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A criterion for quasinormality in $\mathbb{C}^n$

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Abstract. In this article, we give a Zalcman type renormalization result for the quasinormality of a family of holomorphic functions on a domain in $\mathbb{C}^n$ that takes values in a complete complex Hermitian manifold.

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1. Introduction

The convergence of a family of functions always has far reaching consequences. In his path-breaking paper of 1907, Montel [10] gave a result of the convergence of the family of holomorphic functions which states that a sequence of uniformly bounded holomorphic functions has a subsequence that is locally uniformly convergent. Later in 1912, Montel [11] introduced the term *normal family* for a family satisfying this convergence property. In a subsequent paper [12], he introduced the notion of quasinormality of a family of functions in one complex variable. All these ideas are well documented in his influential book [13]. The normality of a family of functions is one of the most fundamental concepts in function theory of one and several complex variables. It has been extensively used in the study of dynamical properties of functions of one or more complex variables. Normality plays a deep role in certain results in complex dynamics. It can be seen that normality is the central part of the definition of the Fatou set of a holomorphic function f, which is defined as the domain of normality of the family of iterations of f. The Fatou set and the Julia set, simply by definition (as the complement of the Fatou set), set up a dichotomy. In a different direction, Beardon and Minda [3] discussed normal families in terms of maps that satisfy certain types of uniform Lipschitz conditions with respect to various conformal metrics and more background materials can be found in [4,13,18]. While all these provide sufficient conditions for normality, Zalcman in [22] proved a striking result that studies consequence of non-normality. Roughly speaking, it says that in an infinitesimal scaling, the family gives a non-constant entire function under the compact-open topology. We state this renormalization result which is known as *Zalcman's Lemma*:

Zalcman's Lemma. A family $\mathcal{F}$ of functions meromorphic (analytic) on the unit disc Δ is not normal if and only if there exist

(*a*) *a number r,* $0 < r < 1$;
(*b*) *points* $z_j, |z_j| < r$;
(*c*) *functions* $\{f_j\} \subseteq \mathcal{F}$;
(*d*) *numbers* $\rho_j \to 0^+$;

such that

$$f_j(z_j + \rho_j\zeta) \to g(\zeta)$$

spherically uniformly (uniformly) on compact subsets of $\mathbb{C}$, *where g is a non-constant meromorphic (entire) function on* $\mathbb{C}$.

This lemma leads to a heuristic principle in function theory. The principle says that any property which forces an entire function to be constant will also force a family of holomorphic functions to be normal. The source is Marty's inequality which gives a necessary and sufficient condition for the normality of a family of holomorphic or meromorphic functions on a domain $\Omega \subset \mathbb{C}$.

It is very natural to explore the extension of Zalcman's Lemma in several complex variables. In [2], Aladro and Krantz gave an analogue of Zalcman's Lemma for families of holomorphic mappings f_j from a hyperbolic domain of $\mathbb{C}^n$ into a complete complex Hermitian manifold M (also see, Lemma 5.1 [7]). Their analysis was completed by Thai, Trang and Huong in [20], which addresses the possibility of compact divergence of the renormalized mappings $g_j(\zeta) = f_j(z_j + \rho_j\zeta)$. In the same paper, Thai *et al.* [20] also defined the concept of Zalcman space. Loosely speaking, a complex space X is a Zalcman space if for each non-normal family of holomorphic mappings of the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ into X, we get a non-constant holomorphic mapping $g : \mathbb{C} \to X$ under the compact-open topology after an infinitesimal scaling. This work is further studied in [19,21]. In this paper, our goal is to prove an analogue of Zalcman's lemma for quasi-normal families in several complex variables (c.f. Theorem 1.1). We have illustrated our results with examples.

The theory of quasinormality is well studied in one complex variable. Chuang, in his text [4], introduced the notion of Q_m-normality ($m \geq 0$) as an extension of quasinormality in the complex plane, Q_0 and Q_1-normality are the usual normality and quasinormality respectively. Loosely speaking, a Q_m-normal family on a domain D is normal outside a subset of D whose m-th order derived set is empty. He introduced the notion of μ_m-point and established some characterizations of Q_m-normality. Roughly speaking, a point $z_0 \in D$ is a μ_1-point of a family $\mathcal{F}$ if the family violates the Marty's criterion on z_0 and μ_2-point is the accumulation point of μ_1-points. Inductively, a μ_m-point is an accumulation point of μ_{m-1}-points. In this paper, we extend the notions of μ_1 and μ_2-points in higher dimensions whereas we could not generalize the notion of μ_m-points for $m \geq 3$ in several variables due to the nature of zeros of holomorphic mappings in higher dimensions. It seems that the 'order of quasinormality' as given in one variable is not plausible in higher dimension. It is interesting to note that using the notion of μ_m-points, Nevo [14] proved a Zalcman type renormalization result for Q_m-normal families on planar domains .

In several complex variables, the theory of quasinormality has its origin in the work of Rutishauser [17] and Fujimoto [6]. Fujimoto [6], extending the work of Rutishauser, introduced the notion of meromorphic convergence. In a recent article, Ivashkovich and Neji [7] discussed several notions of convergence, namely, strong convergence, weak convergence and gamma convergence. It can be seen easily from the definitions that weakly-normal implies quasi-normal. In this paper, we have also given a renormalization result for weakly-normal family of holomorphic mappings. It is instructive to note here a survey article [5] by Dujardin, where he gives a sufficient condition for quasinormality of a family of holomorphic mappings from a complex manifold to a compact Kähler manifold in terms of a suitable sequence of bidegree (1,1) currents.

The main result of this paper provides an analogue of the Zalcman's lemma for the quasi-normal families. Our main result is as follows:

Theorem 1.1. *Let $\Omega \subseteq \mathbb{C}^n$ be a hyperbolic domain. Let M be a complete complex Hermitian manifold of dimension k. Let $\mathcal{F} = \{f_\alpha\}_{\alpha\in A} \subseteq \operatorname{Hol}(\Omega, M)$. The family $\mathcal{F}$ is not quasi-normal if and only if there exist a subset $E \subset \Omega$ which is either a non-analytic subset or the closure $\bar{E}$ has non-empty interior and corresponding to each $p \in E$, there exist*

(*a*) *a sequence of points $\{w_{j,p}\}_{j=1}^{\infty} \subset \Omega$ such that $w_{j,p} \to p$,*
(*b*) *a sequence of functions $\{f_j\} \subset \mathcal{F}$,*
(*c*) *a sequence of positive real numbers $\rho_{j,p} \to 0$ such that*

$$g_j(\zeta) := f_j(w_{j,p} + \rho_{j,p}\xi), \quad \xi \in \mathbb{C}^n \quad (p \in E)$$

satisfies one of the following two assertions:

(*i*) *The sequence $\{g_j\}$ is compactly divergent on $\mathbb{C}^n$.*
(*ii*) *The sequence $\{g_j\}$ converges uniformly on compact subsets of $\mathbb{C}^n$ to a non-constant holomorphic map $g_p : \mathbb{C}^n \to M$.*

Here we give an example to elucidate Theorem 1.1

Example 1.2. Consider the family of holomorphic mappings $\{f_n(z_1, z_2) = z_1^n\}$ from $\mathbb{C}^2$ into $\mathbb{C}$. Clearly, f_n is not normal in $E = \{(z_1, z_2) : |z_1| = 1\}$. Therefore, $\{f_n\}$ is not quasi-normal in $\mathbb{C}^2$. To see this, fix $0 \le \theta < 2\pi$ and consider the sequence $\{z_j\} = \{(z_{1j}, z_{2j})\}$, where $z_{1j} = \mathrm{e}^{i\theta/j}$ and $z_{2j} \in \mathbb{C}$. For this sequence $\{z_j\}$ and $\zeta = (\zeta_1, \zeta_2)$, the sequence $f_j(z_j + \rho_j\zeta)$ converges to a non-constant holomorphic function $\mathrm{e}^{\zeta_1+i\theta}$.

2. Preliminary definitions and main results

Let $\Omega \subset \mathbb{C}^n$ be an open domain and Δ be the unit disc in $\mathbb{C}$. If $z \in \Omega$ and $\xi \in \mathbb{C}^n$, then by the work of Royden [16], the infinitesimal form of the Kobayashi pseudo-metric for Ω at z in the direction ξ is defined as

$$F_K^\Omega(z, \xi) = \inf_f \left\{ \frac{\|\xi\|}{\|f'(0)\|} : f : \Delta \to \Omega \text{ is holomorphic, } f(0) = z, \right.$$
$$\left. \text{and } f'(0) \text{ is a constant multiple of } \xi \right\},$$

where $\|.\|$ represents the Euclidean length. The Kobayashi pseudo-distance between z and w in Ω is defined as

$$K_\Omega(z, w) = \inf_\gamma \int_0^1 F_K^\Omega(\gamma(t), \gamma'(t))\mathrm{d}t,$$

where the infimum is taken over all C^1-curves $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = z$ and $\gamma(1) = w$.

In this work, we shall use the following definition of (Kobayashi) hyperbolicity which, as shown by Royden [16], is equivalent to the original definition.

DEFINITION 2.1 [2,16]

A domain $\Omega \subseteq \mathbb{C}^n$ is called *hyperbolic* at a point $z \in \Omega$ if there is a neighborhood V of z in Ω and a positive constant c such that

$$F_K^\Omega(y, \xi) \geq c \,\|\xi\| \quad \text{for all } y \in V \text{ and all } \xi \in \mathbb{C}^n.$$

We say that Ω is *hyperbolic* if it is hyperbolic at each point.

Let M be a complete complex Hermitian manifold of dimension k and let $\mathscr{T}_p(M)$ denotes the complexified tangent space to M at p. We denote the metric for M at p in the direction of the vector $\xi \in \mathscr{T}_p(M)$ by $E_M(p; \xi)$. Let $\Omega \subseteq \mathbb{C}^n$ be a hyperbolic domain. We denote the set of all holomorphic mappings from Ω into M by $\mathrm{Hol}(\Omega, M)$.

DEFINITION 2.2

Let $\mathcal{F}$ be a family of holomorphic mappings of a domain Ω in $\mathbb{C}^n$ into a complete complex manifold M. $\mathcal{F}$ is said to be a *normal family* on Ω if $\mathcal{F}$ is relatively compact in $\mathrm{Hol}(\Omega, M)$ in the compact open topology.

DEFINITION 2.3

Let X, Y be complex spaces and $\mathcal{F} \subset \mathrm{Hol}(X, Y)$. A sequence $\{f_j\} \subset \mathcal{F}$ is *compactly divergent* if for every compact $K \subset X$ and for every compact $L \subset Y$ there is a number $J = J(K, L)$ such that $f_j(K) \cap L = \emptyset$ for all $j \geq J$. If $\mathcal{F}$ contains no compactly divergent sequences, then $\mathcal{F}$ is called *not compactly divergent.*

Let $\Omega \subseteq \mathbb{C}^n$ be a domain. A subset S of Ω is called a *complex analytic subset* if for any $z \in \Omega$ there exist a neighborhood U of z and holomorphic functions $f_1, \ldots, f_l$ on U such that $S \cap U = \{z \in U : f_1(z) = \ldots = f_l(z) = 0\}$. Notice that analytic subsets are closed and nowhere dense in Ω.

DEFINITION 2.4

A sequence $\{f_j\}$ of holomorphic mappings from a domain $\Omega \subset \mathbb{C}^n$ into a complete complex Hermitian manifold M is said to be *weakly-regular* on Ω if any $z \in \Omega$ has a connected neighborhood U with the property that $\{f_j\}$ converges uniformly on compact subsets of $U \setminus E$ or compactly diverges on $U \setminus E$, where $E \subset U$ is an analytic subset of codimension at least 2.

DEFINITION 2.5

Let $\mathcal{F}$ be a family of holomorphic mappings from a domain Ω in $\mathbb{C}^n$ into a complete complex Hermitian manifold M. $\mathcal{F}$ is said to be a *weakly-normal family* on Ω if any sequence in $\mathcal{F}$ has a weakly-regular subsequence on Ω.

DEFINITION 2.6

A sequence $\{f_j\}$ of holomorphic mappings from a domain $\Omega \subset \mathbb{C}^n$ into a complete complex Hermitian manifold M is said to be *quasi-regular* on Ω if any $z \in \Omega$ has a connected neighborhood U with the property that $\{f_j\}$ converges uniformly on compact subsets of $U \setminus E$ or compactly diverges on $U \setminus E$, where $E \subset U$ is a proper complex analytic subset of U.

DEFINITION 2.7

Let $\mathcal{F}$ be a family of holomorphic mappings from a domain Ω in $\mathbb{C}^n$ into a complete complex Hermitian manifold M. $\mathcal{F}$ is said to be a *quasi-normal family* on Ω if any sequence in $\mathcal{F}$ has a quasi-regular subsequence on Ω.

Theorem A [1,2]. *Let $\Omega \subseteq \mathbb{C}^n$ be a hyperbolic domain. Let M be a complete complex Hermitian manifold of dimension k with metric E_M. Let $\mathcal{F} = \{f_\alpha\}_{\alpha\in A} \subseteq \mathrm{Hol}(\Omega, M)$. If the family $\mathcal{F} = \{f_\alpha\}_{\alpha\in A}$ is a normal family, then for each compact set $L \Subset \Omega$ (i.e., L is relatively compact in Ω), there is a constant C_L such that for all $z \in L$ and all $\xi \in \mathbb{C}^n$, it holds that*

$$\sup_{\alpha\in A} |E_M(f_\alpha(z); (f_\alpha)_*(z).\xi)| \leq C_L F_K^{\Omega}(z, \xi). \tag{2.1}$$

Conversely, if (2.1) holds and if for some $p \in \Omega$, all $f_\alpha(p)$ are in some compact set Q of M, then $\mathcal{F} = \{f_\alpha\}_{\alpha\in A}$ is a normal family.

Aladro and Krantz [2] gave an extension of the Zalcman's lemma to the higher-dimensional setting. A case missing from the analysis in [2] was provided by Thai *et al.* [20]. Their result is as follows:

Theorem B [20]. *Let Ω be a domain in $\mathbb{C}^n$. Let M be a complete complex Hermitian space. Let $\mathcal{F} \subset \mathrm{Hol}(\Omega, M)$. Then the family $\mathcal{F}$ is not normal if and only if there exist sequences $\{p_j\} \subset \Omega$ with $p_j \to p_0 \in \Omega$, $\{f_j\} \subset \mathcal{F}$, $\{\rho_j\} \subset \mathbb{R}$ with $\rho_j > 0$ and $\rho_j \to 0$ such that*

$$g_j(\xi) := f_j(p_j + \rho_j\xi), \qquad \xi \in \mathbb{C}^n$$

satisfies one of the following two assertions:

(*i*) *The sequence $\{g_j\}_{j\geq 1}$ is compactly divergent on $\mathbb{C}^n$.*
(*ii*) *The sequence $\{g_j\}_{j\geq 1}$ converges uniformly on compact subsets of $\mathbb{C}^n$ to a non-constant holomorphic map $g : \mathbb{C}^n \to M$.*

3. Proof of the main result

Before giving the proof of our main result, we give some definitions and lemmas, whose one-dimensional analogue can be found in [4,14]. Throughout section 3, $\Omega \subseteq \mathbb{C}^n$ is a hyperbolic domain and M denotes a complete complex Hermitian manifold. Here we extend the notions μ_1-point and μ_2-point of a sequence in one dimension to a sequence $\{f_j\} \subset \mathrm{Hol}\,(\Omega, M)$ in higher dimension.

DEFINITION 3.1

Let $\Omega \subseteq \mathbb{C}^n$ be a hyperbolic domain. Let M be a complete complex Hermitian manifold of dimension k. Consider a sequence $\{f_j\} \subset \mathrm{Hol}(\Omega, M)$. A point $p_0 \in \Omega$ is said to be a *μ_1-point* of $\{f_j\}$, if for each subset $K \Subset \Omega$ containing p_0,

$$\lim_{j\to\infty} \sup_{p\in K, \|\xi\|=1} |E_M(f_j(p); (f_j)_*(p)\cdot\xi)| = \infty.$$

(1) A point p_0 is called a *μ_2-point* of $\{f_j\}$ if there exists an analytic set $K \subset \Omega$ of codimension at most 1, containing p_0, such that each point of K is a μ_1-point of $\{f_j\}$.
(2) We say p_0 is a *q-point* of $\{f_j\}$ if there exists a subset $K \subset \Omega$, containing p_0, such that closure $\bar{K} = \Omega$ and each point of K is a μ_1-point of $\{f_j\}$.
(3) We say p_0 is a *λ-point* of $\{f_j\}$ if there exists a non-analytic subset $K \subset \Omega$ containing p_0 such that each point of K is a μ_1-point of $\{f_j\}$.

Example 3.2. Let $\{f_n\}$ be a family of holomorphic mappings from $\mathbb{C}^2$ onto itself such that $f_n(z) = nz$, where $z = (z_1, z_2)$. Then $z = (0, 0)$ is a μ_1-point of $\{f_n\}$.

Example 3.3. Let $\{f_n\}$ be a family of holomorphic mappings defined on the polydisc $D = \{(z_1, z_2) : |z_1| < 1 \text{ and } |z_2| < 1\}$ such that $f_n(z_1, z_2) = nz_1z_2$. Then each point of $E = \{(z_1, z_2) : z_1z_2 = 0\}$ is a μ_2-point of $\{f_n\}$.

Example 3.4. Let $\{f_n\}$ be a family of holomorphic mappings defined on $\mathbb{C}^2$ such that $f_n(z_1, z_2) = \mathrm{e}^{nz_1}$. Then each point of $E = \{(z_1, z_2) : \mathfrak{Re}\, z_1 = 0\}$ is a λ-point of $\{f_n\}$.

Lemma 3.5. *Let $\Omega \subseteq \mathbb{C}^n$ be a hyperbolic domain. Let M be a complete complex Hermitian manifold of dimension k. A family $\mathcal{F} \subset \mathrm{Hol}(\Omega, M)$ is normal in Ω if and only if each sequence $\{f_j\}$ of $\mathcal{F}$ has no μ_1-point in Ω.*

Proof. Suppose $\mathcal{F}$ is normal. Then by Theorem A there is no μ_1-point for any sequence $\{f_j\}$ of $\mathcal{F}$.

Let $\mathcal{F} \subset \mathrm{Hol}(\Omega, M)$ such that every sequence $\{f_n\}$ of functions of $\mathcal{F}$ has no μ_1-point in Ω. Then (2.1) holds and $\{f_n\}$ is not compactly divergent. Otherwise, there is a point $p_0 \in \Omega$ such that we can not find a ball $\Gamma = \{p : \|p - p_0\| < r\}$, $\Gamma \Subset \Omega$ and a number $N > 0$ such that for $j \geq 1$, we have

$$|E_M(f_j(p); (f_j)_*(p).\xi)| \leq N \quad \text{in } \bar{\Gamma}.$$

Take two sequences of positive real numbers $\{r_k\} \to 0$ and $\{N_k\} \to \infty$ such that the ball $\bar{\Gamma}_k = \{p : \|p - p_0\| \le r_k\}$ is contained in Ω. Then there is an integer $j_1 \ge 1$ such that

$$\sup_{p\in\bar{\Gamma}_1, \|\xi\|=1} |E_M(f_{j_1}(p); (f_{j_1})_*(p) \cdot \xi)| > N_1.$$

Next there is an integer $j_2 > j_1$ such that

$$\sup_{p\in\bar{\Gamma}_2, \|\xi\|=1} |E_M(f_{j_2}(p); (f_{j_2})_*(p) \cdot \xi)| > N_2.$$

Continuing in this manner, we get a sequence of integers $\{j_k\}$, $(k = 1, 2, \ldots)$ such that for $k \ge 1$, we have

$$\sup_{p\in\bar{\Gamma}_k, \|\xi\|=1} |E_M(f_{j_k}(p); (f_{j_k})_*(p) \cdot \xi)| > N_k.$$

Now consider a ball $\Gamma : \|p - p_0\| < r$ such that $\bar{\Gamma} \Subset \Omega$. Let $k_0 \ge 1$ be an integer such that $r_k < r$ for $k \ge k_0$. Then for $k \ge k_0$, we have

$$\begin{aligned} N_k &< \sup_{p\in\bar{\Gamma}_k, \|\xi\|=1} |E_M(f_{j_k}(p); (f_{j_k})_*(p) \cdot \xi)| \\ &\le \sup_{p\in\bar{\Gamma}, \|\xi\|=1} |E_M(f_{j_k}(p); (f_{j_k})_*(p) \cdot \xi)|. \end{aligned}$$

Hence

$$\lim_{k\to\infty} \sup_{p\in\bar{\Gamma}, \|\xi\|=1} |E_M(f_{j_k}(p); (f_{j_k})_*(p) \cdot \xi)| = \infty.$$

This implies p_0 is a μ_1-point of the sequence $\{f_j\}$ which is a contradiction. □

Lemma 3.6. *Let $\Omega \subseteq \mathbb{C}^n$ be a hyperbolic domain. Let M be a complete complex Hermitian manifold of dimension k. A family $\mathcal{F} \subset \mathrm{Hol}(\Omega, M)$ is weakly-normal in Ω if and only if each sequence $\{f_j\}$ of $\mathcal{F}$ has neither a μ_2-point nor a λ-point in Ω.*

Proof. Suppose that $\mathcal{F}$ is weakly-normal in Ω. Let $\{f_j\}$ be a sequence of functions of $\mathcal{F}$. Then we can extract a weakly-regular subsequence $\{f_{j_k}\}$ from $\{f_j\}$. On the contrary, we assume that $\{f_j\}$ has a μ_2-point p_0 in Ω. Since $\mathcal{F}$ is weakly-normal, therefore we can find a neighborhood U_0 of p_0 in Ω such that $\{f_{j_k}\}$ converges uniformly on compact subsets of $U_0 \setminus E$, or diverges compactly on $U_0 \setminus E$, where E is an analytic subset of U_0 of codimension at least 2. For each $p' \in U_0 \setminus E$, $\{f_{j_k}\}$ converges or diverges compactly and hence $\{f_{j_k}\}$ is normal in $U_0 \setminus E$. So by Lemma 3.5, $\{f_{j_k}\}$ has no μ_1-point in $U_0 \setminus E$. But by definition of μ_2-point, there exist an analytic set $K \subset \Omega$ of codimension at most 1, such that each point of K is a μ_1-point of $\{f_{j_k}\}$ and hence of $\{f_j\}$, which is a contradiction. A similar argument can be given if p_0 is a λ-point.

Conversely, suppose that $\mathcal{F}$ has neither a μ_2-point nor a λ-point and $\mathcal{F}$ is not weakly-normal on Ω. Consider a set $K \Subset \Omega$. Let $\{f_j\}$ be a sequence of functions of $\mathcal{F}$. Then we can not extract a subsequence which is weakly-regular in K. It follows from Lemma 3.5

that $\{f_j\}$ must have μ_1-points in K. Also, the set V of all μ_1-points contains either a non-empty analytic subset $V_1 \subset \Omega$ of codimension at most 1 or a non-analytic set $V_2 \subset \Omega$, otherwise $\{f_j\}$ constitutes a weakly-normal family. Since V_1 is a set of codimension at most 1, then for $p \in V_1$ there exists a neighborhood N_1 of p and each point of $N_1 \cap V_1$ is a μ_1-point of $\{f_j\}$. Thus p is a μ_2-point of $\{f_j\}$. Also, V_2 is a non-analytic set. Then for $p \in V_2$, there exists a neighborhood N_2 of p and each point of $N_2 \cap V_2$ is a μ_1-point of $\{f_j\}$. Thus p is a λ-point of $\{f_j\}$. In either case, we get a contradiction. □

We can prove the following result on similar lines.

Lemma 3.7. *Let $\Omega \subseteq \mathbb{C}^n$ be a hyperbolic domain. Let M be a complete complex Hermitian manifold of dimension k. A family $\mathcal{F} \subset \mathrm{Hol}(\Omega, M)$ is quasi-normal in Ω if and only if each sequence $\{f_j\}$ of $\mathcal{F}$ has neither a q-point nor a λ-point in Ω.*

We now give a local version of Zalcman's lemma for normal families. Lemma 3.8 is a local version of Theorem B. In what follows, the term *normality* (*or quasi-normality*) *at a point* z_0 will be normality (or quasi-normality) on an open neighborhood of the point z_0.

Lemma 3.8. *Let $\Omega \subseteq \mathbb{C}^n$ be a hyperbolic domain. Let M be a complete complex Hermitian manifold of dimension k. Let $\mathcal{F} = \{f_\alpha\}_{\alpha \in A} \subseteq \mathrm{Hol}(\Omega, M)$. The family $\mathcal{F}$ is not normal at $p_0 \in \Omega$ if and only if there exist*

(*a*) *a sequence $\{p_j\} \subset \Omega$ such that $p_j \to p_0$;*
(*b*) *a sequence of functions $\{f_j\} \subset \mathcal{F}$;*
(*c*) *a sequence of positive real numbers $\rho_j \to 0$ such that*

$$g_j(\xi) := f_j(p_j + \rho_j \xi), \quad \xi \in \mathbb{C}^n$$

satisfies one of the following two assertions:

(*i*) *The sequence $\{g_j\}$ is compactly divergent on $\mathbb{C}^n$.*
(*ii*) *The sequence $\{g_j\}$ converges uniformly on compact subsets of $\mathbb{C}^n$ to a non-constant holomorphic map $g : \mathbb{C}^n \to M$.*

Proof. Assume that $\mathcal{F}$ is not normal at p_0. Then by Theorem A, there exists a compact set $K_0 \subset \{p : \|p - p_0\| \leq \rho\} = K_1$ for some $\rho > 0$ and a sequence $f_j \subset \mathcal{F}$, $\{q_j\} \subset K_0$ and $\{\xi_j\} \subset \mathbb{C}^n$, such that

$$|E_M(f_j(q_j); (f_j)_*(q_j) \cdot \xi_j)| \geq j F_K^{\Omega}(q_j, \xi_j). \tag{3.1}$$

Let $k_0 \in \mathbb{N}$ be such that $\dfrac{1}{\sqrt{k_0}} < \rho$. Then for $k \geq k_0$, there is $f_k \in \mathcal{F}$ with

$$|E_M(f_k(q_k); (f_k)_*(q_k) \cdot \xi_k)| \geq k F_K^{\Omega}(q_k, \xi_k), \quad \text{for all } k \geq k_0 \text{ and } q_i \in \left\{p : \|p - p_0\| \leq \frac{1}{2\sqrt{k}}\right\}. \tag{3.2}$$

Now define for $k \geq k_0$,

$$g_k(p) = f_k\left(p_0 + \frac{p}{\sqrt{k}}\right).$$

Each g_k is defined on $B(0,1)=\{p:\|p\|<1\}$ and satisfies

$$\begin{aligned}|E_M(g_k(q_k);(f_k)_*(q_k)\cdot\xi_k|&=\left|E_M\left(f_k\left(p_0+\frac{q_k}{\sqrt{k}}\right);\frac{1}{\sqrt{k}}(f_k)_*\left(p_0+\frac{q_k}{\sqrt{k}}\right)\cdot\xi_k\right)\right|\\&\geq\sqrt{k}F_K^{\Omega}(q_k,\xi),\end{aligned}\tag{3.3}$$

therefore $\{g_k\}$ is not normal in $B(0,1)$. Now by Theorem B, there exist

(1) a compact set $K\Subset B(0,1)$,
(2) a sequence $\{p_j^*\}\subset K$,
(3) a sequence $\{g_{k_j}\}\subset\{g_k\}$,
(4) a sequence of positive real numbers $\rho_j^*\to 0$

such that $h_{k_j}(\xi)=g_{k_j}\left(p_j^*+\rho_j^*\xi\right)$, $\xi\in\mathbb{C}^n$ either compactly diverges on $\mathbb{C}^n$ or converges uniformly on compact subsets of $\mathbb{C}^n$ to a non-constant holomorphic map $g:\mathbb{C}^n\to M$. This is same as $f_{k_j}\left(p_0+\frac{p_j^*}{\sqrt{k_j}}+\frac{\rho_j^*}{\sqrt{k_j}}\xi\right)$, $\xi\in\mathbb{C}^n$ either compactly diverges on $\mathbb{C}^n$ or converges uniformly on compact subsets of $\mathbb{C}^n$ to a non-constant holomorphic map $g:\mathbb{C}^n\to M$. Now set

$$p_j=\frac{p_j^*}{\sqrt{k_j}}+p_0\quad\text{and}\quad\rho_j=\frac{\rho_j^*}{\sqrt{k_j}}.$$

Since the sequence $\{p_j^*\}\subset K$ is bounded, its limit points will be of finite modulus (in fact, of modulus less than one), and the subsequence $\{k_j\}$ of natural numbers converges to ∞ as j goes to ∞. Consequently, $\frac{p_j^*}{\sqrt{k_j}}$ converges to zero and we get that p_j converges to p_0. Also, ρ_j converges to 0. This proves the necessity part of the lemma.

Conversely, assume that the conditions of the lemma are satisfied and suppose, on the contrary, that $\mathcal{F}$ is normal at p_0. Then by Theorem A, for compact subsets K_0 and K_1 with $p_0\in K_0\Subset K_1\Subset\Omega$, there exists a number $N>0$ such that

$$\sup_{p\in K_1,\|\zeta\|=1}|E_M(f(p);(f)_*(p)\cdot\zeta)|\leq N,\quad\text{for each } f\in\mathcal{F}.\tag{3.4}$$

Now, suppose $g_j(\xi)=f_j(p_j+\rho_j\xi)$ converges uniformly on compact subsets of $\mathbb{C}^n$ to a non-constant holomorphic map $g:\mathbb{C}^n\to M$. We have

$$\begin{aligned}|E_M(g_j(\xi);g_j'(\xi)\cdot\zeta)|&=|E_M(f_j(p_j+\rho_j\xi);\rho_j(f_j)_*(p_j+\rho_j\xi)\cdot\zeta)|\\&\leq\rho_jN.\end{aligned}\tag{3.5}$$

Taking the limit, we get

$$\lim_{j\to\infty}|E_M(g_j(\xi);g_j'(\xi).\zeta)|=|E_M(g(\xi);g'(\xi)\cdot\zeta)|=0.$$

Then $g'(\xi)=0$ for every $\xi\in\mathbb{C}^n$, therefore g is a constant function which is a contradiction.

Next, suppose that $g_j(\xi) = f_j(p_j + \rho_j \xi)$ is compactly divergent. Since the family $\mathcal{F}$ is normal, without any loss of generality, we may assume that the sequence $\{f_j\} \to f$. And we get $g_j(\xi) \to f(p_0)$, which is not possible as $\{g_j\}$ is compactly divergent. This completes the proof. □

Example 3.9. Let $D = \{(z_1, z_2) : |z_1| < 1 \text{ and} |z_2| < 1\}$ be the polydisc in $\mathbb{C}^2$. We consider a family of holomorphic mappings $\{f_n\}$ from D into $\mathbb{C}$, where $f_n(z_1, z_2) = \mathrm{e}^{nz_1z_2}$ for all $n \in \mathbb{N}$. Since $\{f_n\}$ has no subsequence which is convergent at any point in the set $E = \{\{(\mathfrak{Re}(z_1), 0) \times (0, \mathfrak{Im}(z_2))\} \cup \{(0, \mathfrak{Im}(z_1)) \times (\mathfrak{Re}(z_2), 0)\}\} \cap D$ so $\{f_n\}$ is not normal in D. As $\{f_n\}$ is not normal at $(0, 0)$, we get a sequence $\{p_n\}$ in D such that $p_n = \left(\frac{z_1^0}{\sqrt{n}}, \frac{z_2^0}{\sqrt{n}}\right)$, where (z_1^0, z_2^0) is a fixed point in D. Notice that $\{p_n\} \to (0, 0)$. Also, we have a sequence of positive real numbers $\{\rho_n\} \to 0$, where $\rho_n = \frac{1}{\sqrt{n}}$ such that for all $\xi = (z_1, z_2) \in \mathbb{C}^2$, we have

$$g_n(p_n + \rho_n \xi) = f_n(p_n + \rho_n \xi) \to \mathrm{e}^{(z_1^0 + z_1)(z_2^0 + z_2)}.$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose all the conditions of the theorem are satisfied. Since E is either a non-analytic subset or the closure $\bar{E}$ has non-empty interior, then for each $p_0 \in E$, we can get a sequence p_j in E such that $p_j \to p_0$. By Lemma 3.8, $\mathcal{F}$ is not normal at p_0. Since p_0 is an arbitrary point of E and E is either a dense subset or a non-analytic subset of Ω, $\mathcal{F}$ is not quasi-normal in Ω.

Conversely, suppose $\mathcal{F}$ is not a quasi-normal family in Ω. Then by Lemma 3.7, there exists a sequence $S' = \{h_j\}$ of $\mathcal{F}$ which has either a q-point or a $\lambda-$point $p_0 \in \Omega$. This implies that there exists a subset $V \Subset \Omega$ which is either dense in Ω or a non-analytic subset containing p_0 so that each point of V is a μ_1-point of S'. Since V is either dense or non-analytic, we can choose a sequence of positive real numbers $\{r_i\}$ such that $\{r_i\} \to 0$ and for each open ball $B(p_0, r_i) = \{p \in \Omega : \|p - p_0\| < r_i\}$, the set $V \cap B(p_0, r_i)$ has at least one μ_1-point. Now we proceed inductively to get the conditions of the theorem.

Step 1. There exists

(A_1) a μ_1-point $p_1 \in \Omega$ such that $p_1 \in V \cap B(p_0, r_1)$. So S' is not normal at p_1. Therefore, by Lemma 3.8 there exist
(B_1) a sequence $\{w_{j,1}\} \subset \Omega$ such that $\{w_{j,1}\} \to p_1$;
(C_1) a subsequence $S_1 = \{h_{j,1}\}$ of S';
(D_1) a sequence of positive real numbers $\{\rho_{j,1}\} \to 0$ such that

$h_{j,1}(w_{j,1} + \rho_{j,1}\xi),\ \xi \in \mathbb{C}^n$ either compactly diverges on $\mathbb{C}^n$ or converges uniformly on compact subsets of $\mathbb{C}^n$ to a non-constant holomorphic map $g_1 : \mathbb{C}^n \to M$.

Step 2. Since p_0 is also a q-point or a λ-point of S_1, there exists

(A_1) a μ_1-point $p_2 \in \Omega$, $p_2 \neq p_1$ (by Definition 3.1) such that $p_2 \in V \cap B(p_0, r_2)$, $0 < r_2 < r_1$. So S_1 is not normal at p_2. Therefore, by Lemma 3.8, there exist
(B_1) a sequence $\{w_{j,2}\} \subset \Omega$ such that $\{w_{j,2}\} \to p_2$;
(C_1) a subsequence $S_2 = \{h_{j,2}\}$ of S_1;
(D_1) a sequence of positive real numbers $\{\rho_{j,2}\} \to 0$ such that

$h_{j,2}(w_{j,2}+\rho_{j,2}\xi)$, $\xi\in\mathbb{C}^n$, either compactly diverges on $\mathbb{C}^n$ or converges uniformly on compact subsets of $\mathbb{C}^n$ to a non-constant holomorphic map $g_2:\mathbb{C}^n\to M$.

Continuing in this manner, we get sequences $\{p_j\}\to p_0$, $\{w_{i,j}\}$, $\{\rho_{i,j}\}$, $\{g_j\}$ and $\{h_{i,j}\}$. Now we use the Cantor's diagonal method and, choose $E=V$; $f_i=h_{i,i}$; $w_{i,p_o}=w_{i,i}$; $\rho_{i,p_0}=\rho_{i,i}$. Then for each $j\geq 1$, $\{f_i\}_{i=j}^{\infty}$ is a subsequence of S_j and $f_i(w_{i,p_0}+\rho_{i,p_0}\xi)$, $\xi\in\mathbb{C}^n$, either compactly divergent on $\mathbb{C}^n$ or converges uniformly on compact subsets of $\mathbb{C}^n$ to a non-constant holomorphic map $g_{p_0}:\mathbb{C}^n\to M$. This completes the proof of the theorem.

□

For *weakly-normal* family, we propose the following theorem.

Theorem 3.10. *Let $\Omega\subseteq\mathbb{C}^n$ be a hyperbolic domain. Let M be a complete complex Hermitian manifold of dimension k. Let $\mathcal{F}=\{f_\alpha\}_{\alpha\in A}\subseteq \mathrm{Hol}(\Omega,M)$. The family $\mathcal{F}$ is not weakly-normal if and only if there exist a subset $E\subset\Omega$ which is not locally contained in an analytic subset of codimension 2, and corresponding to each $p\in E$, there exist*

(*a*) *a sequence of points $\{w_{j,p}\}_{j=1}^{\infty}\subset\Omega$ such that $w_{j,p}\to p$,*
(*b*) *a sequence of functions $\{f_j\}\subset\mathcal{F}$,*
(*c*) *a sequence of positive real numbers $\rho_{j,p}\to 0$ such that*

$$g_j(\zeta):=f_j(w_{j,p}+\rho_{j,p}\xi),\quad \xi\in\mathbb{C}^n\quad (p\in E)$$

satisfies one of the following two assertions:

(*i*) *The sequence $\{g_j\}$ is compactly divergent on $\mathbb{C}^n$.*
(*ii*) *The sequence $\{g_j\}$ converges uniformly on compact subsets of $\mathbb{C}^n$ to a non-constant holomorphic map $g_p:\mathbb{C}^n\to M$.*

The proof of Theorem 3.10 is merely a formality. It can be proven on similar lines as in the proof of Theorem 1.1 using Lemma 3.6 instead of Lemma 3.7.

The following examples elucidate Theorem 3.10.

Example 3.11. Let $\{f_n\}$ be a family of holomorphic mappings defined on the polydisc $D=\{(z_1,z_2):|z_1|<1 \text{ and} |z_2|<1\}$ such that $f_n(z_1,z_2)=nz_1z_2$. Then $\{f_n\}$ is not weakly-normal on D, as $\{f_n\}$ converges compactly in $D\setminus E$, where $E=\{(z_1,z_2):z_1z_2=0\}$ is an analytic subset of codimension 1 of D. Let $(z_1^0,z_2^0)\in E$ be any arbitrary point. Without loss of generality, we take $z_1^0=0$. Then we get a sequence $\{p_n\}\to(0,z_2^0)$ of points in E, where $p_n=\left(0,z_2^0+\frac{1}{\sqrt{n}}\right)$. Also, we have a sequence of positive real numbers $\{\rho_n\}\to 0$, where $\rho_n=\frac{1}{n}$ such that for all $\xi=(z_1,z_2)\in\mathbb{C}^2$, we get

$$g_n(p_n+\rho_n\xi)=f_n(p_n+\rho_n\xi_n)\to z_2^0z_1.$$

Example 3.12. Let $\{f_n\}$ be a family of holomorphic mappings defined on the polydisc $D=\{(z_1,z_2):|z_1|<1 \text{ and } |z_2|<1\}$ such that $f_n(z_1,z_2)=\cos(nz_1z_2)$. Then $\{f_n\}$ is not weakly-normal on D as $\{f_n\}$ is not compactly convergent in any open subset of D containing $E=\{(z_1,z_2):z_1z_2=0\}$, which is of codimension 1. Let (z_1^0,z_2^0) be any arbitrary point of E. Without loss of generality, we take $z_2^0=0$. Then we get a sequence

$\{p_n\} \to (z_1^0, 0)$ of points in E, where $p_n = \left(z_1^0 + \frac{1}{\sqrt{n}}, 0\right)$. Also, we have a sequence of positive real numbers $\{\rho_n\} \to 0$, where $\rho_n = \frac{1}{n}$ such that for all $\xi = (z_1, z_2) \in \mathbb{C}^2$, we obtain

$$g_n(p_n + \rho_n \xi) = f_n(p_n + \rho_n \xi) \to \cos(z_1^0 z_2).$$

Remark 3.13. One can observe that the notions of normality and quasinormality are the local phenomena, and therefore our results holds true even if hyperbolicity is dropped.

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