h_1 = h_1ea_3e = h_1a_3$, $a_5h_2 = (ea_5e)h_2 = h_2(ea_5e) = h_2a_5$ and $a_6h_3 = (ea_6e)h_3 = h_3(ea_6e) = h_3a_6$.

Hence we have proved that either a_3 or a_5 or a_6 is in C. □

Theorem 3.5. *Let R be a prime ring of characteristic different from 2, $f(x_1,\ldots,x_n)$ a non-central multilinear polynomial over C and d a nonzero derivation of R. Suppose that for some $b, c, p, q \in R$, $b[d(u),u] + p[d(u),u]q + [d(u),u]c = 0$ for all $u \in f(R)$. Then one of the following holds*:

(1) *$b, p, pq + c \in C$ and $b + pq + c = 0$;*
(2) *$b + pq, q, c \in C$ and $b + pq + c = 0$.*

Proof. Let d be an inner derivation of R, that is, $d(x) = [a,x]$ for all $x \in R$. By hypothesis, R satisfies

$$b[[a,f(r)],f(r)] + p[[a,f(r)],f(r)]q + [[a,f(r)],f(r)]c = 0, \tag{3}$$

that is,

$$\begin{aligned}
&baf(r)^2 - 2bf(r)af(r) + bf(r)^2a + paf(r)^2q\\
&\quad - 2pf(r)af(r)q + pf(r)^2aq\\
&\quad + af(r)^2c - 2f(r)af(r)c + f(r)^2ac = 0
\end{aligned}$$

for all $r = (r_1,\ldots,r_n) \in R^n$. Since d is nonzero, $a \notin C$. In this case, by Lemma 3.4, we have either $p \in C$ or $q \in C$.

Case i. Let $p \in C$. Then by hypothesis, R satisfies

$$b[[a,f(r)],f(r)] + [[a,f(r)],f(r)](pq + c) = 0.$$

By Lemma 3.3 in [2], b, $pq + c$ and $(b + pq + c)a$ are in C. Since $a \notin C$, we conclude that $b + pq + c = 0$. This is our conclusion (1).

Case ii. Let $q \in C$. By hypothesis, R satisfies

$$(b + pq)[[a, f(r)], f(r)] + [[a, f(r)], f(r)]c = 0.$$

By Lemma 3.3 in [2], $b + pq$, c and $(b + pq + c)a$ are in C. Since $a \notin C$, we conclude that $b + pq + c = 0$. This is our conclusion (2).

Next, suppose that d is an outer derivation of R. By using Fact 5 and Kharchenko's theorem [11], we can replace $d(x_i)$ with y_i and then R satisfies

$$\begin{aligned} &b[f^d(x_1,\dots,x_n) + \sum_i f(x_1,\dots,y_i,\dots,x_n), f(x_1,\dots,x_n)] \\ &\quad + p[f^d(x_1,\dots,x_n) + \sum_i f(x_1,\dots,y_i,\dots,x_n), f(x_1,\dots,x_n)]q \\ &\quad + [f^d(x_1,\dots,x_n) + \sum_i f(x_1,\dots,y_i,\dots,x_n), f(x_1,\dots,x_n)]c = 0. \end{aligned}$$

In particular, R satisfies blended component

$$\begin{aligned} &b[\textstyle\sum_i f(x_1,\dots,y_i,\dots,x_n), f(x_1,\dots,x_n)] \\ &\quad + p[\textstyle\sum_i f(x_1,\dots,y_i,\dots,x_n), f(x_1,\dots,x_n)]q \\ &\quad + [\textstyle\sum_i f(x_1,\dots,y_i,\dots,x_n), f(x_1,\dots,x_n)]c = 0. \end{aligned} \tag{4}$$

Since R is noncommutative, we choose $a' \in R$ such that $a' \notin C$. Replacing $[a', x_i]$ in place of y_i in equation (4), we get

$$b[[a', f(r)], f(r)] + p[[a', f(r)], f(r)]q + [[a', f(r)], f(r)]c = 0$$

for all $r = (r_1,\dots,r_n) \in R^n$, which is the same as equation (3). Then by the same argument as above, we have our conclusions. □

In particular, for right-sided annihilator condition, we have the following.

COROLLARY 3.6

Let R be a prime ring of characteristic different from 2, $f(x_1,\dots,x_n)$ a non-central multilinear polynomial over C and d a nonzero derivation of R. Suppose that for some $a \in R$, $[d(u), u]a = 0$ for all $u \in f(R)$. Then $a = 0$.

In particular, for two-sided annihilator condition, we have the following.

COROLLARY 3.7

Let R be a prime ring of characteristic different from 2, $f(x_1,\dots,x_n)$ a non-central multilinear polynomial over C and d a nonzero derivation of R. Suppose that for some $a, b \in R$, $a[d(u), u]b = 0$ for all $u \in f(R)$. Then either $a = 0$ or $b = 0$.

Putting $p = 0$ and $q = 0$ in Theorem 3.5, we have the inner part of Theorem 5.3 of [2]. More precisely, we obtain the following.

COROLLARY 3.8

Let R be a prime ring of characteristic different from 2, $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over C and d a nonzero derivation of R. Suppose that for some $b, c \in R$, $b[d(u), u] + [d(u), u]c = 0$ for all $u \in f(R)$. Then $b = -c \in C$.

Replacing b by s^2, c by t^2, $p = 2s$ and $q = t$ in Theorem 3.5, we obtain the following.

COROLLARY 3.9

Let R be a prime ring of characteristic different from 2 and $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over C. If d is a nonzero derivation of R, and F is an inner generalized derivation of R such that

$$F^2([d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)]) = 0$$

for all $x_1, \ldots, x_n \in R$, then there exists $a \in U$ such that $F(x) = xa$ for all $x \in R$ or $F(x) = ax$ for all $x \in R$, with $a^2 = 0$.

In the next section, we will extend Corollary 3.9 to the arbitrary generalized derivation. Now we are ready to prove the main theorem.

4. The proof of the main theorem

Lee [12] proved that every generalized derivation can be uniquely extended to a generalized derivation of U, and thus all generalized derivations of R will be implicitly assumed to be defined on the whole U. In particular, Lee proved that every generalized derivation g on a dense right ideal of R can be uniquely extended to U and has the form $g(x) = ax + d(x)$ for some $a \in U$ and a derivation d of R.

Theorem 4.1. *Suppose that R is a prime ring of characteristic different from 2 and $f(x_1, \ldots, x_n)$ is a non-central multilinear polynomial over C. If d is a nonzero derivation of R, and F is a generalized derivation of R such that*

$$F^2([d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)]) = 0$$

for all $x_1, \ldots, x_n \in R$, then there exists $a \in U$ such that $F(x) = xa$ for all $x \in R$ or $F(x) = ax$ for all $x \in R$, with $a^2 = 0$.

Proof. In light of [12, Theorem 3], we may assume that there exist $b \in U$ and derivation δ of U such that $F(x) = bx + \delta(x)$ and so, $F^2(x) = b^2x + 2b\delta(x) + \delta(b)x + \delta^2(x)$. Since R and U satisfy the same generalized polynomial identities (see Fact 2) as well as the same differential identities (see Fact 4), without loss of generality, we have

$$F^2[d(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] = 0$$

for all $r_1, \ldots, r_n \in U$. If F is an inner generalized derivation of R, then assume that $F(x) = bx + xc$ for all $x \in R$, with some $b, c \in U$. In this case, by the hypothesis

$$b^2[d(r), r] + 2b[d(r), r]c + [d(r), r]c^2 = 0$$

for all $r \in f(R)$. Then by Theorem 3.5, one of the following holds:

(i) $b^2, b, 2bc + c^2 \in C$ and $b^2 + 2bc + c^2 = 0$, that is $(b + c)^2 = 0$. In this case, $F(x) = x(b + c)$ for all $x \in R$ with $(b + c)^2 = 0$.
(ii) $b^2 + 2bc, c, c^2 \in C$ and $b^2 + 2bc + c^2 = 0$, that is, $(b + c)^2 = 0$. In this case, $F(x) = (b + c)x$ for all $x \in R$ with $(b + c)^2 = 0$.

Now, we assume that F is outer. By the hypothesis, U satisfies

$$b^2[d(r), r] + 2b\delta([d(r), r]) + \delta(b)[d(r), r] + \delta^2([d(r), r]) = 0 \tag{5}$$

for all $r \in f(R)$.

Case I. Let d and δ be C-dependent modulo inner derivations of U, that is, $\alpha d + \beta\delta = ad_q$, where $\alpha, \beta \in C$, $q \in U$ and $ad_q(x) = [q, x]$ for all $x \in U$. If $\alpha = 0$, then δ must be inner and so F is inner, a contradiction. Hence $\alpha \neq 0$, and hence $d = \lambda\delta + ad_p$, where $\lambda = -\alpha^{-1}\beta$ and $p = \alpha^{-1}q$.

Then by the hypothesis, it follows that

$$\begin{aligned} &b^2[\lambda\delta(r) + [p, r], r] + 2b\delta([\lambda\delta(r) + [p, r], r]) \\ &\quad + \delta(b)[\lambda\delta(r) + [p, r], r] \\ &\quad + \delta^2([\lambda\delta(r) + [p, r], r]) = 0 \end{aligned} \tag{6}$$

for all $r \in f(R)$.

Using Fact 5, substitute the values of $\delta(f(r_1, \ldots, r_n))$, $\delta^2(f(r_1, \ldots, r_n))$ and $\delta^3(f(r_1, \ldots, r_n))$ in equation (6). Then by Kharchenko's theorem [11], we can replace $\delta(r_i)$ with y_i, $\delta^2(r_i)$ with w_i and $\delta^3(r_i)$ with z_i in equation (6) and then U satisfies the blended component

$$[\lambda \sum_i f(r_1, \ldots, z_i, \ldots, r_n), f(r_1, \ldots, r_n)] = 0.$$

We choose $q \in U$ such that $q \notin C$ and replace z_i by $[q, r_i]$. Then U satisfies

$$[\lambda q, f(r_1, \ldots, r_n)]_2 = 0.$$

By [13, Theorem], $\lambda q \in C$. Since $q \notin C$, $\lambda = 0$. Hence by equation (6),

$$b^2[[p, r], r] + 2b\delta([[p, r], r]) + \delta(b)[[p, r], r] + \delta^2([[p, r], r]) = 0 \tag{7}$$

for all $r \in f(R)$.

Putting the values of $\delta(f(r_1, \ldots, r_n))$ and $\delta^2(f(r_1, \ldots, r_n))$ in equation (7), then again by Kharchenko's theorem [11], we can replace $\delta(r_i)$ with y_i and $\delta^2(r_i)$ with w_i in (7), and then U satisfies the blended component

$$[[p, \sum_i f(r_1, \ldots, w_i, \ldots, r_n)], f(r_1, \ldots, r_n)]$$
$$+ [[p, f(r_1, \ldots, r_n)], \sum_i f(r_1, \ldots, w_i, \ldots, r_n)] = 0.$$

By taking $w_1 = r_1$ and $w_2 = \cdots = w_n = 0$, U satisfies

$$2[[p, f(r_1, \ldots, r_n)], f(r_1, \ldots, r_n)] = 0.$$

Since $\mathrm{char}(R) \neq 2$, by [13, Theorem] $p \in C$. This gives that $d = 0$, a contradiction.

Case II. Let d and δ be C-independent modulo inner derivations of U. Then by applying Fact 5 and Kharchenko's theorem [11] to equation (5), we can replace $d(r_i)$ with y_i, $\delta(r_i)$ with z_i, $\delta d(r_i)$ with s_i, $\delta^2(r_i)$ with t_i and $\delta^2 d(r_i)$ with u_i. Then U satisfies the blended component

$$[\sum_i f(r_1, \ldots, u_i, \ldots, r_n), f(r_1, \ldots, r_n)] = 0.$$

In particular, replacing u_i with $[q, r_i]$ for some $q \notin C$, U satisfies

$$[q, f(r_1, \ldots, r_n)]_2 = 0.$$

Again by [13, Theorem], $q \in C$, a contradiction. □

COROLLARY 4.2

Let R be a prime ring of characteristic different from 2 with extended centroid C and $f(x_1, \ldots, x_n)$ a multilinear polynomial over C. If d and δ are two nonzero derivations of R such that

$$\delta^2([d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)]) = 0$$

for all $x_1, \ldots, x_n \in R$, then $f(x_1, \ldots, x_n)$ is central-valued on R.

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