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Combinatorial identities for tenth order mock theta functions

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Abstract. In this paper, the open problem posed by Sareen and Rana (*Proc. Indian Acad. Sci. (Math. Sci.)* **126** (2016) 549–556) is addressed. Here, we interpret two tenth order mock theta functions combinatorially in terms of lattice paths. Then we extend enumeration of one of these with Bender–Knuth matrices; the other by using Frobenius partitions. The combinatorial interpretation of one of these mock theta functions in terms of Frobenius partitions gives an answer to the open problem. Finally, we establish bijections between different classes of combinatorial objects which lead us to one 4-way and one 3-way combinatorial identity.

Keywords. Mock theta functions; lattice paths; color partitions; Bender–Knuth matrices; Frobenius partitions.

1991 Mathematics Subject Classification. 05A15, 05A17, 05A19, 11P81.

1. Introduction and definitions

The last gift of Ramanujan to the mathematical world is 'The Mock Theta Function'. Ramanujan informed Hardy about these functions in his last letter which he wrote just three months before his death. Ramanujan quoted: "I discovered very interesting functions recently which I call the 'mock theta functions'." Ramanujan listed seventeen mock theta functions of order three, five and seven respectively. Recently Choi [8] considered two identities analytically found in the 'lost' notebook of Ramanujan [14] involving four functions and called them mock theta functions of order ten which are given by

$$\psi_R(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q:q^2)_{n+1}}, \tag{1.1}$$

$$\phi_R(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q:q^2)_{n+1}}, \tag{1.2}$$

where the following standard notations are adopted:

$$(a;q)_\infty = (1-a)(1-aq)\cdots$$
$$(a;q)_n = \prod_{i=0}^{n}(1-aq^{i-1}), \quad |q|<1.$$

In literature, we see that several mathematicians have established connections between mock theta functions, partition identities and different combinatorial parameters (see, for instance, [1,4,5,10–13,15]). Using $(n+t)$-color partitions, Sareen and Rana [16] interpreted Ramanujan's mock theta functions (1.1)–(1.2) combinatorially. They further extended their result for the mock theta function (1.1) using Frobenius partitions. They also succeeded in establishing a bijection between the n-color partitions class and Frobenius partitions class for the mock theta function (1.1). But for the mock theta function (1.2), they posed an open problem in their respective paper [16]: "Is it possible to obtain a similar extension and bijection for the mock theta function (1.2)?" Our main objective of this paper is to address this problem. In this paper, we provide combinatorial interpretation of the mock theta function (1.2) using generalized Frobenius partitions and successfully establish bijection between $(n+1)$-color partitions class and generalized Frobenius partitions class for the mock theta function (1.2). This result is discussed in detail in section 2. In section 3, we further extend these combinatorial identities to 3-way combinatorial identities by interpreting mock theta functions (1.1)–(1.2) using lattice paths. We will also establish bijections between certain classes of $(n+t)$-color partitions and the lattice paths. Lastly, in section 4, we use Bender–Knuth matrices to further extend our result for the mock theta function (1.1) to 4-way combinatorial identity. We conclude the paper by posing an open problem.

Before we state our main results, we shall first recall some definitions:

DEFINITION 1.1 [2]

A *partition with* '$(n+t)$ *copies of* n', $t \geq 0$, is a partition in which a part of size n, $n \geq 0$ can come in $(n+t)$ different colors denoted by subscripts: $n_1, n_2, \ldots, n_{n+t}$. Note that zeros are permitted if and only if t is greater than or equal to one. Also, zeros are not permitted to repeat in any partition.

Remark 1.1. We note that if we take $t=0$, then these are nothing but the n-color partitions.

DEFINITION 1.2

The *weighted difference* of two parts g_k, h_l $(g \geq k)$ is defined by $g-k-h-l$ and is denoted by $((g_k - h_l))$.

Example 1.1. There are twelve $(n+1)$-color partitions of 2:

$$2_1,\quad 2_1+0_1,\quad 1_1+1_1,\quad 1_1+1_1+0_1,$$
$$2_2,\quad 2_2+0_1,\quad 1_2+1_1,\quad 1_2+1_1+0_1,$$
$$2_3,\quad 2_3+0_1,\quad 1_2+1_2,\quad 1_2+1_2+0_1.$$

DEFINITION 1.3 [6]

A two-rowed array of non-negative integers

$$\begin{pmatrix} p_1 & p_2 & \cdots & p_r \\ q_1 & q_2 & \cdots & q_r \end{pmatrix},$$

where $p_1 \geq p_2 \geq \cdots \geq p_r \geq 0$, $q_1 \geq q_2 \geq \cdots \geq q_r \geq 0$ is known as a generalized Frobenius partition or more simply an F-partition of ν if $p_1 + p_2 + \cdots + p_r + q_1 + q_2 + \cdots + q_r + r = \nu$.

For example, $\nu = 28 = 4 + (6 + 5 + 2 + 0) + (5 + 3 + 2 + 1)$ and the corresponding Frobenius symbol is

$$\begin{pmatrix} 6 & 5 & 2 & 0 \\ 5 & 3 & 2 & 1 \end{pmatrix}.$$

DEFINITION 1.4

A plane partition δ of a positive integer ν is an array

$$\begin{array}{|cccc} \hline a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

of nonnegative integers for which $\sum_{i,j} a_{i,j} = \nu$ and rows and columns are arranged in non-increasing order. The non-zero entries $a_{i,j}$ are called the parts of δ.

Remark 1.2. In [2], it is observed that the number of n-color partitions of ν equals the number of plane partitions of ν.

Bender and Knuth [7] proved the following theorem.

Theorem 1.1 [7]. *There is a* 1–1 *correspondence between plane partitions of* ν, *on the one hand, and infinite matrices* $\gamma_{u,v}$ $(u, v \geq 1)$ *of nonnegative integer entries such that* $\sum_{t\geq 1} t(\sum_{u+v=t+1} \gamma_{u,v}) = \nu$, *on the other.*

Note 1.1. For definition and other details of the 1–1 correspondence of this theorem, the reader is referred to [7].

Corresponding to every nonnegative integer ν, we shall call the matrices of the above theorem BK_ν-matrices (BK for Bender and Knuth). These are infinite matrices but will be represented in the sequel by the largest possible square matrices whose last row (or column) is non-zero. Thus, for example, the six BK_3-matrices are represented by

$$(3),\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},\ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We state here three more definitions.

DEFINITION 1.5

We define a matrix $U_{i,j}$ as an infinite matrix whose (i, j)-th entry is 1 and all other entries are zeros. We call the $U_{i,j}$ the units of BK_ν-matrices.

DEFINITION 1.6

The following order is defined on the set of all units of BK_ν-matrices. If $k+l < g+h$, then $U_{k,l} < U_{g,h}$, and if $k + l = g + h$, then $U_{k,l} < U_{g,h}$, where $k < g$. Thus, the units satisfy the order: $U_{1,1} < U_{1,2} < U_{2,1} < U_{1,3} < U_{2,2} < U_{3,1} < U_{1,4} < U_{2,3} < U_{3,2} < \cdots$.

DEFINITION 1.7

The order difference of two units $U_{g,h}, U_{k,l}$, where $(g+h \geq k+l)$ is defined by $h-l-2k$ and is denoted by $[[U_{g,h} - U_{k,l}]]$.

Note 1.2. Representation of a BK_ν-matrix as the linear combination of the units $U_{i,j}$ is called the standard factorization of that BK_ν-matrix.

Corteel and Mallet [9] described *lattice paths* as follows.

DEFINITION 1.8

All paths will be of finite length lying in the first quadrant. They will begin on the Y-axis and terminate on the X-axis. The following four unitary steps are allowed at each step:

North–East NE: From (i, j) to $(i + 1, j + 1)$.
South–East SE: From (i, j) to $(i + 1, j - 1)$, only allowed if $j > 0$.
South S: From (i, j) to $(i, j - 1)$, only allowed if $j \geq 1$.
Horizontal (*East*) *H*: From $(i, 0)$ to $(i + 1, 0)$.

All our lattice paths are either empty or terminate with a South-East step, i.e. from $(i, 1)$ to $(i + 1, 0)$ or South step, i.e. from (i, j) to $(i, j - 1)$.

In describing lattice paths, the following terminologies are used:

Peak: Either a vertex on the Y-axis which is followed by a S step or SE step or a vertex preceded by a NE step and followed by a S step (in which case, it is called a NE–S peak) or by a SE step (in which case, it is called NE–SE peak).
Valley: A vertex preceded by a S step or SE step and followed by a NE step. Note that a S step or SE step followed by H step and then followed by a NE step does not constitute a valley.
Mountain: A section of the path which starts on either the X- or Y-axis, which ends on the X-axis and which does not touch the X-axis anywhere in between the end points. Every mountain has at least one peak and may have more than one.
Plain: A section of the path consisting of only H steps which starts at a vertex preceded by a SE step and ends at a vertex followed by a NE step. Note that a sequence of consecutive horizontal steps which is immediately preceded by a S step and immediately followed by a SE step or a NE step does not constitute a plain.
Height: The height of a vertex is its Y-coordinate.

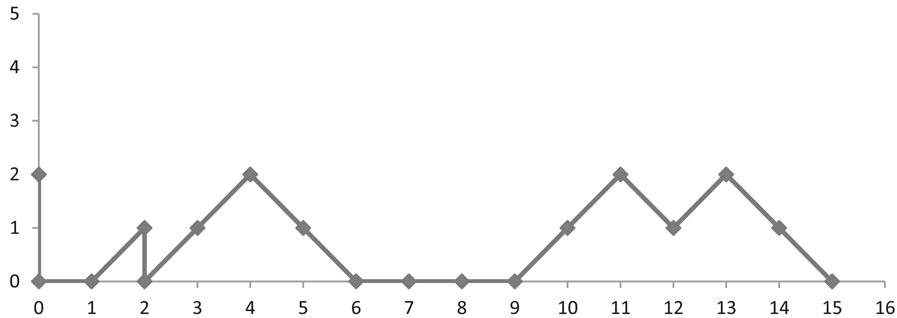


Figure 1. A weighted lattice path with weight 30.

Weight: The weight of a vertex is its X-coordinate.
Weight of a lattice path: This is the sum of the weights of its peaks.

Remark 1.3. When the lattice paths have no S *steps*, then the definition of the lattice paths is as given by Agarwal and Bressoud [3] (see Figure 1).

Example 1.2. In this example, there is a S *step* of depth 2 at the beginning of the path which is followed by a H *step* of length 1. Then there is a peak of height 1 followed by a S *step* of depth 1 which is followed by a peak of height 2, further followed by a plain of length 3. Then there are two peaks of height 2 along with a valley at height 1. The weight of this path is $0 + 2 + 4 + 11 + 13 = 30$.

Sareen and Rana [16] gave the following combinatorial interpretations of the mock theta functions (1.1)–(1.2).

Theorem 1.2. *For $\nu \geq 1$, let $A_1(\nu)$ denote the number of n-color partitions of ν such that*

(*i*) *the weighted difference of any two consecutive parts is -1,*
(*ii*) *for some i, i_i is a part.*

Let $B_1(\nu)$ denote the number of F-partitions of ν such that

(*i*) *For any two adjacent columns $\begin{pmatrix} p_i \\ q_i \end{pmatrix}$ and $\begin{pmatrix} p_{i+1} \\ q_{i+1} \end{pmatrix}$, we have*
  (a) *If $p_i \leq q_i$, then $p_{i+1} > q_{i+1}$ and $p_i = p_{i+1}$, for $1 \leq i \leq (r-2)$,*
  (b) *If $p_i > q_i$, then $p_{i+1} \leq q_{i+1}$ and $q_i = q_{i+1}$, for $1 \leq i \leq (r-1)$.*

(*ii*) $p_r = 0$.

Then $A_1(\nu) = B_1(\nu)$, for all $\nu \geq 1$ and

$$\sum_{\nu=1}^{\infty} A_1(\nu)q^{\nu} = \sum_{\nu=1}^{\infty} B_1(\nu)q^{\nu} = \psi_R(q). \tag{1.3}$$

Theorem 1.3. *For $\nu \geq 0$, let $A_2(\nu)$ denote the number of $(n+1)$-color partitions of ν such that*

(*i*) *the weighted difference of any two consecutive parts is -1,*
(*ii*) *for some i, i_{i+1} is a part.*

Then

$$\sum_{\nu=0}^{\infty} A_2(\nu)q^{\nu} = \phi_R(q).$$

2. Generalized F-partitions and combinatorial identities

In this section, we shall prove that the mock theta function (1.2) have its combinatorial counterpart for generalized F-partitions in the form of the following theorem.

Theorem 2.1. *Let $B_2(\nu)$ denote the number of F-partitions of ν such that*

(*i*) *For any two adjacent columns $\begin{pmatrix} p_i \\ q_i \end{pmatrix}$ and $\begin{pmatrix} p_{i+1} \\ q_{i+1} \end{pmatrix}$, we have*

- (a) *If $p_i < q_i + 2$, then $p_{i+1} \geq q_{i+1} + 2$ and $p_i = p_{i+1}$, for $1 \leq i \leq (r-1)$.*
- (b) *If $p_i \geq q_i + 2$, then $p_{i+1} < q_{i+1} + 2$ and $q_i = q_{i+1}$, for $1 \leq i \leq (r-1)$.*

(*ii*) $p_r = 0$.

Then $A_2(\nu) = B_2(\nu)$ for all $\nu \geq 0$ and

$$\sum_{\nu=0}^{\infty} A_2(\nu)q^{\nu} = \sum_{\nu=0}^{\infty} B_2(\nu)q^{\nu} = \phi_R(q). \tag{2.1}$$

Remark 2.1. In the definition of F-partitions, we allow more generalized values for q_i's, here, $q_1 \geq q_2 \geq \cdots \geq q_r \geq -1$ for the interpretation of $\phi_R(q)$ in terms of F-partitions.

Example 2.1. For $\nu = 8$, $B_2(6) = 6$ and the relevant eight F-partitions are

$$\begin{pmatrix} 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 3 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ 1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}.$$

2.1 *Proof of Theorem* 2.1

To prove this theorem, first we define a set of $(n+2)$-color partitions enumerated by $M_2(\nu)$ such that

(1) the weighted difference of any two consecutive parts is -3,
(2) for some i, i_{i+2} is a part.

One can easily establish a 1–1 correspondence between the partitions enumerated by $A_2(\nu)$ and $M_2(\nu)$ (simply increase the size of each subscript by one of all the parts of the partitions enumerated by $A_2(\nu)$).

Remark 2.2. Note that in a partition enumerated by $M_2(\nu)$, the subscripts of all the parts are greater than 1.

Proof. Now, to prove this theorem, it will suffice to establish a bijection between F-partitions enumerated by $B_2(\nu)$ and $(n+2)$-color partitions enumerated by $M_2(\nu)$. We do this by mapping each column $\binom{p}{q}$ of the Frobenius symbol to a single part g_k of an $(n+2)$-color partition. The mapping is

$$\phi : \binom{p}{q} \to \begin{cases} (p+q+1)_{q-p+3} & \text{if } p < q+2, \\ (p+q+1)_{p-q} & \text{if } p \geq q+2. \end{cases} \tag{2.2}$$

The inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : g_k \to \begin{cases} \begin{pmatrix} (g-k+2)/2 \\ (g+k-4)/2 \end{pmatrix} & \text{if } g \equiv k (\text{mod } 2), \\ \begin{pmatrix} (g+k-1)/2 \\ (g-k-1)/2 \end{pmatrix} & \text{if } g \not\equiv k (\text{mod } 2). \end{cases} \tag{2.3}$$

Now for any two adjacent columns $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ in the Frobenius symbol with $\phi \binom{p}{q} = g_k$ and $\phi \binom{r}{s} = h_l$ as defined in (2.2), we have

$$((g_k - h_l)) = \begin{cases} 2q - 2s - 3 & \text{if } p \geq q+2, r < s+2, \\ 2p - 2r - 3 & \text{if } p < q+2, r \geq s+2. \end{cases} \tag{2.4}$$

Clearly (2.4) and condition (i) of Theorem 2.1 imply the condition (1) of the $(n+2)$-color partitions enumerated by $M_2(\nu)$. Now, from condition (ii) of Theorem 2.1, we have $p_r = 0$ and then using equation (2.2), we have

$$\phi : \binom{p_r}{q_r} \to (q+1)_{q+3} \quad \text{if } p \leq q+2 \tag{2.5}$$

which is of the form i_{i+2}. Thus condition (2) on partitions enumerated by $M_2(\nu)$ is also satisfied.

It is worthy to note that the weighted difference between any two consecutive parts, say, g_k, h_l, of a partition enumerated by $M_2(\nu)$ is -3 which implies that if $g \equiv k$ (mod 2), then $h \not\equiv l$ (mod 2) and vice versa. Now to see the reverse implication, let $g_k,\ h_l$ be any two consecutive parts in $(n+2)$-color partition enumerated by $M_2(\nu)$ such that $\phi^{-1} : g_k = \binom{p}{q}$ and $\phi^{-1} : h_l = \binom{r}{s}$. Then in view of (2.3),

$$p - r = \begin{cases} \frac{((g_k - h_l))+3}{2} & \text{if } g \equiv k, h \not\equiv l (\text{mod } 2), \\ \frac{((g_k - h_l))-3}{2} + k + l & \text{if } g \not\equiv k, h \equiv l (\text{mod } 2), \end{cases} \tag{2.6}$$

$$q - s = \begin{cases} \frac{((g_k - h_l))-3}{2} + k + l & \text{if } g \equiv k, h \not\equiv l (\text{mod } 2), \\ \frac{((g_k - h_l))+3}{2} & \text{if } g \not\equiv k, h \equiv l (\text{mod } 2). \end{cases} \tag{2.7}$$

Furthermore,

$$p - q = \begin{cases} 3 - k & \text{if } g \equiv k (\text{mod } 2), \\ k & \text{if } g \not\equiv k (\text{mod } 2), \end{cases} \tag{2.8}$$

$$r - s = \begin{cases} 3 - l & \text{if } h \equiv l (\text{mod } 2), \\ l & \text{if } h \not\equiv l (\text{mod } 2). \end{cases} \tag{2.9}$$

Now (2.6), (2.7), (2.8) and (2.9) along with condition (1) on the $(n+2)$-color partitions enumerated by $M_2(\nu)$ confirms the validity of condition (i) of Theorem 2.1. Also, the first line of (2.3) and condition (2) further justifies the condition (ii) of Theorem 2.1. This completes the proof of $A_2(\nu) = B_2(\nu)$. □

3. Lattice paths and combinatorial identities

In this section, we further extend the combinatorial identities (1.3)–(2.1) for the mock theta functions (1.1)–(1.2) to 3-way combinatorial identities by using lattice paths as a combinatorial tool.

Theorem 3.1. *Let $C_1(\nu)$ denote the number of lattice paths of weight ν which start at* (0,0), *such that*

(*i*) *they have no valley above height* 0,
(*ii*) *they have no plain at all.*

Then $A_1(\nu) = B_1(\nu) = C_1(\nu)$ for all $\nu \geq 1$, and

$$\sum_{\nu=1}^{\infty} A_1(\nu) q^{\nu} = \sum_{\nu=1}^{\infty} B_1(\nu) q^{\nu} = \sum_{\nu=1}^{\infty} C_1(\nu) q^{\nu} = \psi_R(q). \tag{3.1}$$

Theorem 3.2. *Let $C_2(\nu)$ denote the number of lattice paths of weight ν which start at* (0,1), *such that*

(*i*) *they have no valley above height* 0,
(*ii*) *they have no plain at all.*

Then $A_2(\nu) = B_2(\nu) = C_2(\nu)$ for all $\nu \geq 0$, and

$$\sum_{\nu=0}^{\infty} A_2(\nu) q^{\nu} = \sum_{\nu=0}^{\infty} B_2(\nu) q^{\nu} = \sum_{\nu=0}^{\infty} C_2(\nu) q^{\nu} = \phi_R(q). \tag{3.2}$$

As Theorems 3.1–3.2 have similar proofs, so we will discuss the detailed proof of Theorem 3.1 and provide an outline of the proof of the remaining theorem.

3.1 *Proof of Theorem* 3.1

We will prove this theorem in two steps. First, we will show that the right-hand side of (1.1) generates the lattice paths enumerated by $C_1(\nu)$. Then we will establish a bijection between n-color partitions enumerated by $A_1(\nu)$ and the lattice paths enumerated by $C_1(\nu)$.

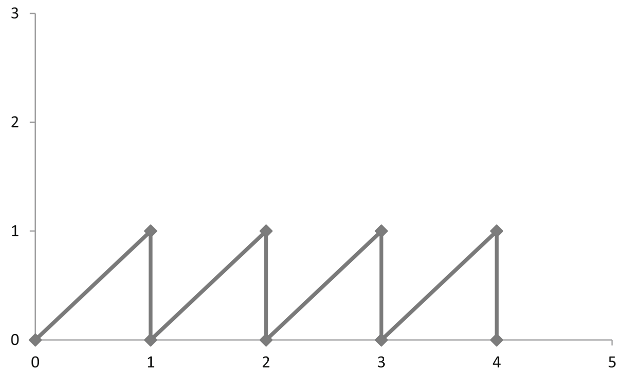


Figure 2. Weighted lattice path for $m = 4$.

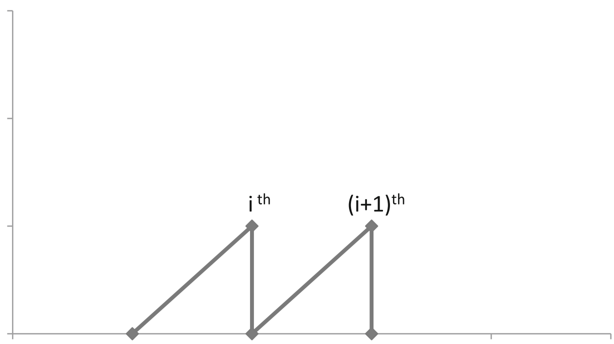


Figure 3. i-th and $(i + 1)$-th peak.

Proof.

Step I: We shall prove that

$$\sum_{\nu=1}^{\infty} C_1(\nu)q^{\nu} = \sum_{m=1}^{\infty} \frac{q^{m(m+1)/2}}{(q;q^2)_m}. \tag{3.3}$$

In $\frac{q^{m(m+1)/2}}{(q;q^2)_m}$, the factor $q^{m(m+1)/2}$ generates the lattice path of m peaks each of height 1 starting at $(0, 0)$ and terminating at $(m, 0)$.

If $m = 4$, the path begins as in Figure 2.

Here we consider two successive peaks, say, i-th and $(i + 1)$-th. Their corresponding coordinates are $(i, 1)$ and $(i + 1, 1)$ respectively.

The factor $1/(q; q^2)_m$ generates m nonnegative multiples of $(2i - 1)$, $1 \leq i \leq m$, say, $\beta_1 \times 1$, $\beta_2 \times 3, \ldots, \beta_m \times (2m - 1)$. This is encoded by having the i-th peak grow to height $\beta_{m-i+1} + 1$ by inserting NE–SE steps. Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two. Figure 3 changes to Figure 4(a) or 4(b) or 4(c) depending upon whether $\beta_{m-i} = \beta_{m-i+1}$ or $\beta_{m-i} > \beta_{m-i+1}$ or $\beta_{m-i} < \beta_{m-i+1}$.

Every lattice path enumerated by $C_1(\nu)$ is uniquely generated in this manner.

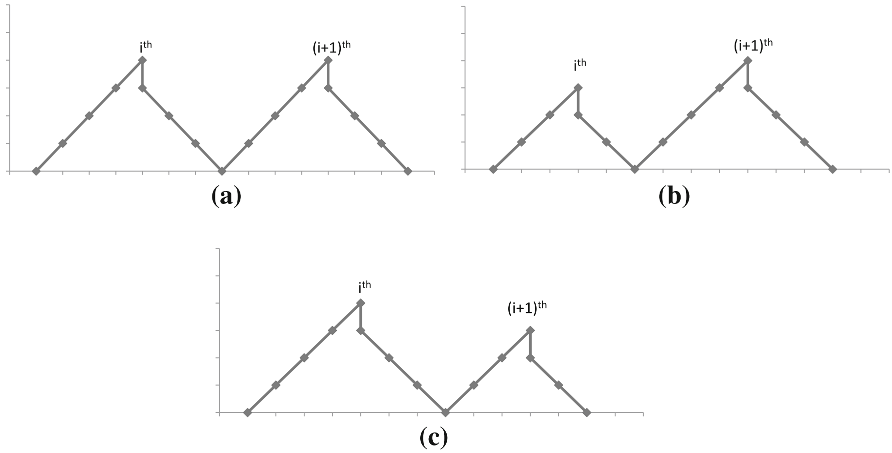


Figure 4. i-th and $(i+1)$-th peak.

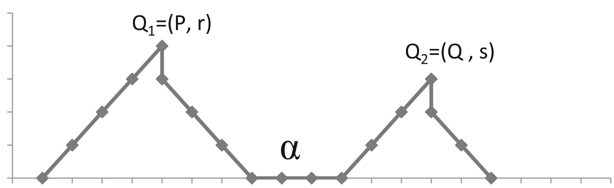


Figure 5. A plain of length α in between two peaks.

Step II: We now establish a 1–1 correspondence between the associated lattice paths enumerated by $C_1(\nu)$ and the n-color partitions enumerated by $A_1(\nu)$.

We do this by encoding each lattice path as the sequence of weights of the peaks with each weight subscripted by the height of the respective peak. Thus, if we denote the two peaks in Figure 4 by P_r and Q_s respectively, then

$$\begin{aligned} P &= i + 2(\beta_m + \beta_{m-1} + \cdots + \beta_{m-i+2}) + \beta_{m-i+1}, \\ r &= \beta_{m-i+1} + 1, \\ Q &= i + 2(\beta_m + \beta_{m-1} + \cdots + \beta_{m-i+1}) + \beta_{m-i}, \\ s &= \beta_{m-i} + 1. \end{aligned}$$

The weighted difference of these two parts is $((Q_s - P_r)) = Q - P - r - s = -1$. Also, corresponding to the first peak $P = 1 + \beta_1,\ r = 1 + \beta_1$. Clearly it corresponds to part i_i in the corresponding n-color partition enumerated by $A_1(\nu)$.

To see the reverse implication, we consider two n-color parts of a partition enumerated by $A_1(\nu)$, say, P_r and Q_s with $Q \geq P$. Clearly, $r \leq P$ and $s \leq Q$.

Let $Q_1 \equiv (P, r)$ and $Q_2 \equiv (Q, s)$ be the corresponding peaks in the associated lattice path. Suppose there is a plain of length α in between these two peaks (see Figure 5).

The length of the plain between the two peaks is $\alpha = Q - s - P - r + 1 = ((Q_s - P_r)) + 1 = 0$. Finally, we show that there can not be a valley above height 0.

Suppose, there is a valley V of height h between the peaks Q_1 and Q_2 (see Figure 6).

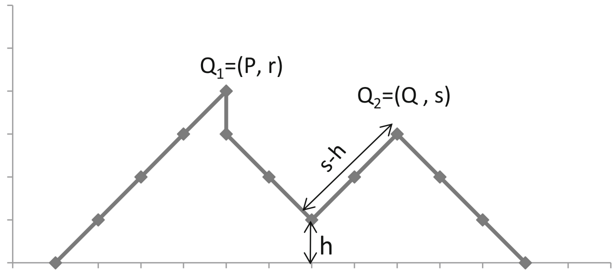


Figure 6. A valley V of height h.

In this case, there is a descent of $r-h-1$ from Q_1 to V and an ascent of $s-h$ from V to Q_2. This implies that $Q = P + (r-h-1) + (s-h) \Rightarrow Q - P - r - s = -2h - 1 \Rightarrow ((Q_s - P_r)) = -2h - 1$. But since the weighted difference is -1, therefore, h=0. This completes the bijection between the lattice paths enumerated by $C_1(\nu)$ and the n-color partitions enumerated by $A_1(\nu)$. □

3.2 *Outline of the proof of Theorem 3.2*

An appeal to Theorem 3.1, the extra factor q^m puts one S step from (0,1) to (0,0). So, in this case, the path begins with $(m+1)$ peaks starting from (0, 1) and ending at $(m, 0)$. Also, the factor $\frac{1}{(1-q^{2m+1})}$ introduces a nonnegative multiple of $2m+1$, say $\beta_{m+1} \times (2m+1)$. This is encoded by having the first peak grow to height $\beta_{m+1} + 1$ in the NE direction.

To illustrate the constructed bijections, we consider the generalized F-partitions enumerated by $B_2(3)$, the lattice paths enumerated by $C_2(3)$ and the corresponding $(n+1)$-color partitions enumerated by $A_2(3)$ in the Table 1.

4. Combinatorial interpretation using Bender–Knuth matrices

This section is fully devoted to interpret identity (1.1) in terms of Bender–Knuth matrices. This will further extend Theorem 3.1 to a new 4–way combinatorial identity.

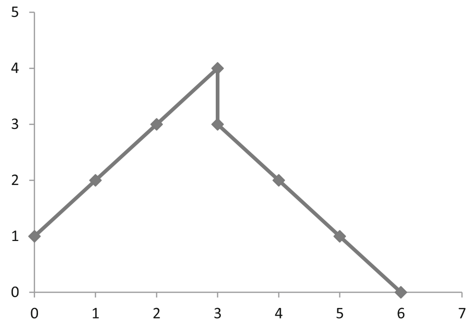
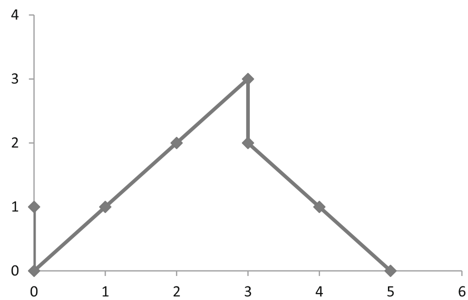
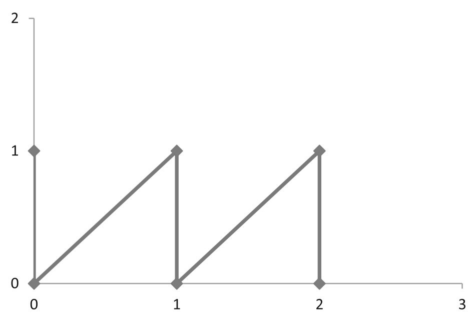
Theorem 4.1. *Let $D_1(\nu)$ denote the number of* BK_ν*-matrices X such that, in the standard factorization of X, the order difference between any two consecutive units $U_{g,h}$ and $U_{k,l}$ is -1, and if $U_{i,j}$ is the only or least unit, then $j = 1$. Then*

$$A_1(\nu) = B_1(\nu) = C_1(\nu) = D_1(\nu), \textit{ for all } \nu \geq 1.$$

Example 4.1. $D_1(6) = 4$. The relevant BK_ν-matrices are

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&1\\ 0&0&0\\ 0&0&0 \end{pmatrix}.$$

Table 1. n-Color partitions enumerated by $A_2(3)$, lattice paths enumerated by $C_2(3)$ and generalized F-partitions enumerated by $B_2(3)$.

n-Color partitions enum. by $A_2(3)$	Lattice paths enum. by $C_2(3)$	F-partitions enum. by $B_2(3)$
3_4		$\begin{pmatrix}0\\2\end{pmatrix}$
3_3+0_1		$\begin{pmatrix}3 & 0\\-1 & -1\end{pmatrix}$
$2_1+1_1+0_1$		$\begin{pmatrix}1 & 1 & 0\\0 & -1 & -1\end{pmatrix}$

Proof of the Theorem 4.1. We shall prove that if X is a BK_ν-matrix enumerated by $D_1(\nu)$, then the n-color partition $\chi(X)$ is enumerated by $A_1(\nu)$. Here, the mapping is given by

$$\chi : U_{g,h} \longrightarrow (g+h-1)_g,$$

and the inverse mapping

$$\chi^{-1} : m_i \longrightarrow U_{i,m-i+1}.$$

Conversely, if π is an n-color partition enumerated by $A_1(\nu)$, then the BK_ν-matrix $\chi^{-1}(\pi)$ is enumerated by $D_1(\nu)$.

Let $X = a_{1,1}U_{1,1} + a_{1,2}U_{1,2} + \cdots + a_{2,1}U_{2,1} + a_{2,2}U_{2,2} \ldots$ be a BK_ν-matrix enumerated by $D_1(\nu)$, where $a_{i,j}$ are nonnegative integers which denote the multiplicities of $U_{i,j}$. Now in view of the condition on ordered difference, i.e. the order difference is -1, the entries in X cannot exceed 1, i.e. each $a_{i,j} = 1$ or 0.

Let $U_{g,h},\ U_{k,l}\ (g + h \geq k + l)$ be two consecutive units of a BK_ν-matrix X enumerated by $D_1(\nu)$ which correspond to two consecutive n-color parts $m_i,\ n_j$ of $\chi(X)$. Then $m_i = (g + h - 1)_g$ and $n_j = (k + l - 1)_k$.

Since $g + h \geq k + l \ \Rightarrow m \geq n$,

$$\begin{aligned}((m_i - n_j)) &= (g + h - 1) - g - (k + l - 1) - k = h - l - 2k \\ &= [[U_{g,h} - U_{k,l}]] = -1.\end{aligned}$$

This shows that in $\chi(X)$, the weighted difference between any two consecutive parts is -1.

Furthermore, if $U_{g,h}$ is the only or the least unit of X, then $\chi(U_{g,h}) = m_i$ will be the only or the least part of $\chi(X)$, and since in $U_{g,h},\ h = 1$, we see that $m_i = \chi(U_{g,h}) = (g + h - 1)_g = (g)_g$ which is of the form i_i. Thus $\chi(X)$ is enumerated by $A_1(\nu)$.

To see the reverse implication, let π be an n-color partition of ν enumerated by $A_1(\nu)$. We shall prove that the BK_ν-matrix $\chi^{-1}(\pi)$ is enumerated by $D_1(\nu)$.

Let $m_i,\ n_j\ (m \geq n)$ be two consecutive parts of π such that $\chi^{-1}(m_i) = U_{g,h}$ and $\chi^{-1}(n_j) = U_{k,l}$. Then $U_{g,h} = U_{i,m-i+1}$ and $U_{k,l} = U_{j,n-j+1}$.

Since $m \geq n$, we have $g + h = m + 1 \geq n + 1 = k + l \Rightarrow (g + h) \geq (k + l)$ and

$$\begin{aligned}[[U_{g,h} - U_{k,l}]] = [[U_{i,m-i+1} - U_{j,n-j+1}]] &= (m - i + 1) - (n - j + 1) - 2j \\ &= m - n - i - j = ((m_i - n_j)) = -1.\end{aligned}$$

Now, if m_i is the only or the least part of π, then $\chi^{-1}(\pi) = U_{g,h}$ will be the only or the least unit in $\chi^{-1}(\pi)$, and since $m - i = 0$, we have $\chi^{-1}(m_i) = U_{i,m-i+1} = U_{i,1}$. This implies $h = 1$. This completes the proof of Theorem 4.1. □

5. Conclusion

We have given 4-way combinatorial identity for one of the tenth-order mock theta function (1.1) using n-color partitions, lattice paths, F-partitions and Bender–Knuth matrices, and a 3-way combinatorial identity for the other one (1.2) using $(n+1)$-color partitions, lattice paths and F-partitions. It is worth exploring the possibility of finding the combinatorial interpretations for the mock theta functions (1.1) and (1.2) using combinatorial objects such as overpartitions, and then establish combinatorial identities.

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