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Augmentation quotients for Burnside rings of some finite p-groups

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Abstract. Let G be a finite group, $\Omega(G)$ be its Burnside ring and $\Delta(G)$ the augmentation ideal of $\Omega(G)$. Denote by $\Delta^n(G)$ and $Q_n(G)$ the n-th power of $\Delta(G)$ and the n-th consecutive quotient group $\Delta^n(G)/\Delta^{n+1}(G)$, respectively. This paper provides an explicit $\mathbb{Z}$-basis for $\Delta^n(\mathcal{H})$ and determine the isomorphism class of $Q_n(\mathcal{H})$ for each positive integer n, where $\mathcal{H} = \langle g, h | \, g^{p^m} = h^p = 1, h^{-1}gh = g^{p^{m-1}+1}\rangle$, p is an odd prime.

Keywords. Finite p-group; Burnside ring; augmentation ideal; augmentation quotient.

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1. Introduction

Let G be a finite group. A G-*set* is a finite set X together with an action of G on X:

$$G \times X \longrightarrow X, \quad (g, x) \mapsto gx, \quad g \in G, x \in X. \tag{1}$$

Two G-sets X and Y are said to be *isomorphic* (denoted by $X \cong Y$), if there exists a bijective map $f : X \longrightarrow Y$ such that

$$f(gx) = gf(x), \quad \forall\, g \in G,\ x \in X. \tag{2}$$

It is easy to verify that isomorphism of G-sets is an equivalence relation. The equivalence classes are called isomorphism classes. The isomorphism class of X is denoted by $[X]$. The sum $[X] + [Y]$ of two isomorphism classes $[X]$ and $[Y]$ is defined by

$$[X] + [Y] = [X \sqcup Y], \tag{3}$$

where $X \sqcup Y$ is the disjoint union of X and Y, which is also a G-set in the canonical way.

The *Burnside ring* $\Omega(G)$ is the group completion of the monoid (under sum) of isomorphism classes of G-sets. Its multiplication is induced by cartesian product of finite sets. Note that the cartesian product $X \times Y$ is a G-set via the co-ordinating action. By [8], $\Omega(G)$

is a commutative ring with an identity element. Its underlying group is a finitely generated free abelian group with the isomorphism classes of transitive G-sets as basis. Hence, by [8], its free rank is equal to the number of conjugacy classes of subgroups of G.

The number of fixed points of G-set induces a ring homomorphism

$$\phi : \Omega(G) \longrightarrow \mathbb{Z}. \tag{4}$$

This homomorphism is called the *augmentation map*. Its kernel $\Delta(G)$ is called the *augmentation ideal* of $\Omega(G)$. Denote by $\Delta^n(G)$ the n-th power of $\Delta(G)$. The n-th *augmentation quotient group* of $\Omega(G)$ is defined as

$$Q_n(G) = \Delta^n(G)/\Delta^{n+1}(G).$$

The problem of determining the structure of $\Delta^n(G)$ and $Q_n(G)$ is an interesting topic in group ring theory. In [15], Wu and Tang determined the isomorphism class of $Q_n(G)$ for all finite abelian groups and for any positive integer n. Tang *et al.* [10] settled the problem completely for certain nonabelian 2-groups. Chang [6] provided an explicit $\mathbb{Z}$-basis of $\Delta^n(\mathcal{D})$ and determined the isomorphism class of $Q_n(\mathcal{D})$ for each positive integer n, where $\mathcal{D}$ is the generalized dihedral group of a finite abelian group of odd order. However, the isomorphism class of $Q_n(G)$ remained unclear for other non-abelian groups.

Let p be a fixed odd prime. For each positive integer m greater than or equal to 2, it is well known that there is exactly one isomorphism class of a nonabelian p-group of order p^{m+1} which have a cyclic subgroup of index p. Denote this nonabelian p-group by $\mathcal{H}$. Its presentation is

$$\mathcal{H} = \langle a, b \,|\, a^{p^m} = b^p = 1, b^{-1}ab = a^{p^{m-1}+1} \rangle. \tag{5}$$

The goal of this article is to provide an explicit $\mathbb{Z}$-basis for $\Delta^n(\mathcal{H})$ and determine the isomorphism class of $Q_n(\mathcal{H})$ for each positive integer n. Wen and Chang [14] settled this problem for $m = 2$. Thus we assume that $m \geqslant 3$ in the sequel.

The result also yields $\mathrm{Tor}_1^{\Omega(\mathcal{H})}(\Omega(\mathcal{H})/\Delta^n(\mathcal{H}), \Omega(\mathcal{H})/\Delta(\mathcal{H}))$ because, for any finite group G, $Q_n(G) \cong \mathrm{Tor}_1^{\Omega(G)}(\Omega(G)/\Delta^n(G), \Omega(G)/\Delta(G))$.

Two related problems of recent interest have been to investigate the augmentation ideals and their consecutive quotients for integral group rings and representation rings of finite groups. These problems have been well studied in [1–5,9,11–13] and [7].

2. Preliminaries

In this section, we provide some useful results about $\Omega(G)$, $\Delta^n(G)$, $Q_n(G)$ and finitely generated free abelian groups. Indeed, Lemmas 2.1 and 2.2, Theorem 2.3 and Corollary 2.4 were found in [8] and, Lemmas 2.5 and 2.6 were proved in [6].

Let X be a G-set. There is an equivalence relation on X given by stating that x is related to y if there exists $g \in G$ with $gx = y$. The equivalence classes are called *orbits*. It is easy to see that the orbits of x are

$$Gx = \{gx \mid g \in G\}. \tag{6}$$

A G-set is *transitive* if it has only one orbit. For instance, if $x \in X$, the orbit Gx is a transitive G-set. As another example, if K is a subgroup of G, the set G/K of the left cosets of K is a transitive G-set under $g(hK) = (gh)K$.

There is a standard form for each transitive G-set. The *stabilizer* of an element $x \in X$ is the set

$$G_x = \{g \in G \mid gx = x\}. \tag{7}$$

It is easy to verify that the stabilizer G_x is a subgroup of G. The following lemma shows that every transitive G-set is isomorphic to G/K for some subgroup K of G.

Lemma 2.1. *For any G-set X and $x \in X$,*

$$Gx \cong G/G_x, \quad gx \mapsto gG_x. \tag{8}$$

Note that $G_{gx} = gG_x g^{-1}$ for each $x \in X$ and $g \in G$. Moreover, any isomorphism of G-sets $f : X \longrightarrow Y$ preserves stabilizers $G_x = G_{f(x)}$. From these two facts, we get the following lemma.

Lemma 2.2. *Let K and L be two subgroups of G. Then $G/K \cong G/L$ if and only if K and L are conjugate in G.*

By assembling the above lemmas, we have proved the following theorem.

Theorem 2.3. *Let $\mathcal{K}$ be a set of subgroups of G, one chosen from each conjugacy class of subgroups of G. Then $\Omega(G)$ is, additively, the free abelian based on $\{[G/K] \mid K \in \mathcal{K}\}$.*

Recall that $\Delta(G)$ is the kernel of $\phi : \Omega(G) \longrightarrow \mathbb{Z}$ which sends $[X]$ to the number $\#(X)$ of fixed points of X. Brief calculations show that

$$\#(G/K) = \begin{cases} 1, & \text{if } K = G, \\ 0, & \text{if } K < G. \end{cases} \tag{9}$$

From this, we get the following corollary.

CORALLARY 2.4

For each finite group G, the underlying group of $\Delta(G)$ is the free abelian group based on $\{[G/K] \mid K \in \mathcal{K}, K < G\}$.

The multiplication in $\Omega(G)$ is completely determined by the product

$$[G/K][G/L] = [(G/K) \times (G/L)], \tag{10}$$

where K, L are the subgroups of G. The following lemma tackles this product.

Lemma 2.5. *Let K be a subgroup of G and L be a normal subgroup of G. Then*

$$[G/K][G/L] = \frac{|G| \cdot |K \cap L|}{|K| \cdot |L|}[G/(K \cap L)] = \frac{|G|}{|KL|}[G/(K \cap L)]. \tag{11}$$

It is easy to see that $\Delta^n(G)$ is a finitely generated free abelian group for any positive integer n. However, it is usually difficult to write down explicitly a basis of $\Delta^n(G)$, even for a finite p-group. The following lemma sheds some light on its free rank.

Lemma 2.6. *For each positive integer n, $Q_n(G)$ is a finite abelian group, hence $\Delta^n(G)$ has the same free rank as $\Delta(G)$.*

At last, we recall a classical result about finitely generated free abelian groups.

Lemma 2.7. *Let F be a finite generated free abelian group of rank r. If the r elements $g_1, \ldots, g_r$ generate F, then they form a basis of F.*

3. Necessary tools

In this section, we construct a basis of $\Delta(\mathcal{H})$ as a free abelian group. Then we determine the multiplication in $\Delta(\mathcal{H})$.

Recall that the presentation of $\mathcal{H}$ is $\langle a, b | a^{p^m} = b^p = 1, b^{-1}ab = a^{p^{m-1}+1}\rangle$. The following lemma determines all the proper subgroups of $\mathcal{H}$. For convenience, we fix the following notation:

- for any positive integer j, set $\bar{j} = \{0, 1, \ldots, j\}$, $\underline{j} = \{1, \ldots, j\}$,
- denote by N_i the cyclic subgroup of $\mathcal{H}$ which is generated by a^{p^i}, $i \in \bar{m}$,
- for any subset $\Gamma \subset \Omega(\mathcal{H})$, denote by $\mathbb{Z}\Gamma$ the set of all $\mathbb{Z}$-linear combinations of elements of Γ,
- denote by C_p the cyclic group of order p.

Lemma 3.1. *Let K be a proper subgroup of $\mathcal{H}$.*

(1) *If $K \subset N_0$, then $K = N_i$ for some integer $i \in \bar{m}$. Moreover, K is a normal subgroup of $\mathcal{H}$.*

(2) *If $K \not\subset N_0$, then there are two integers k, l with $k \in \underline{m}$, $l \in \overline{p-1}$ such that*

$$K = \bigcup_{j=0}^{p-1} (ba^{lp^{k-1}})^j N_k = \bigcup_{j=0}^{p-1} b^j a^{jlp^{k-1}} N_k. \tag{12}$$

Proof.

(1) It is a direct corollary of the presentation of $\mathcal{H}$.

(2) Note that KN_0 is a subgroup of $\mathcal{H}$ which contains N_0 properly. This implies $KN_0 = \mathcal{H}$ since N_0 is a maximal subgroup of $\mathcal{H}$. Due to item (1), there exists an integer $k \in \bar{m}$ such that $K \cap N_0 = N_k$. Then we have the following group isomorphism:

$$\begin{aligned} K/N_k \ &\cong \ KN_0/N_0 = \mathcal{H}/N_0 \ \cong \ \langle b\rangle, \\ b^u a^v N_k \mapsto \ &\ b^u a^v N_0 = b^u N_0 \ \mapsto \ b^u, \end{aligned} \tag{13}$$

where $\langle b\rangle$ is the cyclic subgroup of $\mathcal{H}$ which is generated by b. Let ba^vN_k be the inverse image of b under the above isomorphism. Then K/N_k is a cyclic group of order p which is generated by ba^vN_k. From this, it follows that

$$K=\bigcup_{j=0}^{p-1}(ba^vN_k)^j=\bigcup_{j=0}^{p-1}(ba^v)^jN_k, \tag{14}$$

where $(ba^vN_k)^p=(ba^v)^pN_k=N_k$, i.e. $(ba^v)^p$ lies in N_k. We claim $k\neq 0$, otherwise N_0 is properly contained in K, which is a contradiction since N_0 is a maximal subgroup. Without loss of generality, we can assume that $v\in\overline{p^k-1}$ since ba^vN_k depends only on the residue class of v modulo p^k. To complete the proof, we need the following assertion.

Assertion 3.2. For any natural numbers w and j, we have

$$(ba^w)^j=b^ja^{w\left[\frac{j(j-1)}{2}p^{m-1}+j\right]}. \tag{15}$$

Proof. We prove the assertion by induction on j. When $j=0$, it is trivial. If $j\geqslant 1$, assume that the assertion holds for $j-1$ and any natural number w. Note that $a^wb=ba^{w(p^{m-1}+1)}$. Hence

$$\begin{aligned}(ba^w)^j&=b(a^wb)^{j-1}a^w=b(ba^{w(p^{m-1}+1)})^{j-1}a^w\\&=bb^{j-1}a^{w(p^{m-1}+1)\left[\frac{(j-1)(j-2)}{2}p^{m-1}+j-1\right]}a^w\\&=b^ja^{w\left[\left(\frac{(j-1)(j-2)}{2}+j-1\right)p^{m-1}+j-1\right]}a^v\\&=b^ja^{w\left[\frac{j(j-1)}{2}p^{m-1}+j\right]}\end{aligned} \tag{16}$$

as required. □

We now return to the proof of Lemma 3.1. Due to Assertion 3.2, we get

$$(ba^v)^p=b^pa^{v\left[\frac{p(p-1)}{2}p^{m-1}+p\right]}=a^{vp}. \tag{17}$$

From this and the fact that $(ba^v)^p$ lies in N_k, it follows that v is a multiple of p^{k-1}. Set $v=lp^{k-1}$. Then $l\in\overline{p-1}$. A brief calculation shows that, for any natural number j,

$$(ba^{lp^{k-1}})^jN_k=b^ja^{lp^{k-1}\left[\frac{j(j-1)}{2}p^{m-1}+j\right]}N_k=b^ja^{jlp^{k-1}}N_k.$$

Thus the lemma is proved. □

Due to Theorem 2.3, we need a set of representatives of all conjugacy classes of subgroups of $\mathcal{H}$ to construct a basis of $\Omega(\mathcal{H})$. By Lemma 3.1, each subgroup of $\mathcal{H}$ contained in N_0 is normal. Set

$$M_{kl}=\bigcup_{j=0}^{p-1}(ba^{lp^{k-1}})^jN_k,\quad k\in\underline{m},\, l\in\mathbb{N}. \tag{18}$$

It is easy to verify that M_{kl} is a subgroup of $\mathcal{H}$. For later use, we remind that, for a fixed integer $k \in \underline{m}$, M_{kl} depends only on the residue class of l modulo p. The following lemma determines their conjugacy classes.

Lemma 3.3. *For any natural number l, we have*

(1) *M_{kl} is a normal subgroup of $\mathcal{H}$ if $k \in \underline{m-1}$,*
(2) *M_{ml} is conjugate to M_{m0}.*

Proof.

(1) Let $k \in \underline{m-1}$. Then short calculations show

$$\begin{aligned} a^{-1}(ba^{lp^{k-1}})a &= ba^{-(p^{m-1}+1)}a^{lp^{k-1}}a \\ &= ba^{lp^{k-1}-p^{m-1}} \\ &= ba^{lp^{k-1}}(a^{p^k})^{-p^{m-1-k}}, \end{aligned} \tag{19}$$

$$\begin{aligned} b^{-1}(ba^{lp^{k-1}})b &= ba^{lp^{k-1}(p^{m-1}+1)} \\ &= ba^{lp^{k-1}}(a^{p^k})^{lp^{m-2}}. \end{aligned} \tag{20}$$

This implies both $a^{-1}(ba^{lp^{k-1}})a$ and $b^{-1}(ba^{lp^{k-1}})b$ belong to M_{kl}. Thus M_{kl} is a normal subgroup since M_{kl} is generated by N_k and $ba^{lp^{k-1}}$.

(2) Note that N_m is the trivial subgroup, so M_{ml} is generated by $ba^{lp^{m-1}}$. Brief calculations show

$$a^{-l}(ba^{lp^{m-1}})a^l = ba^{-l(p^{m-1}+1)}a^{lp^{m-1}}a^l = b, \tag{21}$$

which implies M_{ml} is conjugate to M_{m0}. □

Thanks to Lemmas 3.1 and 3.3, we get a basis of $\Omega(\mathcal{H})$, hence a basis of $\Delta(\mathcal{H})$. For convenience, denote $[\mathcal{H}/N_i]$, $[\mathcal{H}/M_{kl}]$ and $[\mathcal{H}/\mathcal{H}]$ by α_i, β_{kl} and ε, respectively, where $i \in \bar{m}$, $k \in \underline{m}$, l is a natural number.

Theorem 3.4. *The underlying group of $\Omega(\mathcal{H})$ is the free abelian group with basis*

$$\{\alpha_i | i \in \bar{m}\} \cup \left\{\beta_{kl} | k \in \underline{m-1}, l \in \overline{p-1}\right\} \cup \{\beta_{m0}, \varepsilon\}. \tag{22}$$

Proof. It is easy to verify that each element of $\mathcal{H}$ has a unique expression as $b^u a^v$, $u \in \overline{p-1}$, $v \in \overline{p^m-1}$. By this, one can easily verify that

$$\{N_i | i \in \bar{m}\} \cup \{M_{kl} | k \in \underline{m-1}, l \in \overline{p-1}\} \cup \{M_{m0}, \mathcal{H}\} \tag{23}$$

is a set of representatives of all conjugacy classes of subgroups of $\mathcal{H}$. Then the theorem follows from Theorem 2.3. □

CORALLARY 3.5

$\Delta(\mathcal{H})$ is, additively, the free abelian group based on

$$\{\alpha_i | i \in \bar{m}\} \cup \{\beta_{kl} | k \in \underline{m-1}, l \in \overline{p-1}\} \cup \{\beta_{m0}\}. \tag{24}$$

Now we determine the multiplication in $\Delta(\mathcal{H})$.

Lemma 3.6. *For any integers* $i, j \in \bar{m}$, $k \in \underline{m}$, $r \in \underline{m-1}$ *and* $l, s \in \overline{p-1}$, *we have*

(1) $\alpha_i\alpha_j = p^{\min\{i,j\}+1}\alpha_{\max\{i,j\}}$,
(2) $\alpha_i\beta_{kl} = p^{\min\{i,k\}}\alpha_{\max\{i,k\}}$,
(3) $\beta_{kl}\beta_{rs} = \begin{cases} p^r\beta_{kl}, & \text{if } \{k > r, s = 0\} \text{ or } \{k = r, l = s\}, \\ p^{r-1}\alpha_k, & \text{if } \{k > r, s \in \underline{p-1}\} \text{ or } \{k = r, l \neq s\}, \end{cases}$
(4) $\beta_{m0}^2 = p^{m-1}\beta_{m0} + (p^{m-1} - p^{m-2})\alpha_m$.

Proof. Lemma 3.6(1), (2) and (3) are direct corollaries of Lemma 2.5. For (4), it is easy to verify that

$$\mathcal{H}/M_{m0} = \{a^v M_{m0} | v \in \overline{p^m - 1}\}. \tag{25}$$

Let $x = (a^v M_{m0}, a^w M_{m0}) \in \mathcal{H}/M_{m0} \times \mathcal{H}/M_{m0}$. From (21), it follows that

$$\mathcal{H}_x = (a^v M_{m0} a^{-v}) \cap (a^w M_{m0} a^{-w}) = M_{mv} \cap M_{mw}, \tag{26}$$

where $\mathcal{H}_x$ is the stabilizer of x. Recall that, for any natural number l, M_{ml} depends only on the residue class of l modulo p. This implies

$$\mathcal{H}_x = \begin{cases} M_{mv}, & \text{if } v \equiv w \pmod{p}, \\ \{1\} = N_m, & \text{if } v \not\equiv w \pmod{p}. \end{cases} \tag{27}$$

Thus there are exactly p^{2m-1} elements of $\mathcal{H}/M_{m0} \times \mathcal{H}/M_{m0}$ whose orbits are isomorphic to $\mathcal{H}/M_{m0}$, and the orbits of the rest elements are isomorphic to $\mathcal{H}/N_m$. Note that the cardinalities of $\mathcal{H}/M_{m0}$ and $\mathcal{H}/N_m$ are p^m and p^{m+1}, respectively. Then the lemma follows. □

4. Main results

In this section, we construct an explicit $\mathbb{Z}$-basis for $\Delta^n(\mathcal{H})$ and determine the isomorphism class of $Q_n(\mathcal{H})$ for each positive integer n. For later use, we remind the readers that for a fixed $k \in \underline{m}$, β_{kl} depends only on the residue class of l modulo p. Moreover, β_{ml} equals β_{m0} for any natural number l.

Theorem 4.1. *For any positive integer* n, $\Delta^{n+1}(\mathcal{H})$ *is, additively, the free abelian group based on*

$$\{p^n\alpha_0\} \cup \{p^{n-1}\alpha_i | i \in \underline{m}\} \cup \{p^n\beta_{kl} | k \in \underline{m-1}, l \in \overline{p-1}\} \cup \{p^n\beta_{m0}\}. \tag{28}$$

Proof. Note that (28) has the same cardinality as (24), so due to Lemmas 2.6, and 2.7 and Corollary 3.5, we just need to show it generates $\Delta^n(\mathcal{H})$. We prove this by induction on n. When $n = 1$, brief calculations show

$$\Delta^2(\mathcal{H}) = \Delta(\mathcal{H})\Delta(\mathcal{H})$$

$$
\begin{aligned}
&= \mathbb{Z}\{\alpha_i\alpha_j | i, j \in \bar{m}, i \leqslant j\} + \mathbb{Z}\{\alpha_i\beta_{kl} | i \in \bar{m}, k \in \underline{m}, l \in \overline{p-1}\} \\
&\quad + \mathbb{Z}\{\beta_{kl}\beta_{rs} | k \in \underline{m}, r \in \underline{m-1}, l, s \in \overline{p-1}, k \geqslant r\} + \mathbb{Z}\{\beta_{m0}^2\} \\
&= \mathbb{Z}\{p^{i+1}\alpha_j | i, j \in \bar{m}, i \leqslant j\} + \mathbb{Z}\{p^{\min\{i,k\}}\alpha_{\max\{i,k\}} | i \in \bar{m}, k \in \underline{m}\} \\
&\quad + \mathbb{Z}\{p^r\beta_{kl} | k \in \underline{m}, r \in \underline{m-1}, l \in \overline{p-1}, k \geqslant r\} \\
&\quad + \mathbb{Z}\{p^{r-1}\alpha_k | k \in \underline{m}, r \in \underline{m-1}, k \geqslant r\} \\
&\quad + \mathbb{Z}\{p^{m-1}\beta_{m0} + (p^{m-1} - p^{m-2})\alpha_m\} \\
&= \mathbb{Z}\{p\alpha_j | j \in \bar{m}\} + \mathbb{Z}\{\alpha_k | k \in \underline{m}\} + \mathbb{Z}\{p\beta_{kl} | k \in \underline{m}, l \in \overline{p-1}\} \\
&\quad + \mathbb{Z}\{\alpha_k | k \in \underline{m}\} + \mathbb{Z}\{p^{m-1}\beta_{m0} + (p^{m-1} - p^{m-2})\beta_{m0}\} \\
&= \mathbb{Z}\{p\alpha_0\} + \mathbb{Z}\{\alpha_i | i \in \underline{m}\} + \mathbb{Z}\{p\beta_{kl} | k \in \underline{m-1}, l \in \overline{p-1}\} \\
&\quad + \mathbb{Z}\{p\beta_{m0}\}, \qquad (29)
\end{aligned}
$$

as required. When $n \geqslant 2$, assume the theorem holds for $n-1$, which means the underlying group of $\Delta^n(\mathcal{H})$ is the free abelian group with basis

$$
\begin{aligned}
&\{p^{n-1}\alpha_0\} \cup \{p^{n-2}\alpha_j | j \in \underline{m}\} \cup \{p^{n-1}\beta_{rs} | r \in \underline{m-1}, s \in \overline{p-1}\} \cup \\
&\{p^{n-1}\beta_{m0}\}. \qquad (30)
\end{aligned}
$$

From this, it follows that

$$
\begin{aligned}
\Delta^{n+1}(\mathcal{H}) &= \Delta(\mathcal{H})\Delta^n(\mathcal{H}) \\
&= \mathbb{Z}\{p^{n-1}\alpha_i\alpha_0 | i \in \bar{m}\} + \mathbb{Z}\{p^{n-2}\alpha_i\alpha_j | i \in \bar{m}, j \in \underline{m}\} \\
&\quad + \mathbb{Z}\{p^{n-1}\alpha_i\beta_{rs} | i \in \bar{m}, r \in \underline{m-1}, s \in \overline{p-1}\} \\
&\quad + \mathbb{Z}\{p^{n-1}\alpha_i\beta_{m0} | i \in \bar{m}\} + \mathbb{Z}\{p^{n-1}\beta_{kl}\alpha_0 | k \in \underline{m}, l \in \overline{p-1}\} \\
&\quad + \mathbb{Z}\{p^{n-2}\beta_{kl}\alpha_j | j, k \in \underline{m}, l \in \overline{p-1}\} \\
&\quad + \mathbb{Z}\{p^{n-1}\beta_{kl}\beta_{rs} | k \in \underline{m}, r \in \underline{m-1}, l, s \in \overline{p-1}\} \\
&\quad + \mathbb{Z}\{p^{n-1}\beta_{kl}\beta_{m0} | k \in \underline{m}, l \in \overline{p-1}\} \\
&= \mathbb{Z}\{p^{n-1}\alpha_0^2\} + \mathbb{Z}\{p^{n-2}\alpha_i\alpha_j | i \in \bar{m}, j \in \underline{m}\} \\
&\quad + \mathbb{Z}\{p^{n-1}\beta_{kl}\alpha_0 | k \in \underline{m}, l \in \overline{p-1}\} \\
&\quad + \mathbb{Z}\{p^{n-2}\beta_{kl}\alpha_j | j, k \in \underline{m}, l \in \overline{p-1}\} \\
&\quad + \mathbb{Z}\{p^{n-1}\beta_{kl}\beta_{rs} | k \in \underline{m}, r \in \underline{m-1}, l, s \in \overline{p-1}\} + \mathbb{Z}\{p^{n-1}\beta_{m0}^2\} \\
&= \mathbb{Z}\{p^n\alpha_0\} + \mathbb{Z}\{p^{n+i-1}\alpha_j | i \in \overline{m}, j \in \underline{m}, i \leqslant j\} \\
&\quad + \mathbb{Z}\{p^{n-1}\alpha_k | k \in \underline{m}\} + \mathbb{Z}\{p^{n+k-2}\alpha_j | j, k \in \underline{m}, j \geqslant k\} \\
&\quad + \mathbb{Z}\{p^{n+r-1}\beta_{kl} | k \in \underline{m}, r \in \underline{m-1}, l \in \overline{p-1}, k \geqslant r\} \\
&\quad + \mathbb{Z}\{p^{n+r-2}\alpha_k | k \in \underline{m}, r \in \underline{m-1}, k \geqslant r\} \\
&\quad + \mathbb{Z}\{p^{n-1}(p^{m-1}\beta_{m0} + (p^{m-1} - p^{m-2})\alpha_m)\} \\
&= \mathbb{Z}\{p^n\alpha_0\} + \mathbb{Z}\{p^{n-1}\alpha_j | j \in \underline{m}\} + \mathbb{Z}\{p^{n-1}\alpha_k | k \in \underline{m}\} \\
&\quad + \mathbb{Z}\{p^{n-1}\alpha_j | j \in \underline{m}\} + \mathbb{Z}\{p^n\beta_{kl} | k \in \underline{m}, l \in \overline{p-1}\} \\
&\quad + \mathbb{Z}\{p^{n-1}\alpha_k | k \in \underline{m}\} + \mathbb{Z}\{p^{m-2}(p^n\beta_{m0} + (p^n - p^{n-1})\alpha_m)\}
\end{aligned}
$$

$$= \mathbb{Z}\{p^n\alpha_0\} + \mathbb{Z}\{p^{n-1}\alpha_i | i \in \underline{m}\} + \mathbb{Z}\{p^n\beta_{kl} | k \in \underline{m-1}, l \in \overline{p-1}\} + \mathbb{Z}\{p^n\beta_{m0}\}.$$

Thus the theorem is proved. □

Theorem 4.2. *For any positive integer n,*

$$Q_n(\mathcal{H}) \cong \begin{cases} (C_p)^{(m-1)p+2}, & n = 1, \\ (C_p)^{(m-1)p+m+2}, & n \geqslant 2. \end{cases} \tag{31}$$

Proof. It is a direct corollary of Theorem 4.1. □

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