



Regularity of binomial edge ideals of certain block graphs

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Abstract. We prove that the regularity of binomial edge ideals of graphs obtained by gluing two graphs at a free vertex is the sum of the regularity of individual graphs. As a consequence, we generalize certain results of Zafar and Zahid (Electron J Comb 20(4), 2013). We obtain an improved lower bound for the regularity of trees. Further, we characterize trees which attain the lower bound. We prove an upper bound for the regularity of certain subclass of block-graphs. As a consequence, we obtain sharp upper and lower bounds for a class of trees called lobsters.

Keywords. Binomial edge ideals; Castelnuovo–Mumford regularity; block graph; tree.

Mathematics Subject Classification. 13D02, 05E40.

1. Introduction

Let G be a simple graph on the vertex set $[n]$. Let $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ be the polynomial ring in $2n$ variables, where K is a field. Then the ideal J_G generated by $\{x_i y_j - x_j y_i \mid (i, j) \text{ is an edge in } G\}$ is called the binomial edge ideal of G . This was introduced by Herzog *et al.* [8] and independently by Ohtani [12]. Recently, there have been many results relating to the combinatorial data of graphs with the algebraic properties of the corresponding binomial edge ideals, see [1, 2, 6, 11, 14, 15, 18]. In particular, there have been active research connecting algebraic invariants of the binomial edge ideals such as Castelnuovo–Mumford regularity, depth, Betti numbers, etc., with combinatorial invariants associated with graphs such as length of maximal induced path, number of maximal cliques, matching number. For example, Matsuda and Murai [11] proved that $\ell \leq \text{reg}(S/J_G) \leq n - 1$, where ℓ is the length of the longest induced path in G . They conjectured that if $\text{reg}(S/J_G) = n - 1$, then G is a path of length n . In [10], Kiani *et al.* proved the conjecture. Chaudhry *et al.* [2] proved that if T is a tree whose longest induced path has length ℓ , then $\text{reg}(S/J_T) = \ell$ if and only if T is a caterpillar. Therefore, the trees that attain the minimal or maximal regularity have been characterized. However, for most

of the graph classes, the Matsuda–Murai bounds are far from being tight. Saeedi *et al.* [14] proved that if G is a closed graph, then $\text{reg}(S/J_G) \leq c(G)$, where $c(G)$ is the number of maximal cliques in G . Here, a graph is said to be closed if its binomial edge ideal has a quadratic Gröbner basis. They generalized this result to the case of binomial edge ideal of a pair of a closed graph and a complete graph, and conjectured that for any graph G , $\text{reg}(S/J_G) \leq c(G)$ [15]. In [9], they proved the conjecture for generalized block graph. In [3], Ene and Zarojanu proved that if G is a chordal graph with the property that any two distinct maximal cliques intersect in at most one vertex, then $\text{reg}(S/J_G) \leq c(G)$.

Though the bound obtained for block graphs by Saeedi Madani and Kiani is sharp, there are several subclasses of block graphs, including trees, where the upper bound is more than the actual regularity (for example, caterpillar [2]). In this article, we study the regularity of binomial edge ideals of certain classes of block graphs, and in particular trees.

In [13], Rauf and Rinaldo studied binomial edge ideals of graphs obtained by gluing two graphs at free vertices. We extend their arguments to observe that the regularity of the binomial edge ideal of a graph obtained by gluing two graphs at free vertices is equal to the sum of the regularities of the binomial edge ideals of the individual graphs, Theorem 3.1. As a consequence, we obtain precise expressions for the regularities of several classes of trees and block graphs (Corollaries 3.2, 3.3 and 3.4).

The lower bound for the regularity of a binomial edge ideal given by Matsuda and Murai [11] namely, the length of the longest induced path, is the best lower bound known as of now. By using inductive application of Theorem 3.1, we obtain a lower bound for the regularity of binomial edge ideals of trees in terms of the number of internal vertices (Theorem 4.1). We characterize trees which attain the lower bound in terms of the presence of a specific tree as a subgraph (Theorem 4.2).

We then move on to study certain subclasses of block graphs and obtain improved upper bounds for their regularity (Theorems 4.4 and 4.5). As a consequence, we get an upper bound for the regularity of the binomial edge ideals of lobsters (see section 2 for definition) (Corollary 4.6). We also obtain a precise expression for the regularity of binomial edge ideals of a subclass of lobsters, called pure lobsters, in Corollary 4.3.

2. Preliminaries

In this section, we set up the basic definitions and notation.

Let G be a finite simple graph. A vertex x of G is said to be a cut vertex if $G \setminus \{x\}$ has strictly more connected components than G . A block of G is a maximal subgraph without a cut vertex. A graph G is a *block graph* if every block of G is a complete graph.

Let T be a tree and $L(T) = \{v \in V(T) \mid \deg(v) = 1\}$ be the set of all leaves of T . We say that a tree T is a *caterpillar* if $T \setminus L(T)$ is either empty or is a simple path. Similarly, a tree T is said to be a *lobster*, if $T \setminus L(T)$ is a caterpillar, [4]. Observe that every caterpillar is also a lobster. A longest path in a lobster is called a *spine* of the lobster. Note that given any spine, every edge of a caterpillar is incident to it. With respect to a fixed spine P , the pendant edges incident with P are called *whiskers*. It can be seen that every non-leaf vertex u not incident on a fixed spine P of a lobster forms the center of a star $(K_{1,m}, m \geq 2)$. Each such star is said to be a *limb* with respect to P . More generally, given a vertex v on any simple path P , we can attach a star $(K_{1,m}, m \geq 2)$ with center u by identifying exactly one of the leaves of the star with v . Such a star is called a *limb attached to P* .

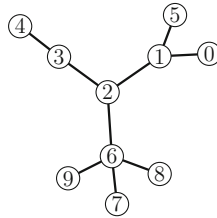


Figure 1. Graph G on 10 vertices.

Note that the limbs and whiskers depend on the spine. Whenever a spine is fixed, we will refer to them simply as limb and whisker.

Example 2.1. Let G denote the graph on 10 vertices (Figure 1).

In this example, G has many longest induced paths. The path induced by the vertices $\{0, 1, 2, 3, 4\}$, $\{0, 1, 2, 6, 9\}$ are two such (there are more) paths. Let P denote the path induced by the vertices $\{0, 1, 2, 3, 4\}$. Then $(1, 5)$ is a whisker with respect to P . Also the subgraph induced by the vertices $\{2, 6, 7, 8, 9\}$ is a limb with respect to P . If we consider $\{0, 1, 2, 6, 9\}$ as spine P , then $\{(1, 5), (6, 7), (6, 8)\}$ are whiskers with respect to P and the path induced by $\{2, 3, 4\}$ is a limb.

We now describe the construction of a useful exact sequence introduced by Ene *et al.* [6].

2.1 Ene–Herzog–Hibi process

Let G be a block graph, $\Delta(G)$ be the clique complex of G and F_1, \dots, F_r be a leaf order on the facets of $\Delta(G)$. Assume that $r > 1$. Let $v \in V(G)$ be the unique vertex in F_r such that $F_r \cap F_j \subseteq \{v\}$ for all $j < r$. Let G' be the graph obtained by adding necessary edges to G so that $N(v) \cup \{v\}$ is a clique. Let G'' be the graph induced on $G \setminus \{v\}$ and H be the graph induced on $G' \setminus \{v\}$. Then there exists an exact sequence

$$0 \rightarrow S/J_G \rightarrow S/J_{G'} \oplus S/J_{G''} \rightarrow S/J_H \rightarrow 0. \tag{1}$$

We call G' , G'' and H to be the graphs obtained by applying EHH-process on G with respect to v . This exact sequence has been found extremely useful in inductive arguments in the study of homological properties of the binomial edge ideals.

3. Regularity via gluing

In this section, we describe the process of gluing and use it to obtain precise regularity expressions for certain classes of graphs. Let G be a graph and v be a cut vertex in G . Let G_1, \dots, G_k be the components of $G \setminus \{v\}$ and $G'_i = G[V(G_i) \cup \{v\}]$, the subgraph of G induced by $V(G_i) \cup \{v\}$. Then, G'_1, \dots, G'_k is called the *split* of G at v and we say that G is obtained by *gluing* G_1, \dots, G_k at v .

Theorem 3.1. *Let G_1 and G_2 be the split of a graph G at v . If v is a free vertex in both G_1 and G_2 , then*

$$\operatorname{reg}(S/J_G) = \operatorname{reg}(S/J_{G_1}) + \operatorname{reg}(S/J_{G_2}).$$

Proof. Let G_1 and G_2 be graphs on the vertices $\{1, \dots, n\}$ and $\{n+1, \dots, n+m\}$ respectively. Assume that n is a free vertex in G_1 and $n+m$ is a free vertex in G_2 . Let G be the graph obtained by identifying vertices n and $n+m$ in $G_1 \cup G_2$, i.e., $v = n = n+m$. Let $G' = G_1 \cup G_2$ and $S' = K[x_1, \dots, x_{n+m}, y_1, \dots, y_{n+m}]$. Then it can be easily seen that $S/J_G \cong S'/(J_{G'} + (x_n - x_{n+m}, y_n - y_{n+m}))$. From the proof of Theorem 2.7 in [13], it follows that $(x_n - x_{n+m}, y_n - y_{n+m})$ is a regular sequence on $S'/J_{G'}$. Hence the assertion follows. \square

As an immediate consequence, we have the following:

COROLLARY 3.2

Let $G = G_1 \cup \dots \cup G_k$ be such that

- (1) for $i \neq j$, if $G_i \cap G_j \neq \emptyset$, then $G_i \cap G_j = \{v_{ij}\}$, for some vertex v_{ij} which is a free vertex in G_i as well as G_j ;
- (2) for distinct i, j, k , $G_i \cap G_j \cap G_k = \emptyset$.

Then $\operatorname{reg} S/J_G = \sum_{i=1}^k \operatorname{reg} S/J_{G_i}$.

Recall that for a (generalized) block graph G , $\operatorname{reg} S/J_G \leq c(G)$ [9]. We obtain a subclass of block graphs which attain this bound.

COROLLARY 3.3

If G is a block graph such that no vertex is contained in more than two maximal cliques, then $\operatorname{reg} S/J_G = c(G)$.

Proof. We use induction on $c(G)$. If $c(G) = 1$, then G is a complete graph and hence $\operatorname{reg} S/J_G = 1$. Now assume that $c(G) > 1$. Consider any cut vertex v of G . Let G_1 and G_2 be the split of G at $\{v\}$. Then, $c(G) = c(G_1) + c(G_2)$. Now the result follows from Corollary 3.2 and induction hypothesis. \square

In [18], Zafar and Zahid considered special classes of graphs called \mathcal{G}_3 and \mathcal{T}_3 and obtained the regularities of the corresponding binomial edge ideals. We generalize their results.

COROLLARY 3.4

- (1) Let P_1, \dots, P_s be paths of lengths r_1, \dots, r_s respectively. Let G be the graph obtained by identifying a leaf of P_i with the i -th vertex of the complete graph K_s . Then $\operatorname{reg}(S/J_G) = 1 + \sum_{i=1}^s r_i$.

- (2) Let P_1, \dots, P_k be paths of lengths r_1, \dots, r_k respectively. Let G be the graph obtained by identifying a leaf of P_i with the i -th leaf of the star $K_{1,k}$. Then, $\text{reg}(S/J_G) = 2 + \sum_{i=1}^k r_i$.

Proof. Both the assertions follow from Theorem 3.1. □

4. Regularity of block graphs

In this section, we study the regularity of binomial edge ideals of certain block graphs, and, in particular, trees. We first obtain a lower bound for the regularity. We then consider a class of graphs, called lobsters, which are a generalization of caterpillars. We generalize a result of Chaudhry *et al.* [2] to obtain sharp upper bounds for the regularity of binomial edge ideals of lobsters. It was shown by Matsuda and Murai [11, Corollary 2.3] that for any graph G , $\text{reg}(S/J_G) \geq \ell$, where ℓ is the length of the longest induced path in G . Below we prove a much improved lower bound, for the class of trees.

For a tree T , let $\text{iv}(T) := \#\{\text{internal vertices of } T\}$. Given a tree T , it is easy to see that one can construct T from the trivial graph by adding vertices v_i to T_{i-1} at step i to get T_i so that v_i is a leaf in the tree T_i . Any such ordering of vertices is called a *leaf ordering*.

Theorem 4.1. *For a tree T , $\text{reg}(S/J_T) \geq \text{iv}(T) + 1$.*

Proof. Let v_1, \dots, v_r be a leaf ordering of the vertices of G , and let G_i be the subgraph of G induced by v_1, \dots, v_i . Let m_i denote the number of internal vertices of G_i . We argue by induction on i . If $i = 2$, then G_2 is an edge and hence $\text{reg}(S/J_{G_2}) = 1$. Therefore, the result holds. Assume the result for G_i . Then G_{i+1} is obtained by adding a leaf v_{i+1} to some vertex v of G_i . If v is a leaf in G_i , then v is a free vertex in G_i , and hence by Theorem 3.1, $\text{reg}(S/J_{G_{i+1}}) = \text{reg}(S/J_{G_i}) + 1$. Further, v becomes a new internal vertex in G_{i+1} , i.e., $m_{i+1} = m_i + 1$, and therefore the result holds. If v is an internal vertex in G_i , then $m_{i+1} = m_i$ and since G_i is an induced subgraph of G_{i+1} , $\text{reg}(S/J_{G_{i+1}}) \geq \text{reg}(S/J_{G_i}) \geq m_i + 1 = m_{i+1} + 1$ as required. □

Let T be the tree given in Figure 2. It follows from [2, Theorem 4.1] and Theorem 3.1 that $\text{reg}(S/J_T) = 26 = \text{iv}(T) + 1$, while the longest path of T has length 15.

It is interesting to note that the graph \mathcal{J} , which we call *Jewel*, is the smallest tree for which $\text{reg}(S/J_{\mathcal{J}}) > \text{iv}(\mathcal{J}) + 1$ (Figure 3). In fact, we can make the gap between the

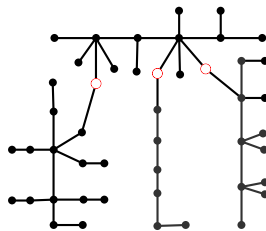


Figure 2. Tree T .

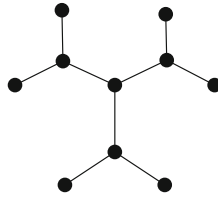


Figure 3. \mathcal{J} : Jewel.

regularity and the number of internal vertices arbitrarily large by attaching edge disjoint copies of Jewel to leaves of any arbitrary tree. For example, Figure 5, which is two copies of the jewel superimposed together, has regularity 12, much larger than the number of internal vertices which is 7.

We now characterize trees which attain the minimal regularity.

Theorem 4.2. *A tree T contains Jewel as a subgraph if and only if $\text{reg}(S/J_T) \geq \text{iv}(T) + 2$.*

Proof. Suppose T is a tree on $[n]$ containing Jewel, \mathcal{J} , as a subgraph. Note that there is a leaf ordering v_1, \dots, v_n such that $V(\mathcal{J}) = \{v_1, \dots, v_{10}\}$. Recall that $\text{reg}(S/J_{\mathcal{J}}) = 6 = \text{iv}(\mathcal{J}) + 2$. Let G_i denote the subgraph of T on the vertex set $\{v_1, \dots, v_i\}$, $i \geq 10$. For each $i \geq 10$, $\text{reg}(S/J_{G_{i+1}}) = \text{reg}(S/J_{G_i}) + 1$ if the neighbor of v_{i+1} is a leaf in G_i and $\text{reg}(S/J_{G_{i+1}}) \geq \text{reg}(S/J_{G_i})$ otherwise. Note also that the neighbor of v_{i+1} is a leaf in G_i if and only if $\text{iv}(G_{i+1}) = \text{iv}(G_i) + 1$. Since $\text{reg}(S/J_{G_{10}}) = \text{iv}(G_{10}) + 2$, we get that $\text{reg}(S/J_{G_i}) \geq \text{iv}(G_i) + 2$ for all $i \geq 10$. Hence the assertion follows.

Conversely, suppose $\text{reg}(S/J_T) \geq \text{iv}(T) + 2$. First assume that T does not have a vertex of degree 2. If T is a caterpillar, then by [2], $\text{reg}(S/J_T) = \text{iv}(T) + 1$. Therefore, T is not a caterpillar. Then it contains the Y graph ($K_{1,3}$ attached with a leaf at each of its leaf vertices) as a subgraph [17, Theorem 2.2.19]. Since T does not have vertices of degree 2, each degree 2 vertex in the Y graph must have one more neighbour in T , which induces a Jewel in T .

Now assume that T contains a vertex of degree 2. Let T' be a minimal (with respect to the number of vertices) subgraph of T so that $\text{reg}(S/J_{T'}) \geq \text{iv}(T') + 2$. If T' has no vertex of degree 2, then T' contains a jewel. Suppose T' contains degree 2 vertex, say v . Let T_1 and T_2 be the split of T' at $\{v\}$. Note that $\text{iv}(T') = \text{iv}(T_1) + \text{iv}(T_2) + 1$. By Theorem 3.1, $\text{reg}(S/J_{T'}) = \text{reg}(S/J_{T_1}) + \text{reg}(S/J_{T_2}) \geq \text{iv}(T_1) + \text{iv}(T_2) + 3$. Hence there exists $i \in \{1, 2\}$ such that $\text{reg}(S/J_{T_i}) \geq \text{iv}(T_i) + 2$. Since T_i is a subgraph of T' , this contradicts the minimality of T' . Hence T' does not contain a degree 2 vertex. Therefore, T' , and thus T contains a jewel. \square

Below, we obtain a class of trees which attain the lower bound. For a lobster, a limb of the form $K_{1,2}$ is called a *pure limb*. A lobster with only pure limbs and no whiskers is called a *pure lobster*.

COROLLARY 4.3

If G is a pure lobster with spine length ℓ and t pure limbs attached to the spine, then $\text{reg}(S/J_G) = \ell + t$.

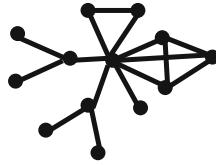


Figure 4. $G \in \mathcal{G}(2, 2, 1)$.

Proof. Since in a pure lobster, only vertices that have degree 3 or more are in the spine, it can not contain the Jewel graph as a subgraph. Therefore by Theorems 4.1 and 4.2, we have $\text{reg}(S/J_G) = \text{iv}(G) + 1 = \ell + t$. \square

In Theorem 3.1, it was shown that the regularity of the graph obtained by gluing two graphs at a free vertex is sum of the regularities of these two graphs. Naturally, one tends to ask what happens to the regularity if we glue more graphs at a free vertex. We partially answer this question in the next theorem. Let $\mathcal{G}(m, n, w)$ be the family of graphs obtained by identifying a free vertex each of $K_{1,r_1}, \dots, K_{1,r_m}$, where $r_i \geq 3$, n cliques on at least three vertices and w whiskers (Figure 4).

Theorem 4.4. *If $G \in \mathcal{G}(m, n, w)$, $n \geq 2$, then $\text{reg}(S/J_G) = n + 2m$.*

Proof. We prove by induction on m . Let $m = 0$. If $w = 0$, then the result follows from Kiani and Saeedi Madani [9]. Suppose $w \geq 1$. Let v denote the vertex which is common to all the cliques and whiskers. Let G' be the clique on $V(G)$, G'' be the graph induced on $V(G) \setminus \{v\}$ and H be the graph $G' \setminus \{v\}$. Then $\text{reg}(S/J_{G'}) = \text{reg}(S/J_H) = 1$. Since G'' is a collection of n disjoint cliques and w isolated vertices, $\text{reg}(S/J_{G''}) = n$. Therefore, the assertion follows from the exact sequence:

$$0 \rightarrow S/J_G \rightarrow S/J_{G'} \oplus S/J_{G''} \rightarrow S/J_H \rightarrow 0. \tag{2}$$

Now assume that $m \geq 1$. Let $\{u\}$ be a leaf vertex in G and $\{u, v\} \in E(G)$. Let G' be the graph obtained by adding necessary edges to G so that $N[v]$ is a clique. Let G'' be the induced subgraph of G on $V(G) \setminus \{v\}$. Let H be the induced subgraph of G' on $V(G') \setminus \{v\}$. Therefore, we have the exact sequence (2). Note that $G', H \in \mathcal{G}(m - 1, n + 1, w)$ and $G'' \in \mathcal{G}(m - 1, n, w)$. Therefore, by induction hypothesis, $\text{reg}(S/J_{G'}) = \text{reg}(S/J_H) = n + 2m - 1$ and $\text{reg}(S/J_{G''}) = n + 2m - 2$. Therefore, from the short exact sequence, we get $\text{reg}(S/J_G) \leq n + 2m$. Note that G contains n vertex disjoint edges and m vertex disjoint paths of length 2. Therefore, $\text{reg}(S/J_G) \geq n + 2m$. \square

We now consider another subclass of block graphs and obtain an improved upper bound on the regularity of binomial edge ideals of those graphs.

Theorem 4.5. *Let G be the union $P \cup C_1 \cup \dots \cup C_r \cup L_1 \cup \dots \cup L_t \cup e_1 \cup \dots \cup e_w$, where P is a longest induced path on the vertices $\{v_0, \dots, v_\ell\}$, C_1, \dots, C_r are the maximal cliques on at least three vertices, L_1, \dots, L_t be limbs and e_1, \dots, e_w are whiskers such that $e_i \cap \{v_0, v_\ell\} = \emptyset$ and for all $A, B \in \{C_1, \dots, C_r, L_1, \dots, L_t, e_1, \dots, e_w\}$ with $A \neq B$,*

$$(1) \ A \cap B \subset P \text{ and } |A \cap B| \leq 1;$$

$$(2) |A \cap P| = 1.$$

Then $\text{reg}(S/J_G) \leq \ell + 2t + r$.

Proof. Without loss of generality, we assume that there are no degree 2 vertices in $\{v_0, \dots, v_\ell\}$. Further, we may assume that there is a clique, say C_i , such that $C_i \cap P = \{v_\ell\}$. If not, then v_ℓ is a leaf in G . Attach a clique C' to $\{v_\ell\}$, to get a graph G_1 having $\text{reg}(S/J_{G_1}) = \text{reg}(S/J_G) + 1$ (by gluing theorem).

We prove the assertion by induction on t . Let $t = 0$. We argue this case by induction on ℓ . Suppose $\ell = 0$. Since $e_i \cap \{v_0, v_\ell\} = \emptyset$, $w = 0$. Hence $G \in \mathcal{G}(0, r, 0)$. Hence the result follows from Theorem 4.4.

Let $\ell = 1$. Suppose $v_0 \in C_1$. Let G' , G'' and H be the graphs obtained by applying EHH-process on G with respect to v_0 . Then G'' is a block graph with exactly r -cliques, G' and H are block graphs with at most r -cliques. Therefore, it follows from the exact sequence (1) and [9, Theorem 3.5] that

$$\text{reg}(S/J_G) \leq \max\{\text{reg}(S/J_{G'}), \text{reg}(S/J_{G''}), \text{reg}(S/J_H) + 1\} \leq r + 1.$$

Now, suppose $\ell \geq 2$. Without loss of generality, assume that $C_1 \cap \{v_{\ell-1}, v_\ell\} \neq \emptyset$. Let G' , G'' and H be the graphs obtained by applying EHH-process on G with respect to $v_{\ell-1}$. Then G' is the union of path P' of length at most $\ell - 1$ containing $\{v_0, \dots, v_{\ell-2}\}$ and cliques, $\{C'_1, C_2, \dots, C_r\}$ and a subset of whiskers $\{e_1, \dots, e_w\}$. Hence by induction, $\text{reg}(S/J_{G'}) \leq \ell - 1 + r$. Similarly $\text{reg}(S/J_H) \leq \ell - 1 + r$ and $\text{reg}(S/J_{G''}) \leq \ell - 1 + r$. Therefore, it follows from the exact sequence (1) that $\text{reg}(S/J_G) \leq \ell + r$.

Let us assume that $t \geq 1$. Since there are no degree 2 vertices in G , each limb $L_i = K_{1, \mu_i}$ for some $\mu_i \geq 3$. Consider the limb L_t . Suppose $L_t \cap P = \{v\}$ and $N_{L_t}(v) = \{u\}$. Let G' , G'' and H be the graphs obtained by applying EHH-process on G with respect to u . Then G' and H both have spines of length $\ell, r + 1$ cliques, $t - 1$ limbs and w whiskers. Also, G'' has spine of length ℓ, r cliques, $t - 1$ limbs and w whiskers. Therefore, by induction, $\text{reg}(S/J_{G'}), \text{reg}(S/J_H) \leq \ell + (r + 1) + 2(t - 1)$, and $\text{reg}(S/J_{G''}) \leq \ell + r + 2(t - 1)$. Hence, from the exact sequence (1), it follows that $\text{reg}(S/J_G) \leq \ell + (r + 1) + 2(t - 1) + 1 = \ell + r + 2t$ as required. \square

Note that the graphs G considered in Theorem 4.5 are block graphs, and hence by Theorem 3.5 of [9], one has $\text{reg}(S/J_G) \leq c(G)$, where $c(G)$ is the number of cliques in G . In the case where $L_i = K_{1, r_i}$ for $r_i \geq 3$ and $w > 0$, the bound given above is much smaller to the Saeedi Madani–Kiani bound.

As a consequence of the above theorem, we generalize a result of Chaudhry *et al.* [2] to obtain an upper bound on the regularity of lobster graphs.

COROLLARY 4.6

If G is a lobster with spine P of length ℓ and t limbs P , then $\text{reg}(S/J_G) \leq \ell + 2t$.

Proof. Take $r = 0$ in Theorem 4.5. \square

Example 4.7. This is an example of a lobster which attains the upper bound given in Theorem 4.6.

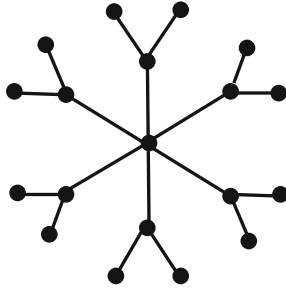


Figure 5. Graph G .

Graph G (Figure 5) has many different longest induced paths. By fixing any one of them, one can see that G has spine length $\ell = 4$, $t = 4$ limbs attached to the spine and 2 whiskers. It can be shown that $\text{reg}(S/J_G) = 12 = \ell + 2t$.

COROLLARY 4.8

Let G be a lobster with spine P of length ℓ , t limbs and r whiskers. Then $\ell + t \leq \text{reg}(S/J_G) \leq \ell + 2t$.

Proof. The upper bound is proved in Corollary 4.6. To prove the lower bound, note that G has a subgraph G' with spine P , t pure limbs and without any whiskers as an induced subgraph. By Corollary 4.3, $\text{reg}(S/J_{G'}) = \ell + t$ as required. \square

From Theorem 4.2, it is clear that the presence of the Jewel graph as a subgraph plays a crucial role in determining the regularity of a tree. It can be seen that $\text{reg}(S/J_T) \geq \text{iv}(T) + j$, where T contains j vertex disjoint copies of the Jewel graph. We believe that understanding the regularity behaviour of a collection of Jewels that share vertices and/or edges can lead to a precise estimation of regularity of trees.

Recently, Herzog and Rinaldo [7] has generalized Theorem 4.1 to certain block graphs.

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