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Congruences for two restricted overpartitions

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Abstract. Congruences for partitions have received a great deal of attention in literature. Recently, Bringmann *et al.* (*Electron J. Combin.* **22(3)** (2015) Paper 3.17, 16 pp.) studied overpartitions with restricted odd differences. In this paper, we present a number of Ramanujan-type congruences for these restricted overpartition functions.

Keywords. Partition; overpartition; Ramanujan-type congruence.

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1. Introduction

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum equals n. Corteel and Lovejoy [2] introduced overpartitions which are ordinary partitions allowing a possible overline designation on the last (or equivalently, the first) occurrence of each distinct part. Let $\bar{p}(n)$ be the number of overpartitions of n. It is known that the generating function of $\bar{p}(n)$ is

$$\sum_{n=0}^{\infty} \bar{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}.$$

Here and in what follows, we adopt the standard q-series notation (cf. [4]).

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

In 2015, Bringmann *et al.* [1] studied a new type of overpartitions with restricted odd differences. Let $\bar{t}(n)$ be the number of overpartitions, where (i) the difference between the successive parts may be odd only if the larger part is overlined, and (ii) if the smallest part is odd, then it is overlined. Moreover, we denote by $\bar{s}(n)$ the number of overpartitions counted by $\bar{t}(n)$ but with the smallest part as odd. Let $\bar{s}_{\pm}(n)$ be the number of overpartitions counted by $\bar{s}(n)$ with the largest part even (resp. odd). Bringmann *et al.* [1] provided the following generating function identities:

$$\sum_{n=0}^{\infty} \bar{t}(n)q^n = \frac{(q^3;q^3)_\infty}{(q;q)_\infty (q^2;q^2)_\infty}, \tag{1.1}$$

$$1 + 3\sum_{n=1}^{\infty}(\bar{s}_+(n) - \bar{s}_-(n))q^n = \frac{(-q^3;q^3)_\infty}{(-q;q)_\infty^3}. \tag{1.2}$$

For convenience, let

$$\sum_{n=0}^{\infty} \bar{\bar{s}}(n)q^n := 1 + 3\sum_{n=1}^{\infty}(\bar{s}_+(n) - \bar{s}_-(n))q^n = \frac{(-q^3;q^3)_\infty}{(-q;q)_\infty^3}.$$

Our main purpose here is to study Ramanujan-type congruences for the foregoing partition functions $\bar{t}(n)$ and $\bar{\bar{s}}(n)$. This paper is organized as follows. In section 2, we introduce some preliminary results. In the next two sections, we will prove some Ramanujan-type congruences for $\bar{t}(n)$ and $\bar{\bar{s}}(n)$, respectively. We finally end this paper by raising a modulo 5 congruence for $\bar{t}(n)$ as an open problem.

2. Preliminaries

Let $f(a,b)$ be Ramanujan's general theta function given by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab| < 1.$$

We now introduce the following Ramanujan's classical theta functions,

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2}, \tag{2.1}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{f_2^2}{f_1}, \tag{2.2}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} = f_1. \tag{2.3}$$

One readily verifies

$$\varphi(-q) = \frac{f_1^2}{f_2}. \tag{2.4}$$

Here and in the sequel, we write $f_k := (q^k;q^k)_\infty$ for positive integers k for notational convenience.

We first require the following 2-dissections.

Lemma 2.1. *It holds that*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{2.5}$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.6}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}, \tag{2.7}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{2.8}$$

Proof. Here (2.6) is the 2-dissection of $\varphi(q)$; see [5, eq. (1.9.4)]. One may obtain (2.5) by replacing q with $-q$ in (2.6). Identity (2.8) is the 2-dissection of $\varphi(q)^2$; see [5, eq. (1.10.1)]. Equation (2.7) can be obtained from (2.8) by replacing q with $-q$. □

Lemma 2.2. *It holds that*

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}, \tag{2.9}$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}, \tag{2.10}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{2.11}$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{2.12}$$

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}. \tag{2.13}$$

Proof. For (2.9), see [5, eq. (30.10.3)]. For (2.10), one may refer to [5, eq. (22.6.2)]. We obtain (2.11) by replacing q with $-q$ in (2.10). One may refer to [5, eq. (22.6.3)] for (2.12). Finally, for (2.13), see [5, eq. (30.12.1)]. □

The following 3-dissections are also necessary.

Lemma 2.3. *It holds that*

$$\varphi(-q) = \varphi(-q^9)(1 - 2qw(q^3)), \tag{2.14}$$

$$\psi(q) = \psi(q^9)\left(\frac{1}{w(q^3)} + q\right), \tag{2.15}$$

$$\frac{1}{\varphi(-q)} = \frac{\varphi(-q^9)^3}{\varphi(-q^3)^4}(1 + 2qw(q^3) + 4q^2 w(q^3)^2), \tag{2.16}$$

$$\frac{1}{\psi(q)} = \frac{\psi(q^9)^3}{\psi(q^3)^4}\left(\frac{1}{w(q^3)^2} - \frac{q}{w(q^3)} + q^2\right), \tag{2.17}$$

where

$$w(q)=\frac{f_1f_6^3}{f_2f_3^3}.$$

Proof. For (2.14) and (2.15), see [5, equations (14.3.4) and (14.3.5)]. Substituting ωq and $\omega^2 q$ for q in (2.14) and multiplying the two results, we obtain (2.16). Here ω is a cube root of unity other than 1. Finally, (2.17) follows from (2.15) in the same way. □

As a useful consequence, we have as follows.

COROLLARY 2.4

It holds that

$$\frac{1}{f_1f_2}=\frac{f_9^3f_{18}^3}{f_3^4f_6^4}\left(\frac{1}{w(q^3)^2}+\frac{q}{w(q^3)}+3q^2-2q^3w(q^3)+4q^4w(q^3)^2\right). \tag{2.18}$$

Proof. We have

$$\begin{aligned}\frac{1}{f_1f_2}&=\frac{1}{\varphi(-q)\psi(q)}\\&=\frac{\varphi(-q^9)^3\psi(q^9)^3}{\varphi(-q^3)^4\psi(q^3)^4}(1+2qw(q^3)\\&\quad+4q^2w(q^3)^2)\left(\frac{1}{w(q^3)^2}-\frac{q}{w(q^3)}+q^2\right)\\&=\frac{f_9^3f_{18}^3}{f_3^4f_6^4}\left(\frac{1}{w(q^3)^2}+\frac{q}{w(q^3)}+3q^2-2q^3w(q^3)+4q^4w(q^3)^2\right).\end{aligned}$$

Lemma 2.5. *It holds that*

$$f_1^3=P(q^3)-3qf_9^3, \tag{2.19}$$

$$\frac{1}{f_1^3}=\frac{f_9^3}{f_3^{12}}(P(q^3)^2+3qf_9^3P(q^3)+9q^2f_9^6), \tag{2.20}$$

where

$$P(q)=f_1a(q) \tag{2.21}$$

$$=f_1\left(1+6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}}-\frac{q^{3n+2}}{1-q^{3n+2}}\right)\right) \tag{2.22}$$

$$=f_1\left(\frac{\psi(q)^3}{\psi(q^3)}+3q\frac{\psi(q^3)^3}{\psi(q)}\right) \tag{2.23}$$

$$= \frac{f_2^6 f_3}{f_1^2 f_6^2} + 3q \frac{f_1^2 f_6^6}{f_2^2 f_3^3}. \tag{2.24}$$

Proof. For (2.19), see [5, eq. (21.3.3)]. One may obtain (2.20) by replacing q with ωq and $\omega^2 q$ in (2.19) and then multiplying the two results. For (2.21), (2.22) and (2.23), see respectively (21.3.7), (21.1.1) and (22.11.6) in [5]. □

Furthermore, we need the p-dissection formula of $f(-q)$.

Lemma 2.6 ([3, *Theorem* 2.2]). *For any prime $p \geq 5$,*

$$f(-q) = (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}) + \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right).$$

We further claim that for $-(p-1)/2 \leq k \leq (p-1)/2$ and $k \neq (\pm p - 1)/6$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}.$$

Here for any prime $p \geq 5$,

$$\frac{\pm p - 1}{6} := \begin{cases} \frac{p-1}{6}, & p \equiv 1 \pmod{6}, \\ \frac{-p-1}{6}, & p \equiv -1 \pmod{6}. \end{cases}$$

3. Congruences for $\bar{t}(n)$

Theorem 3.1. *For $n \geq 0$ and $\alpha \geq 1$, we have*

$$\bar{t}(6n+4) \equiv 0 \pmod{2}, \tag{3.1}$$
$$\bar{t}(6n+6) \equiv 0 \pmod{2}, \tag{3.2}$$
$$\bar{t}(24n+20) \equiv 0 \pmod{2}, \tag{3.3}$$
$$\bar{t}(4^\alpha(6n+2)) \equiv \bar{t}(6n+2) \pmod{2}, \tag{3.4}$$
$$\bar{t}(4n+2) \equiv 0 \pmod{3}, \tag{3.5}$$
$$\bar{t}(48n+6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46) \equiv 0 \pmod{6}, \tag{3.6}$$
$$\bar{t}(96n+4, 36, 52, 68, 84) \equiv 0 \pmod{8}, \tag{3.7}$$
$$\bar{t}(9n+6) \equiv 0 \pmod{9}, \tag{3.8}$$
$$\bar{t}(48n+14, 22, 30, 46) \equiv 0 \pmod{12}. \tag{3.9}$$

Proof. According to (1.1) and (2.9), we have

$$\sum_{n=0}^{\infty}\bar{t}(n)q^n=\frac{1}{f_2}\cdot\frac{f_3}{f_1}=\frac{1}{f_2}\left(\frac{f_4f_6f_{16}f_{24}^2}{f_2^2f_8f_{12}f_{48}}+q\frac{f_6f_8^2f_{48}}{f_2^2f_{16}f_{24}}\right). \tag{3.10}$$

It follows that

$$\sum_{n=0}^{\infty}\bar{t}(2n)q^n=\frac{f_2f_3f_8f_{12}^2}{f_1^3f_4f_6f_{24}}. \tag{3.11}$$

From (3.11) and (2.19), we have

$$\sum_{n=0}^{\infty}\bar{t}(2n)q^n\equiv\frac{f_1^{10}f_3^9}{f_1^7f_3^{10}}=\frac{f_1^3}{f_3}\equiv\frac{P(q^3)+qf_9^3}{f_3}\pmod 2. \tag{3.12}$$

Since there are no terms in which the power of q is 2 modulo 3, so (3.1) follows. We further notice that, with the help of (2.21) and (2.22),

$$\sum_{n=0}^{\infty}\bar{t}(6n)q^n\equiv\frac{P(q)}{f_1}\equiv\frac{f_1a(q)}{f_1}=a(q)\equiv 1\pmod 2,$$

from which (3.2) follows. Furthermore, from (3.12) and (2.12), we see that

$$\sum_{n=0}^{\infty}\bar{t}(6n+2)q^n\equiv\frac{f_3^3}{f_1}=\frac{f_4^3f_6^2}{f_2^2f_{12}}+q\frac{f_{12}^3}{f_4}\pmod 2.$$

Thus, we derive

$$\sum_{n=0}^{\infty}\bar{t}(12n+8)q^n\equiv\frac{f_6^3}{f_2}\pmod 2,$$

from which we get (3.3). Also,

$$\sum_{n=0}^{\infty}\bar{t}(24n+8)q^n\equiv\frac{f_3^3}{f_1}\equiv\sum_{n=0}^{\infty}\bar{t}(6n+2)q^n\pmod 2.$$

Hence,

$$\bar{t}(24n+8)\equiv\bar{t}(6n+2)\pmod 2,$$

which implies (3.4) by induction on α.

Next, by (3.11) and (2.11), we derive that

$$\sum_{n=0}^{\infty}\bar{t}(2n)q^n = \frac{f_2 f_8 f_{12}^2}{f_4 f_6 f_{24}}\cdot\frac{f_3}{f_1^3} = \frac{f_2 f_8 f_{12}^2}{f_4 f_6 f_{24}}\left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}\right).$$

It follows that

$$\sum_{n=0}^{\infty}\bar{t}(4n)q^n = \frac{f_2^5 f_3^2 f_4}{f_1^8 f_{12}} \tag{3.13}$$

and

$$\sum_{n=0}^{\infty}\bar{t}(4n+2)q^n = 3\frac{f_2 f_4 f_6^4}{f_1^6 f_{12}}, \tag{3.14}$$

the latter of which implies (3.5).

In addition, we have from (3.13) and (2.5) with q^3 for q, that

$$\sum_{n=0}^{\infty}\bar{t}(4n)q^n \equiv \frac{f_2 f_4}{f_{12}}\left(\frac{f_6 f_{24}^5}{f_{12}^2 f_{48}^2} - 2q^3\frac{f_6 f_{48}^2}{f_{24}}\right) \pmod 8.$$

It follows that

$$\sum_{n=0}^{\infty}\bar{t}(8n+4)q^n \equiv -2q\frac{f_1 f_2 f_3 f_{24}^2}{f_6 f_{12}} \pmod 8.$$

Together with (2.13), we deduce that

$$\sum_{n=0}^{\infty}\bar{t}(8n+4)q^n \equiv -2q\frac{f_2 f_{24}^2}{f_6 f_{12}}\left(\frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}\right) \pmod 8.$$

Hence

$$\sum_{n=0}^{\infty}\bar{t}(16n+4)q^n \equiv 2q\frac{f_2^4 f_{12}^4}{f_4^2 f_6^3} \equiv 2q\frac{f_{12}^4}{f_6^3} \pmod 8.$$

Since there are no terms on the right-hand side in which the power of q is 0, 2, 3, 4 or 5 modulo 6, we obtain (3.7).

On the other hand, we see from (3.14) that

$$\sum_{n=0}^{\infty}\bar{t}(4n+2)q^n = 3\frac{f_2 f_4 f_6^4}{f_1^6 f_{12}} \equiv 3\frac{f_1^6 f_3^8}{f_1^6 f_3^4} = 3f_3^4 \equiv 3f_{12} \pmod 6.$$

Since there are no terms on the right-hand side in which the power of q is 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 or 11 modulo 12, we obtain (3.6).

In addition, with the help of (3.14), (2.6) and (2.8), we arrive at

$$\sum_{n=0}^{\infty} \bar{t}(4n+2)q^n = 3\frac{f_2 f_4 f_6^4}{f_{12}}\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right)\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right).$$

It follows that

$$\sum_{n=0}^{\infty} \bar{t}(8n+6)q^n = 6\frac{f_2^{17} f_3^4 f_8^2}{f_1^{18} f_4^5 f_6} + 12\frac{f_2^3 f_3^4 f_4^9}{f_1^{14} f_6 f_8^2}. \tag{3.15}$$

Thus, by (2.19), we see that

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{t}(8n+6)q^n &\equiv 6\frac{f_2^{17} f_3^4 f_8^2}{f_1^{18} f_4^5 f_6} \equiv 6\frac{f_1^{50} f_3^4}{f_1^{38} f_3^2} = 6 f_1^{12} f_3^2 \\ &\equiv 6 f_4^3 f_6 = 6 f_6 (P(q^{12}) - 3q^4 f_{36}^3) \pmod{12}.\end{aligned}$$

Since there are no terms in which the power of q is 1, 2, 3 or 5 modulo 6, we arrive at (3.9).

Finally, we show (3.8). It follows from (1.1) and (2.18) that

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{t}(n)q^n &= \frac{f_3}{f_1 f_2} \\ &= \frac{f_9^3 f_{18}^3}{f_3^3 f_6^4}\left(\frac{1}{w(q^3)^2} + \frac{q}{w(q^3)} + 3q^2 - 2q^3 w(q^3) + 4q^4 w(q^3)^2\right).\end{aligned}$$

Hence, by modulo 9,

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{t}(3n)q^n &= \frac{f_3^3 f_6^4}{f_1^3 f_2^4}\left(\frac{1}{w(q)^2} - 2q w(q)\right) \\ &= \frac{f_3^9}{f_1^5 f_2^2 f_6^3} - 2q\frac{f_6^6}{f_1^2 f_2^5} \\ &= \frac{f_3^9}{f_6^3} \cdot \frac{f_1^4}{f_1^9 f_2^2} - 2q f_6^6 \cdot \frac{f_2^4}{f_1^2 f_2^9} \\ &\equiv \frac{f_3^9}{f_6^3} \cdot \frac{1}{f_3^3} \cdot \frac{f_1^4}{f_2^2} - 2q f_6^6 \cdot \frac{1}{f_6^3}\frac{f_2^4}{f_1^2} \\ &= \frac{f_3^6}{f_6^3}\varphi(-q)^2 - 2q f_6^3 \psi(q)^2 \\ &= \frac{f_3^6}{f_6^3}(\varphi(-q^9) - 2q\varphi(-q^9)\omega(q^3))^2\end{aligned}$$

$$- 2qf_6^3\left(\frac{\psi(q^9)}{\omega(q^3)}+q\psi(q^9)\right)^2, \tag{3.16}$$

where, in the last identity, we use (2.14) and (2.15). It follows from (3.16) that, by modulo 9,

$$\begin{aligned}\sum_{n=0}^{\infty}\bar{t}(9n+6)q^n &\equiv 4\frac{f_1^6}{f_3^3}\varphi(-q^3)^2\omega(q)^2 - 4f_2^3\frac{\psi(q^3)^2}{\omega(q)}\\ &= 4\frac{f_1^6}{f_2^3}\left(\frac{f_3^2}{f_6}\right)^2\left(\frac{f_1f_6^3}{f_2f_3^3}\right)^2 - 4f_2^3\left(\frac{f_6^2}{f_3}\right)^2\frac{f_2f_3^3}{f_1f_6^3}\\ &= 4\left(\frac{f_1^8f_6^4}{f_2^5f_3^2} - \frac{f_2^4f_3f_6}{f_1}\right)\\ &= 4\frac{f_1^9f_6^4 - f_2^9f_3^3f_6}{f_1f_2^5f_3^2}\\ &= 4\frac{(f_1^9f_6^3 - f_2^9f_3^3)f_6}{f_1f_2^5f_3^2}\\ &\equiv 4\frac{(f_1^9f_2^9 - f_2^9f_1^9)f_6}{f_1f_2^5f_3^2}\\ &= 0.\end{aligned}$$

This yields (3.8). □

Theorem 3.2. *For $n \geq 0$, $\alpha \geq 1$, and prime $p \geq 5$, we have*

$$\bar{t}(4p^{2\alpha}n + 2(p+2j)p^{2\alpha-1}) \equiv 0 \pmod 6, \tag{3.17}$$

where $j = 1, 2, \ldots, p-1$.

Proof. We see from (3.14) that

$$\sum_{n=0}^{\infty}\bar{t}(4n+2)q^n \equiv 3f_3^4 \equiv 3f_{12} \pmod 6.$$

Thanks to Lemma 2.6, for any prime $p \geq 5$, we extract terms in which the exponent of q is congruent to $(p^2-1)/2$ modulo p and replace q^p by q to obtain

$$\sum_{n=0}^{\infty}\bar{t}(4pn+2p^2)q^n \equiv 3f(-q^{12p}) \pmod 6.$$

We further extract terms in which the exponent of q is divisible by p and replace q^p by q. Then

$$\sum_{n=0}^{\infty} \bar{t}(4p^2n + 2p^2)q^n \equiv 3f(-q^{12}) \pmod 6.$$

Repeating the process, we derive that, for $\alpha \geq 1$,

$$\sum_{n=0}^{\infty} \bar{t}(4p^{2\alpha-1}n + 2p^{2\alpha})q^n \equiv 3f(-q^{12p}) \pmod 6.$$

This immediately leads to

$$\bar{t}(4p^{2\alpha-1}(pn + j) + 2p^{2\alpha}) \equiv 0 \pmod 6,$$

where $j = 1, 2, \ldots, p - 1$. □

4. Congruences for $\bar{\bar{s}}(n)$

Theorem 4.1. *For $n \geq 0$ and $\alpha \geq 1$, we have*

$$\bar{\bar{s}}(2n+1) \equiv 0 \pmod 3, \tag{4.1}$$
$$\bar{\bar{s}}(3n+2) \equiv 0 \pmod 3, \tag{4.2}$$
$$\bar{\bar{s}}(4n+2) \equiv 0 \pmod 3, \tag{4.3}$$
$$\bar{\bar{s}}(6n+6) \equiv 0 \pmod 4, \tag{4.4}$$
$$\bar{\bar{s}}(6n+4) \equiv 0 \pmod 6, \tag{4.5}$$
$$\bar{\bar{s}}(6n+5) \equiv 0 \pmod 9, \tag{4.6}$$
$$\bar{\bar{s}}(18n+15) \equiv 0 \pmod 9, \tag{4.7}$$
$$\bar{\bar{s}}(3^{\alpha}(18n+9)) \equiv \bar{\bar{s}}(18n+9) \pmod 9, \tag{4.8}$$
$$\bar{\bar{s}}(8n+6) \equiv 0 \pmod{12}, \tag{4.9}$$
$$\bar{\bar{s}}(48n+14, 22, 30, 46) \equiv 0 \pmod{24}. \tag{4.10}$$

Proof. It follows from (1.2) and (2.10) that

$$\sum_{n=0}^{\infty} \bar{\bar{s}}(n)q^n = \frac{f_1^3 f_6}{f_2^3 f_3} = \frac{f_1^3}{f_3} \cdot \frac{f_6}{f_2^3} = \frac{f_6}{f_2^3}\left(\frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}\right).$$

Hence,

$$\sum_{n=0}^{\infty} \bar{\bar{s}}(2n)q^n = \frac{f_2^3 f_3}{f_1^3 f_6} \tag{4.11}$$

and

$$\sum_{n=0}^{\infty} \bar{\bar{s}}(2n+1)q^n = -3\frac{f_6^3}{f_1 f_2 f_3}, \tag{4.12}$$

the latter of which implies (4.1).

It follows from (4.11) and (2.11) that

$$\sum_{n=0}^{\infty}\bar{\bar{s}}(2n)q^n=\frac{f_2^3f_3}{f_1^3f_6}=\frac{f_2^3}{f_6}\cdot\frac{f_3}{f_1^3}=\frac{f_2^3}{f_6}\left(\frac{f_4^6f_6^3}{f_2^9f_{12}^2}+3q\frac{f_4^2f_6f_{12}^2}{f_2^7}\right).$$

We therefore have

$$\sum_{n=0}^{\infty}\bar{\bar{s}}(4n+2)q^n=3\frac{f_2^2f_6^2}{f_1^4}, \tag{4.13}$$

which yields (4.3). Referring to (2.8) and (4.13), we deduce that

$$\sum_{n=0}^{\infty}\bar{\bar{s}}(4n+2)q^n=3f_2^2f_6^2\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right). \tag{4.14}$$

Thus,

$$\sum_{n=0}^{\infty}\bar{\bar{s}}(8n+6)q^n=12\frac{f_2^2f_3^2f_4^4}{f_1^8}. \tag{4.15}$$

This gives (4.9). Moreover, we have

$$\sum_{n=0}^{\infty}\bar{\bar{s}}(8n+6)q^n\equiv 12f_4^3f_6=12f_6(P(q^{12})-3q^4f_{36}^3)\pmod{24}.$$

Since there are no terms in which the power of q is $1, 2, 3$ or 5 modulo 6, we arrive at (4.10).

On the other hand, we write (4.11) as

$$\sum_{n=0}^{\infty}\bar{\bar{s}}(2n)q^n=\frac{f_3}{f_6}\cdot\frac{\psi(q)}{\varphi(-q)}.$$

Invoking (2.15) and (2.16), we derive that

$$\sum_{n=0}^{\infty}\bar{\bar{s}}(6n)q^n=\frac{f_1}{f_2}\frac{\varphi(-q^3)^3\psi(q^3)}{\varphi(-q)^4}\left(\frac{1}{w(q)}+4qw(q)^2\right) \tag{4.16}$$

and

$$\sum_{n=0}^{\infty}\bar{\bar{s}}(6n+4)q^n=6\frac{f_1}{f_2}\frac{\varphi(-q^3)^3\psi(q^3)w(q)}{\varphi(-q)^4}. \tag{4.17}$$

Here (4.17) implies (4.5). In addition, we have

$$\sum_{n=0}^{\infty} \bar{\bar{s}}(6n)q^n \equiv \frac{f_2^4 f_3^8}{f_1^8 f_6^4} \equiv 1 \pmod 4,$$

which yields (4.4).

From (4.12), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{\bar{s}}(2n+1)q^n &= -3\frac{f_6^3}{f_3} \cdot \frac{1}{f_1 f_2} \\ &= -3\frac{f_6^3}{f_3} \cdot \frac{f_9^3 f_{18}^3}{f_3^4 f_6^4} \left(\frac{1}{w(q^3)^2} + \frac{q}{w(q^3)} \right. \\ &\qquad \left. + 3q^2 - 2q^3 w(q^3)^2 + 4q^4 w(q^3)^2 \right).\end{aligned}$$

This immediately yields (4.6). We also have

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{\bar{s}}(6n+3)q^n &= -3\frac{f_3^3 f_6^3}{f_1^5 f_2} \left(\frac{1}{\omega(q)} + 4q\omega(q)^2 \right) \\ &= -3\frac{f_3^3 f_6^3}{f_1^5 f_2} \left(\frac{f_2 f_3^3}{f_1 f_6^3} + 4q \frac{f_1^2 f_6^6}{f_2^2 f_3^6} \right) \\ &= -3\frac{f_3^6}{f_1^6} - 12q \frac{f_6^9}{f_1^3 f_2^3 f_3^3} \\ &\equiv -3\frac{f_3^6}{f_3^2} - 3q \frac{f_6^9}{f_3^4 f_6} \pmod 9 \\ &= -3 f_3^4 - 3q\psi(q^3)^4. \end{aligned} \tag{4.18}$$

Since there are no terms on the right-hand side in which the power of q is 2 modulo 3, we derive (4.7). Furthermore, it follows from (4.18) that

$$\sum_{n=0}^{\infty} \bar{\bar{s}}(18n+9)q^n \equiv -3\psi(q)^4 = -3\sum_{n=0}^{\infty} \sigma(2n+1)q^n \pmod 9,$$

where $\sigma(n)$ is the sum of the divisors of n; see [5, eq. (25.1.26)]. So,

$$\bar{\bar{s}}(18n+9) \equiv -3\sigma(2n+1) \pmod 9.$$

It follows that

$$\bar{\bar{s}}(54n+27) \equiv \bar{\bar{s}}(18n+9) \pmod 9.$$

We derive (4.8) by induction on α.

Finally, one may write (1.2) as

$$\sum_{n=0}^{\infty} \bar{\bar{s}}(n)q^n = \frac{f_1^3 f_6}{f_2^3 f_3} = \frac{f_6}{f_3} \cdot \frac{\varphi(-q)}{\psi(q)}$$
$$= \frac{f_6}{f_3}$$
$$\cdot \varphi(-q^9)(1-2qw(q^3)) \cdot \frac{\psi(q^9)^3}{\psi(q^3)^4}\left(\frac{1}{w(q^3)^2} - \frac{q}{w(q^3)} + q^2\right), \tag{4.19}$$

where we use (2.14) and (2.17) in the last identity. Hence

$$\sum_{n=0}^{\infty} \bar{\bar{s}}(3n+2)q^n = 3\frac{f_2}{f_1}\frac{\varphi(-q^3)\psi(q^3)^3}{\psi(q)^4} = 3\frac{f_1^3 f_6^5}{f_2^7 f_3}, \tag{4.20}$$

which implies (4.2). □

Theorem 4.2. *For $n \geq 0$, $\alpha \geq 1$, and any prime $p \geq 5$, we have*

$$\bar{\bar{s}}(8p^{2\alpha}n + 2(p+4j)p^{2\alpha-1}) \equiv 0 \pmod{6}, \tag{4.21}$$

where $j = 1, 2, \ldots, p-1$.

Proof. It follows from (4.14) that

$$\sum_{n=0}^{\infty} \bar{\bar{s}}(8n+2)q^n = 3\frac{f_2^{14} f_3^2}{f_1^{12} f_4^4} \equiv 3f_6 \pmod{6}.$$

The rest of the proof is similar to that of Theorem 3.2. By employing Lemma 2.6, we obtain that, for any prime $p \geq 5$,

$$\sum_{n=0}^{\infty} \bar{\bar{s}}(8pn + 2p^2)q^n \equiv 3f(-q^{6p}) \pmod{6}$$

and

$$\sum_{n=0}^{\infty} \bar{\bar{s}}(8p^2 n + 2p^2)q^n \equiv 3f(-q^6) \pmod{6}.$$

We deduce by induction on $\alpha \geq 1$ that

$$\sum_{n=0}^{\infty} \bar{\bar{s}}(8p^{2\alpha-1}n + 2p^{2\alpha})q^n \equiv f(-q^{6p}) \pmod{6}.$$

This immediately yields

$$\bar{\bar{s}}(8p^{2\alpha-1}(pn+j)+2p^{2\alpha}) \equiv 0 \pmod{6},$$

where $j = 1, 2, \ldots, p-1$. □

Theorem 4.3. *For $n \geq 0$, $\alpha \geq 1$, and odd prime p with*

$$\left(\frac{-3}{p}\right) = -1,$$

we have

$$\bar{\bar{s}}(12p^{2\alpha}n + 2(p+6j)p^{2\alpha-1}) \equiv 0 \pmod{9}, \tag{4.22}$$

where $j = 1, 2, \ldots, p-1$.

Proof. We see from (4.20) and (2.10) that

$$\sum_{n=0}^{\infty} \bar{\bar{s}}(6n+2)q^n = 3\frac{f_2^3 f_3^5}{f_1^7 f_6}. \tag{4.23}$$

With the help of (2.12) and (4.23), we conclude that

$$\sum_{n=0}^{\infty} \bar{\bar{s}}(6n+2)q^n \equiv 3\frac{f_3^3}{f_1} = 3\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right) \pmod{9}.$$

Thus, we have

$$\sum_{n=0}^{\infty} \bar{\bar{s}}(12n+2)q^n \equiv 3\frac{f_2^3 f_3^2}{f_1^2 f_6} \equiv 3f_1 f_3 \pmod{9}.$$

For a prime $p \geq 5$, and integers k and m with $-(p-1)/2 \leq k, m \leq (p-1)/2$, we consider

$$\frac{3k^2+k}{2} + 3\cdot\frac{3m^2+m}{2} \equiv \frac{p^2-1}{6} \pmod{p},$$

that is,

$$2(6k+1)^2 + 6(6m+1)^2 \equiv 0 \pmod{p}. \tag{4.24}$$

It follows that, for any odd prime p with

$$\left(\frac{-3}{p}\right) = -1,$$

the solution to (4.24) is $k = m = (\pm p - 1)/6$.

Hence, using Lemma 2.6, we arrive at

$$\sum_{n=0}^{\infty} \bar{\bar{s}}(12pn + 2p^2)q^n \equiv 3f(-q^p)f(-q^{3p}) \pmod 9$$

and

$$\sum_{n=0}^{\infty} \bar{\bar{s}}(12p^2n + 2p^2)q^n \equiv 3f(-q)f(-q^3) \pmod 9.$$

We therefore obtain by induction on $\alpha \geq 1$ that

$$\sum_{n=0}^{\infty} \bar{\bar{s}}(12p^{2\alpha-1}n + 2p^{2\alpha})q^n \equiv 3f(-q^p)f(-q^{3p}) \pmod 9.$$

This implies that

$$\bar{\bar{s}}(12p^{2\alpha-1}(pn+j) + 2p^{2\alpha}) \equiv 0 \pmod 9,$$

where $j = 1, 2, \ldots, p-1$. □

5. Final remarks

We also obtain the following modulo 5 congruence for $\bar{t}(n)$. By an algorithm due to Radu and Sellers [6], we may provide a modular form proof. However, this proof is very routine and tedious. We therefore want an elementary proof for this congruence.

Claim 5.1. For $n \geq 0$, we have

$$\bar{t}(45n + 30) \equiv 0 \pmod 5.$$

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