


Gromov hyperbolicity in lexicographic product graphs

WALTER CARBALLOSA^{1,2}, AMAURIS DE LA CRUZ³
and JOSÉ M RODRÍGUEZ^{3,*} 

¹Department of Mathematics and Statistics, Florida International University,
11200 SW 8th Street, Miami, FL 33199, USA

²Department of Mathematics, Miami Dade College, 300 NE Second Ave., Miami,
FL 33132, USA

³Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la
Universidad 30, 28911 Leganés, Madrid, Spain

*Corresponding author.

E-mail: waltercarb@gmail.com; alcruz@math.uc3m.es; amaurisdelac@gmail.com;
jomaro@math.uc3m.es

MS received 14 May 2017; revised 19 July 2017; accepted 26 July 2017;
published online 17 December 2018

Abstract. If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X . The space X is δ -*hyperbolic* (in the Gromov sense) if any side of T is contained in a δ -neighborhood of the union of the two other sides, for every geodesic triangle T in X . If X is hyperbolic, we denote by $\delta(X)$ the sharp hyperbolicity constant of X , i.e. $\delta(X) = \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\}$. In this paper, we characterize the lexicographic product of two graphs $G_1 \circ G_2$ which are hyperbolic, in terms of G_1 and G_2 : the lexicographic product graph $G_1 \circ G_2$ is hyperbolic if and only if G_1 is hyperbolic, unless if G_1 is a trivial graph (the graph with a single vertex); if G_1 is trivial, then $G_1 \circ G_2$ is hyperbolic if and only if G_2 is hyperbolic. In particular, we obtain the sharp inequalities $\delta(G_1) \leq \delta(G_1 \circ G_2) \leq \delta(G_1) + 3/2$ if G_1 is not a trivial graph, and we characterize the graphs for which the second inequality is attained.

Keywords. Lexicographic product graphs; geodesics; Gromov hyperbolicity; infinite graphs.

Mathematics Subject Classification. Primary: 05C76, 05C10; Secondary: 05C35, 05C63, 05C12.

1. Introduction

Hyperbolic spaces play an important role in geometric group theory and in the geometry of negatively curved spaces (see [1, 22, 23]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature bounded away from 0, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [1, 22, 23]).

The different kinds of products of graphs are an important research topic in graph theory, applied mathematics and computer science. Some large graphs are composed from some

existing smaller ones by using several products of graphs, and many properties of such large graphs are strongly associated with that of the corresponding smaller ones. In particular, the lexicographic product of graphs has been extensively investigated in relation to a wide range of subjects (see, for example, [32,45,51,57,58] and references therein).

The first few works on Gromov hyperbolic spaces deal with finitely generated groups (see [23]). Initially, Gromov spaces were applied to the study of automatic groups in the science of computation (see, for example, [39]). Indeed, hyperbolic groups are strongly geodesically automatic, i.e., there is an automatic structure on the group [17].

The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. For example, it has been shown empirically in [52] that the internet topology embeds with better accuracy into a hyperbolic space than into an Euclidean space of comparable dimension (formal proofs that the distortion is related to hyperbolicity can be found in [55]); the same holds for many complex networks, see [34]. A few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers (see [19–21,33]). Another important application of these spaces is the study of the spread of viruses through the internet (see [29,30]). Furthermore, hyperbolic spaces are useful in secure transmission of information on the network (see [28–30,38]).

Several investigators have shown interest in proving that the metrics used in geometric function theory are Gromov hyperbolic. For instance, the Klein–Hilbert and Kobayashi metrics are Gromov hyperbolic (under particular conditions on the domain of definition, see [3,5,31]), the Gehring–Osgood j -metric is Gromov hyperbolic, and the Vuorinen j -metric is not Gromov hyperbolic except in the punctured space (see [24]). Also, in [35], the hyperbolicity of the conformal modulus metric μ and the related so-called Ferrand metric λ^* , have been studied. Gromov hyperbolicity of the quasihyperbolic and the Poincaré metrics is also the subject of [2,4,10,25,26,42–44,48–50,54].

If $\gamma : [a, b] \rightarrow X$ is a continuous curve in a metric space (X, d) , we can define the *length* of γ as

$$L(\gamma) := \sup \left\{ \sum_{i=1}^n d(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 < t_1 < \dots < t_n = b \right\}.$$

The curve γ is a *geodesic* if we have $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$ (then γ is equipped with an arc-length parametrization). The metric space X is said to be *geodesic* if for every couple of points in X , there exists a geodesic joining them. We denote by $[xy]$ any geodesic joining x and y ; this notation is ambiguous, since in general, we do not have uniqueness of geodesics, but it is very convenient. Consequently, any geodesic metric space is connected. If the metric space X is a graph, then the edge joining the vertices u and v will be denoted by $[u, v]$.

In order to consider a graph G as a geodesic metric space, identify (by an isometry) any edge $[u, v] \in E(G)$ with the interval $[0, 1]$ in the real line. Then the edge $[u, v]$ (considered as a graph with just one edge) is isometric to the interval $[0, 1]$. Thus, the points in G are the vertices and, also, the points in the interior of any edge of G . In this way, any graph G has a natural distance defined on its points, induced by taking shortest paths in G , and we can see G as a metric graph. Throughout this paper, $G = (V, E)$ denotes a simple connected graph such that every edge has length 1. These properties guarantee that any graph is a geodesic metric space. Note that to exclude multiple edges and loops is not an important loss of generality, since [8, Theorems 8 and 10] reduce the problem of computation of the hyperbolicity constant of graphs with multiple edges and/or loops to the study of simple graphs.

Consider a polygon $J = \{J_1, J_2, \dots, J_n\}$ with sides $J_j \subseteq X$ in a geodesic metric space X . We say that J is δ -thin if for every $x \in J_i$, we have $d(x, \cup_{j \neq i} J_j) \leq \delta$. Let us denote by $\delta(J)$ the sharp thin constant of J , i.e., $\delta(J) := \inf\{\delta \geq 0 : J \text{ is } \delta\text{-thin}\}$. If x_1, x_2, x_3 are three points in X , a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X . We say that X is δ -hyperbolic if every geodesic triangle in X is δ -thin, and we denote by $\delta(X)$ the sharp hyperbolicity constant of X , i.e., $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ -hyperbolic for some $\delta \geq 0$; then X is hyperbolic if and only if $\delta(X) < \infty$. A geodesic *bigon* is a geodesic triangle $\{x_1, x_2, x_3\}$ with $x_2 = x_3$. Therefore, every bigon in a δ -hyperbolic geodesic metric space is δ -thin.

Trivially, any bounded metric space X is $((\text{diam } X)/2)$ -hyperbolic. A normed linear space is hyperbolic if and only if it has dimension one. A geodesic space is 0-hyperbolic if and only if it is a metric tree. If a complete Riemannian manifold is simply connected and their sectional curvatures satisfy $K \leq c$ for some negative constant c , then it is hyperbolic. See the classical references [1, 22] in order to find more background and further results.

We want to remark that the main examples of hyperbolic graphs are the trees. In fact, the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how ‘tree-like’ the space is, since those spaces X with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, for example, [18]). Also, it is well-known that any Gromov hyperbolic space with n points embeds into a tree metric with distortion $O(\delta \log n)$ (see, for example, [22, p. 33]).

A main problem in the theory is to characterize in a simple way the hyperbolic graphs. Given a Cayley graph (of a presentation with solvable word problem), there is an algorithm which allows to decide if it is hyperbolic. However, for a general graph deciding whether or not a space is hyperbolic is a very difficult problem. Therefore, it is interesting to study the hyperbolicity of particular classes of graphs. The papers [9, 11, 12, 14–16, 36, 40, 41, 46, 53, 56] study the hyperbolicity of, respectively, complement of graphs, chordal graphs, strong product graphs, corona and join of graphs, line graphs, Cartesian product graphs, cubic graphs, tessellation graphs, short graphs, median graphs and k -chordal graphs. In [12, 15, 36], the authors characterize the hyperbolic product graphs (for strong product, corona and join of graphs, and Cartesian product) in terms of properties of their factor graphs. The graphs with small hyperbolicity constant, the distortion of the hyperbolicity constant, and the relation of the hyperbolicity constant with several parameters are studied in [6, 13, 38], respectively.

The study of lexicographic product graphs is a subject of increasing interest (see, for example, [32, 45, 51, 57, 58] and references therein). In this paper, we characterize the hyperbolic lexicographic product of two graphs $G_1 \circ G_2$, in terms of G_1 and G_2 : if G_1 has at least two vertices, then $G_1 \circ G_2$ is hyperbolic if and only if G_1 is hyperbolic; besides, if G_1 has a single vertex, then $G_1 \circ G_2$ is hyperbolic if and only if G_2 is hyperbolic (see Theorem 3.18 and Remark 3.19). We also prove the sharp inequalities $\delta(G_1) \leq \delta(G_1 \circ G_2) \leq \delta(G_1) + 3/2$ if G_1 is not a trivial graph, see Theorems 3.2 and 3.14; Example 3.4 provides a family of graphs for which the first inequality is attained; besides, Theorems 3.20 and 3.24 characterize the graphs for which the second inequality is attained.

Furthermore, we obtain the precise value of the hyperbolicity constant for many lexicographic products (see Examples 3.3, 3.4 and Theorem 3.25). In particular, Theorem 3.25 allows to compute, in a simple way, the hyperbolicity constant of the lexicographic product of any tree and any graph.

2. Distances in lexicographic products

In order to estimate the hyperbolicity constant of the lexicographic product of two graphs G_1 and G_2 , we must obtain bounds on the distances between any two arbitrary points in $G_1 \circ G_2$. Besides, we study the geodesics in $G_1 \circ G_2$, relating them with the geodesics in G_1 . The lemmas of this section provide these results.

We will use the lexicographic product definition given in [27].

DEFINITION 2.1

Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. The lexicographic product $G_1 \circ G_2$ of G_1 and G_2 has $V(G_1) \times V(G_2)$ as a vertex set, so that two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \circ G_2$ are adjacent if either $[u_1, u_2] \in E(G_1)$ or $u_1 = u_2$ and $[v_1, v_2] \in E(G_2)$.

Note that the lexicographic product of two graphs is not always commutative (see Figure 1). We use the notation (x, y) for the points of the graph $G_1 \circ G_2$ with $x \in V(G_1)$ or $y \in V(G_2)$. Otherwise, this notation can be ambiguous.

Remark 2.2. The Cartesian and the strong product of two graphs are subgraphs of the lexicographic product of two graphs, i.e., $G_1 \square G_2 \subseteq G_1 \boxtimes G_2 \subseteq G_1 \circ G_2$.

Along this work by *trivial graph*, we mean a graph having just a single vertex, and we denote it by E_1 . If G_1 and G_2 are isomorphic, then we write $G_1 \simeq G_2$.

Remark 2.3. Let G be any graph. Then $G \circ E_1 \simeq G$ and $E_1 \circ G \simeq G$.

In what follows, we denote by π the projection $\pi : G_1 \circ G_2 \rightarrow G_1$. The following result allows to compute the distance between any two vertices of $G_1 \circ G_2$.

Lemma 2.4. Let G_1 be a non-trivial graph and G_2 any graph. Let $(u, v), (u', v')$ be two vertices in $G_1 \circ G_2$. Then

$$d_{G_1 \circ G_2}((u, v), (u', v')) = \begin{cases} \min\{2, d_{G_2}(v, v')\}, & \text{if } u = u', \\ d_{G_1}(u, u'), & \text{if } u \neq u'. \end{cases}$$

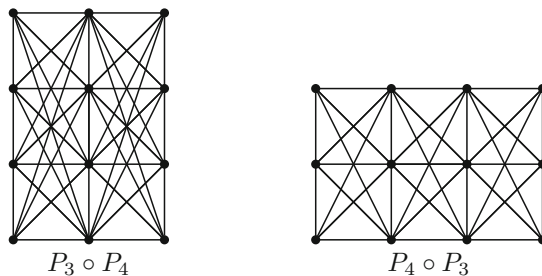


Figure 1. Non commutative lexicographic product of two graphs ($P_3 \circ P_4 \not\cong P_4 \circ P_3$).

Proof. Assume first that $u = u'$, thus $(u, v), (u, v') \in V(\{u\} \circ G_2)$. If $d_{G_2}(v, v') \leq 2$, then $d_{G_1 \circ G_2}((u, v), (u, v')) = d_{G_2}(v, v')$ since a path in $G_1 \circ G_2$ joining (u, v) and (u, v') which is not contained in $\{u\} \circ G_2$ has a vertex out of $\{u\} \circ G_2$, and so, its length is at least 2. If $d_{G_2}(v, v') > 2$, then

$$d_{G_1 \circ G_2}((u, v), (u, v')) = d_{G_1 \circ G_2}((u, v), \{w\} \circ G_2) + d_{G_1 \circ G_2}(\{w\} \circ G_2, (u, v')) = 2,$$

where $[u, w] \in E(G_1)$.

Assume now that $u \neq u'$. If $\gamma := [uu']$ is a geodesic in G_1 joining the points u and u' with $L(\gamma) = k$, then there exist vertices A_1, \dots, A_{k-1} in $\gamma \setminus \{u, u'\}$. Without loss of generality, we can assume that γ meets A_1, \dots, A_{k-1} in this order. If we fix $v_0 \in V(G_2)$, then

$$d_{G_1 \circ G_2}((u, v), (u', v')) \leq d_{G_1 \circ G_2}((u, v), (A_1, v_0)) + \dots + d_{G_1 \circ G_2}((A_{k-1}, v_0), (u', v')) = k.$$

If $d_{G_1 \circ G_2}((u, v), (u', v')) < k$, then there exists a geodesic Γ in $G_1 \circ G_2$ joining (u, v) and (u', v') with $L(\Gamma) = r < k$. Denote by B_1, \dots, B_{r-1} the vertices in $\Gamma \setminus \{(u, v), (u', v')\}$. Without loss of generality, we can assume that Γ meets B_1, \dots, B_{r-1} in this order. Then we have

$$\Gamma := [(u, v), B_1] \cup \left\{ \bigcup_{j=1}^{r-2} [B_j, B_{j+1}] \right\} \cup [B_{r-1}, (u', v')].$$

By Definition 2.1,

$$\gamma_1 := [u, \pi(B_1)] \cup \left\{ \bigcup_{j=1}^{r-2} [\pi(B_j), \pi(B_{j+1})] \right\} \cup [\pi(B_{r-1}), u']$$

is a path joining u and u' in G_1 such that $L(\gamma_1) \leq L(\Gamma) < L(\gamma)$. This is a contradiction, thus

$$d_{G_1 \circ G_2}((u, v), (u', v')) = d_{G_1}(u, u').$$

□

Let X be a metric space, Y a non-empty subset of X and ε a positive number. We call ε -neighborhood of Y in X , denoted by $\mathcal{V}_\varepsilon(Y)$ to the set $\{x \in X : d_X(x, Y) \leq \varepsilon\}$.

Lemma 2.5. Let G_1 be a non-trivial graph and G_2 be any graph. Then $G_1 \circ G_2 \subseteq \mathcal{V}_{3/2}(G_1 \circ \{v\})$ for every $v \in V(G_2)$.

Proof. Let p be any point of $G_1 \circ G_2$. If $p \in V(G_1 \circ G_2)$, then consider any $u_0 \in V(G_1)$ such that $[\pi(p), u_0] \in E(G_1)$. Definition 2.1 gives $d_{G_1 \circ G_2}(p, G_1 \circ \{v\}) \leq$

$d_{G_1 \circ G_2}(p, (u_0, v)) = 1$ for every $v \in V(G_2)$ since G_1 is non-trivial. Assume that $p \notin V(G_1 \circ G_2)$. Let $A \in V(G_1 \circ G_2)$ with $d_{G_1 \circ G_2}(p, A) \leq 1/2$. Hence, we have

$$d_{G_1 \circ G_2}(p, G_1 \circ \{v\}) \leq d_{G_1 \circ G_2}(p, A) + d_{G_1 \circ G_2}(A, G_1 \circ \{v\}) \leq 3/2.$$

□

Lemma 2.6. Let y_1, y_2 be any points in G_2 with $d_{G_2}(y_1, y_2) \leq 5/2$ and x_0 a fixed vertex in G_1 . Then $\gamma := \{x_0\} \times [y_1 y_2]$ is a geodesic in $G_1 \circ G_2$ joining the points (x_0, y_1) and (x_0, y_2) .

Proof. If G_1 is a trivial graph, then $G_1 \circ G_2 \simeq G_2$ and we have the result. Assume that G_1 is a non-trivial graph. Seeking for a contradiction, assume that γ is not a geodesic in $G_1 \circ G_2$. Therefore, there is a geodesic Γ in $G_1 \circ G_2$ joining (x_0, y_1) and (x_0, y_2) which is not contained in $\{x_0\} \circ G_2$. Hence, Γ has a vertex V outside of $\{x_0\} \circ G_2$; thus, we have $2 \leq L(\Gamma) < L(\gamma) \leq 5/2$. We have

$$\Gamma = [(x_0, y_1)(x_0, B_1)] \cup [(x_0, B_1), V] \cup [V, (x_0, B_2)] \cup [(x_0, B_2)(x_0, y_2)],$$

where B_i is a closest vertex to y_i in G_2 , for $i = 1, 2$. Since $\gamma \cup \Gamma$ contains a cycle C with $(x_0, B_1), (x_0, B_2) \in C$ and $L(\gamma) + L(\Gamma) < 5$, we have $L(C) \leq 4$ and $d_{G_2}(B_1, B_2) \leq 2$, and so, we obtain

$$\begin{aligned} d_{G_2}(y_1, y_2) &\leq d_{G_2}(y_1, B_1) + d_{G_2}(B_1, B_2) + d_{G_2}(B_2, y_2) \\ &\leq d_{G_2}(y_1, B_1) + 2 + d_{G_2}(B_2, y_2) = L(\Gamma) < L(\gamma) = d_{G_2}(y_1, y_2). \end{aligned}$$

This is the contradiction we were looking for, and so, γ is a geodesic in $G_1 \circ G_2$. □

COROLLARY 2.7

Let G_1 be a non-trivial graph and G_2 be any graph. Let y_1, y_2 be any points in G_2 with $d_{G_2}(y_1, y_2) > 3$ and x_0 a fixed vertex in G_1 . Then $\{x_0\} \times [y_1 y_2]$ is not a geodesic in $G_1 \circ G_2$.

Proof. Let B_i be the closest vertex to y_i in G_2 , for $i = 1, 2$. Since G_1 is a non-trivial graph, there is a vertex $u_0 \in V(G_1)$ such that $[x_0, u_0] \in E(G_1)$. For any fixed $v_0 \in V(G_2)$, we have

$$\begin{aligned} \Gamma := & [(x_0, y_1)(x_0, B_1)] \cup [(x_0, B_1), (u_0, v_0)] \\ & \cup [(u_0, v_0), (x_0, B_2)] \cup [(x_0, B_2)(x_0, y_2)] \end{aligned}$$

is a path in $G_1 \circ G_2$ joining (x_0, y_1) and (x_0, y_2) . Besides, since $d_{G_2}(y_1, B_1) \leq 1/2$ and $d_{G_2}(y_2, B_2) \leq 1/2$, we have $L(\Gamma) \leq 3 < d_{G_2}(y_1, y_2) = L(\{x_0\} \times [y_1 y_2])$. □

Remark 2.8. Let y_1, y_2 be two midpoints in any graph G_2 with $d_{G_2}(y_1, y_2) = 3$ and x_0 a fixed vertex in any graph G_1 . Then $\{x_0\} \times [y_1 y_2]$ is a geodesic in $G_1 \circ G_2$ joining (x_0, y_1) and (x_0, y_2) .

Lemma 2.9. Let G_1 be a non-trivial graph and G_2 be any graph. If γ is a geodesic in $G_1 \circ G_2$ joining x and y with $L(\gamma) > 3$, then $\pi(\gamma)$ contains at least three vertices in G_1 .

Furthermore, if σ is a path in $G_1 \circ G_2$ joining x and y , then $\pi(\sigma)$ contains at least three vertices in G_1 .

Proof. Since $L(\gamma) > 3$, then γ contains at least three vertices in $G_1 \circ G_2$. Let V_1 and V_2 be the closest vertices to x and y in γ , respectively. Seeking for a contradiction, assume that $\pi(\gamma)$ contains either one or two vertices in G_1 . Since G_1 is a non-trivial graph and $\pi(\gamma)$ contains at most two vertices, Lemma 2.4 gives that $d_{G_1 \circ G_2}(V_1, V_2) = 2$ and $\pi(V_1) = \pi(V_2)$. Furthermore, since $L(\gamma) > 3$, we have either $d_{G_1 \circ G_2}(x, V_1) > 1/2$ or $d_{G_1 \circ G_2}(y, V_2) > 1/2$. Without loss of generality, we can assume that $d_{G_1 \circ G_2}(x, V_1) > 1/2$. Let W be the vertex in $G_1 \circ G_2$ with x in the edge $[V_1, W]$. Then $d_{G_1 \circ G_2}(x, W) < 1/2 < d_{G_1 \circ G_2}(x, V_1)$. Consider now a path $\gamma_1 := [xW] \cup [WV_2] \cup [V_2y]$ joining x and y in $G_1 \circ G_2$. Hence, $L(\gamma_1) < L(\gamma)$ since $d_{G_1 \circ G_2}(W, V_2) \leq 2$. This is the contradiction we were looking for, and then $\pi(\gamma)$ contains at least three vertices in G_1 . Finally, since $L(\sigma) \geq L(\gamma)$ and $\pi(\gamma)$ contains at least three vertices, the proof is straightforward. \square

Lemma 2.10. Let G_1 be a non-trivial graph and G_2 be any graph. Consider a geodesic γ in $G_1 \circ G_2$ joining x and y . If $L(\gamma) > 3$, then $\pi(\gamma)$ is a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$. Besides, if $L(\gamma) = 3$, then $\pi(\gamma)$ contains a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$.

Proof. Assume first that $L(\gamma) > 3$. By Lemma 2.9, $\pi(\gamma)$ contains at least three vertices in G_1 . Denote by V_1, \dots, V_r the vertices of $G_1 \circ G_2$ in γ with $r \geq 3$, and v_1, \dots, v_r their projections in G_1 (there are at least three different vertices). Without loss of generality, we can assume that γ meets V_1, \dots, V_r in this order. Let V'_1, V'_r be two vertices in $G_1 \circ G_2$ such that $x \in [V'_1, V_1]$ and $y \in [V_r, V'_r]$, and denote by v'_1, v'_r their projections in G_1 , respectively. Since $d_{G_1 \circ G_2}(V_1, V_r) \geq 2$ and $d_{G_1 \circ G_2}(x, y) \geq 3$, Lemma 2.4 gives $d_{G_1}(\{v_1, v'_1\}, \{v_r, v'_r\}) \geq 2$.

Seeking for a contradiction, assume that there is a geodesic Γ in G_1 joining $\pi(x)$ and $\pi(y)$ with length less than $L(\pi(\gamma))$. Let us consider $v_i^* := \{v_i, v'_i\} \cap \Gamma$ and $V_i^* \in \{V_i, V'_i\}$ with $\pi(V_i^*) = v_i^*$ for $i \in \{1, r\}$. Now, we have three cases.

(1) $\pi(x) \neq v_1$ and $\pi(y) \neq v_r$. Then $\pi(x) \in [v'_1, v_1]$ and $\pi(y) \in [v_r, v'_r]$. Let $\gamma_1 := [xV_1^*] \cup [V_1^*V_r^*] \cup [V_r^*y] \subset G_1 \circ G_2$. Since $d_{G_1}(v_1^*, v_r^*) \geq 2$, Lemma 2.4 gives $d_{G_1 \circ G_2}(V_1^*, V_r^*) = d_{G_1}(v_1^*, v_r^*)$, and so $L(\gamma_1) = L(\Gamma) < L(\pi(\gamma)) \leq L(\gamma)$. This is the contradiction we were looking for, and so, $\pi(\gamma)$ is a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$.

(2) $\pi(x) = v_1$ and $\pi(y) \neq v_r$ or $\pi(x) \neq v_1$ and $\pi(y) = v_r$. By symmetry, we can assume $\pi(x) = v_1$ and $\pi(y) \neq v_r$. Then $\pi(y) \in [v'_r, v_r]$ and $d_{G_1 \circ G_2}(x, V_1) \leq 1/2$. Let $\gamma_1 := [xV_1] \cup [V_1V_r^*] \cup [V_r^*y] \subset G_1 \circ G_2$. Since $d_{G_1}(v_1, v_r^*) \geq 2$, Lemma 2.4 gives $d_{G_1 \circ G_2}(V_1, V_r^*) = d_{G_1}(v_1, v_r^*)$, and so $L(\gamma_1) = L(\Gamma) + L([xV_1]) < L(\pi(\gamma)) + L([xV_1]) \leq L(\gamma)$. This is the contradiction we were looking for, and so, $\pi(\gamma)$ is a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$.

(3) $\pi(x) = v_1$ and $\pi(y) = v_r$. Then $\pi(\gamma) = \pi([V_1V_r])$. Since $d_{G_1}(v_1, v_r) \geq 2$, Lemma 2.4 gives $d_{G_1 \circ G_2}(V_1, V_r) = d_{G_1}(v_1, v_r)$. Then $L(\pi(\gamma)) = d_{G_1}(v_1, v_r)$, and $\pi(\gamma)$ is a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$.

Assume now that $L(\gamma) = 3$. Then $\pi(\gamma)$ contains either one, two, three or four vertices in G_1 .

If $\pi(\gamma)$ contains a single vertex in G_1 , then γ is contained in $\{v\} \circ G_2$ for some $v \in V(G_1)$. Thus, $\pi(\gamma) = v$ is a geodesic in G_1 joining $\pi(x)$ with $\pi(y)$.

If $\pi(\gamma)$ contains exactly two vertices in G_1 , then x, y are midpoints of edges and $\pi(x) = \pi(y)$.

If $\pi(\gamma)$ contains three or four vertices in G_1 , then $\pi(\gamma)$ contains a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$, and the argument used in the proof of the case $L(\gamma) > 3$ gives that $\pi(\gamma)$ is a geodesic. \square

Remark 2.11. Let γ be a geodesic in $G_1 \circ G_2$ joining x and y . If $L(\gamma) = 3$ and $\pi(\gamma)$ is not a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$, then x, y are midpoints of edges $\pi(x) = \pi(y) \in V(G_1)$ and $\text{diam}(\pi(\gamma)) = 1$.

DEFINITION 2.12

The diameter of the vertices of the graph G , denoted by $\text{diam } V(G)$, is defined as

$$\text{diam } V(G) := \sup\{d_G(u, v) : u, v \in V(G)\},$$

and the diameter of the graph G , denoted by $\text{diam } G$, is defined as

$$\text{diam } G := \sup\{d_G(x, y) : x, y \in G\}.$$

COROLLARY 2.13

Let γ be a geodesic in $G_1 \circ G_2$ joining x and y . If $\pi(\gamma)$ is not a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$, then $\text{diam}(\pi(\gamma)) < 3$.

Notice that, if γ is a geodesic in $G_1 \circ G_2$ joining the points x and y , then it is possible that $\pi(\gamma)$ does not contain a geodesic in G_1 joining the points $\pi(x)$ and $\pi(y)$, as the following example shows.

Example 2.14. Consider G_1 as the cycle graph C_3 with vertices $\{v_1, v_2, v_3\}$ and G_2 as the path graph P_3 with vertices $\{w_1, w_2, w_3\}$ and $E(G_2) = \{[w_1, w_2], [w_2, w_3]\}$. Let x and y be the midpoints of edges $[(v_1, w_1), (v_2, w_1)]$ and $[(v_1, w_3), (v_3, w_3)]$, respectively. We have that $\gamma := [x(v_2, w_1)] \cup [(v_2, w_1), (v_3, w_3)] \cup [(v_3, w_3)y]$ is a geodesic in $G_1 \circ G_2$ joining x and y , but $\pi(\gamma) = [\pi(x)v_2] \cup [v_2, v_3] \cup [v_3\pi(y)]$ does not contain the geodesic in G_1 joining $\pi(x)$ and $\pi(y)$ (note that this geodesic is $[\pi(x)v_1] \cup [v_1\pi(y)]$).

3. Hyperbolicity in lexicographic products

Some bounds for the hyperbolicity constant of the lexicographic product of two graphs are studied in this section. These bounds allow to prove Theorem 3.18, which characterizes the hyperbolic lexicographic products of two graphs.

We say that a subgraph Γ of G is *isometric* if $d_\Gamma(x, y) = d_G(x, y)$ for every $x, y \in \Gamma$. The following result which appeared in [47, Lemma 5] will be useful.

Lemma 3.1. *If Γ is an isometric subgraph of G , then $\delta(\Gamma) \leq \delta(G)$.*

The next theorem shows an important qualitative result: if G_1 is not hyperbolic, then $G_1 \circ G_2$ is not hyperbolic.

Theorem 3.2. *Let G_1 and G_2 two graphs, then $\delta(G_1) \leq \delta(G_1 \circ G_2)$.*

Proof. Since $G_1 \circ \{y\}$ is an isometric subgraph of $G_1 \circ G_2$ for every $y \in V(G_2)$, Lemma 3.1 gives the result. □

Example 3.4 shows that the equality in Theorem 3.2 is attained: $\delta(C_n) = \delta(C_n \circ P_2)$ for $n \geq 5$.

Note that the strong product graph $G \boxtimes P_2$ is isomorphic to $G \circ P_2$ for any graph G . We recall that $\delta(P_n) = 0$ since the path graph P_n is a tree; besides, it is well known that the hyperbolicity constant of the cycle graph C_n is $n/4$, see [47, Theorem 11]. The following results that appeared in [12] give the hyperbolicity constant of some lexicographic product graphs.

Example 3.3. Let P_n be the path graph with $n \geq 2$. Then

$$\delta(P_n \circ P_2) = \begin{cases} 1, & \text{if } n = 2, \\ 5/4, & \text{if } n = 3, \\ 3/2, & \text{if } n \geq 4. \end{cases}$$

Example 3.4. Let C_n be the cycle graph with $n \geq 3$. Then

$$\delta(C_n \circ P_2) = \begin{cases} 1, & \text{if } n = 3, \\ 5/4, & \text{if } n = 4, \\ n/4, & \text{if } n \geq 5. \end{cases}$$

Example 3.5. Let K_m, K_n be the complete graphs with m, n vertices, respectively, and $m, n \geq 2$. Then $K_m \circ K_n$ is isomorphic to K_{mn} and $\delta(K_m \circ K_n) = 1$.

PROPOSITION 3.6

Let G_1 be a non-trivial graph and G_2 be any graph. Consider isometric subgraphs Γ_1, Γ_2 of G_1, G_2 , respectively, with Γ_1 being non-trivial. Then $\Gamma_1 \circ \Gamma_2$ is an isometric subgraph to $G_1 \circ G_2$.

Note that taking Γ_1 as a trivial graph, $\Gamma_1 \circ \Gamma_2$ is not an isometric subgraph to $G_1 \circ G_2$ if $\text{diam } V(\Gamma_2) \geq 3$.

Proof. Since $\Gamma_1 \circ \Gamma_2$ is a subgraph of $G_1 \circ G_2$, we have $d_{\Gamma_1 \circ \Gamma_2}(x, y) \geq d_{G_1 \circ G_2}(x, y)$ for every $x, y \in \Gamma_1 \circ \Gamma_2$. Let x, y be any points of $\Gamma_1 \circ \Gamma_2$. If $x, y \in V(\Gamma_1 \circ \Gamma_2)$, then by Lemma 2.4, we have $d_{G_1 \circ G_2}(x, y) = d_{\Gamma_1 \circ \Gamma_2}(x, y)$ and we obtain the result. Without loss of generality, we can assume that $x, y \notin V(\Gamma_1 \circ \Gamma_2)$. Let $A_1, A_2, B_1, B_2 \in V(\Gamma_1 \circ \Gamma_2)$

with $x \in [A_1, A_2]$, $y \in [B_1, B_2]$. Consider a geodesic γ in $G_1 \circ G_2$ joining x and y with $\gamma := [xA_i] \cup [A_i B_j] \cup [B_j y]$ for some $i, j \in \{1, 2\}$. Then

$$\begin{aligned} d_{\Gamma_1 \circ \Gamma_2}(x, y) &\leq d_{\Gamma_1 \circ \Gamma_2}(x, A_i) + d_{\Gamma_1 \circ \Gamma_2}(A_i, B_j) + d_{\Gamma_1 \circ \Gamma_2}(B_j, y) \\ &= d_{G_1 \circ G_2}(x, y). \end{aligned}$$

Thus, $d_{G_1 \circ G_2}(x, y) = d_{\Gamma_1 \circ \Gamma_2}(x, y)$. \square

Theorem 3.7. *Let G_1 be a non-trivial graph and G_2 be any graph. Then*

$$\begin{aligned} \delta(G_1 \circ G_2) &= \max\{\delta(\Gamma_1 \circ \Gamma_2) : \Gamma_i \text{ is isometric to } G_i \text{ for} \\ &\quad i = 1, 2 \text{ and } \Gamma_1 \text{ non-trivial}\}. \end{aligned}$$

Proof. By Lemma 3.1 and Proposition 3.6, we have $\delta(G_1 \circ G_2) \geq \delta(\Gamma_1 \circ \Gamma_2)$ for any Γ_1, Γ_2 . Besides, since any graph is an isometric subgraph of itself, we obtain the equality by taking $\Gamma_1 = G_1$ and $\Gamma_2 = G_2$. \square

The following results will be useful.

Theorem 3.8 (Theorem 8 in [47]). *In any graph G , the inequality $\delta(G) \leq \frac{1}{2} \text{diam } G$ holds and it is sharp.*

Denote by $J(G)$ the set of vertices and midpoints of edges in G . As usual, by *cycle* we mean a simple closed curve, i.e., a path with different vertices, unless the last one, which is equal to the first vertex.

Theorem 3.9 (Theorem 2.6 in [7]). *For every hyperbolic graph G , $\delta(G)$ is a multiple of $1/4$.*

Theorem 3.10 (Theorem 2.7 in [7]). *For any hyperbolic graph G , there exists a geodesic triangle $T = \{x, y, z\}$ that is a cycle with $x, y, z \in J(G)$ and $\delta(T) = \delta(G)$.*

Theorem 3.11. *If G_1 and G_2 are non-trivial graphs, then $\delta(G_1 \circ G_2) \geq 1$.*

Proof. Since G_i is a non-trivial graph, there is a subgraph P_2^i in G_i isomorphic to an edge, for $i = 1, 2$. Hence, by Example 3.3 and Theorem 3.7, we have $\delta(G_1 \circ G_2) \geq \delta(P_2^1 \circ P_2^2) = 1$. \square

Theorem 3.12. *Let G_2 be any non-trivial graph and G_1 be any graph. If $\text{diam } V(G_1) = 2$, then $\delta(G_1 \circ G_2) \geq 5/4$. If $\text{diam } V(G_1) \geq 3$, then $\delta(G_1 \circ G_2) \geq 3/2$.*

Proof. Assume that $\text{diam } V(G_1) = 2$. Since G_2 is a non-trivial graph, there is a subgraph P_2 in G_2 isomorphic to an edge. Besides, since $\text{diam } V(G_1) = 2$, then there is an isometric subgraph in G_1 isomorphic to a path P_3 with 3 vertices. Example 3.3 and Theorem 3.7 give $5/4 = \delta(P_3 \circ P_2) \leq \delta(G_1 \circ G_2)$.

If $\text{diam } V(G_1) \geq 3$, then a similar argument replacing P_3 by P_4 gives $\delta(G_1 \circ G_2) \geq 3/2$. \square

Theorem 3.13. *If G_1 is any non-trivial graph and G_2 is any graph with $\text{diam } G_2 > 2$, then $\delta(G_1 \circ G_2) \geq 5/4$.*

Proof. Since $\text{diam } G_2 \geq 5/2$, we have that there exist a midpoint $x \in J(G_2) \setminus V(G_2)$ and a vertex $y \in V(G_2)$ such that $d_{G_2}(x, y) = 5/2$. Hence, by Lemma 2.6, we have that $\gamma_1 := \{v_0\} \times [xy]$ is a geodesic in $G_1 \circ G_2$ joining the points (v_0, x) and (v_0, y) for some $v_0 \in V(G_1)$. Without loss of generality, we can assume that $(v_0, x) \in [A_1, A_2]$ such that $A_1 \in \gamma_1$. Denote it by $\gamma_2 := [(v_0, x)A_2] \cup [A_2W] \cup [W(v_0, y)]$, where $W \in V(\{v_1\} \circ G_2)$ with $[v_0, v_1] \in E(G_1)$. Therefore, $L(\gamma_2) = 5/2$ and γ_2 is a geodesic in $G_1 \circ G_2$ joining the points (v_0, x) and (v_0, y) . Now we have a geodesic bigon $B := \{\gamma_1, \gamma_2\}$ in $G_1 \circ G_2$. If p is the midpoint of γ_1 , then $d_{G_1 \circ G_2}(p, \gamma_2) = 5/4$ and we conclude that $\delta(G_1 \circ G_2) \geq \delta(B) = 5/4$. \square

Theorem 3.14. *Let G_1 be any non-trivial graph and G_2 be any graph. Then we have $\delta(G_1 \circ G_2) \leq \delta(G_1) + 3/2$.*

Proof. If G_1 is not hyperbolic, then $\delta(G_1) = \infty$, and so, Theorem 3.2 gives the result (with equality). Assume now that G_1 is hyperbolic. By Theorem 3.10, it suffices to consider geodesic triangles $T = \{x, y, z\}$ in $G_1 \circ G_2$ that are cycles with $x, y, z \in J(G_1 \circ G_2)$. Let $\gamma_1 := [xy]$, $\gamma_2 := [yz]$ and $\gamma_3 := [zx]$. It suffices to prove that $d_{G_1 \circ G_2}(p, \gamma_2 \cup \gamma_3) \leq \delta(G_1) + 3/2$ for every $p \in \gamma_1$. If $d_{G_1 \circ G_2}(p, \{x, y\}) \leq 3/2$, then $d_{G_1 \circ G_2}(p, \gamma_2 \cup \gamma_3) \leq d_{G_1 \circ G_2}(p, \{x, y\}) \leq 3/2$.

Assume that $d_{G_1 \circ G_2}(p, \{x, y\}) > 3/2$; then $L(\gamma_1) > 3$. Let $V_p := (v, w)$ be the closest vertex to p in γ_1 . Consider the canonical projection $\pi : G_1 \circ G_2 \rightarrow G_1 \circ \{w\}$. By Lemma 2.10, $\pi(\gamma_1)$ is a geodesic in $G_1 \circ \{w\}$ joining the points $\pi(x)$ and $\pi(y)$.

If $\pi(\gamma_2)$ and $\pi(\gamma_3)$ are geodesics in $G_1 \circ \{w\}$, then there is a point $\alpha \in \pi(\gamma_2) \cup \pi(\gamma_3)$ such that $d_{G_1 \circ \{w\}}(V_p, \alpha) \leq \delta(G_1)$. Assume that $\alpha \in V(\pi(\gamma_2) \cup \pi(\gamma_3))$. Since $L(\gamma_1) > 3$ and $\gamma_2 \cup \gamma_3$ joins x and y , by Lemma 2.9, $\pi(\gamma_2) \cup \pi(\gamma_3)$ contains at least three vertices; hence, there exists a vertex $(v_\alpha, w) \in V(\pi(\gamma_2) \cup \pi(\gamma_3))$ such that $[\alpha, (v_\alpha, w)] \in E(G_1 \circ \{w\})$. Let V_α be a vertex in $(\{v_\alpha\} \circ G_2) \cap (\gamma_2 \cup \gamma_3)$. Thus, $[\alpha, V_\alpha] \in E(G_1 \circ G_2)$ and

$$d_{G_1 \circ G_2}(p, \gamma_2 \cup \gamma_3) \leq d_{G_1 \circ G_2}(p, V_p) + d_{G_1 \circ \{w\}}(V_p, \alpha) + d_{G_1 \circ G_2}(\alpha, V_\alpha) \leq \delta(G_1) + 3/2.$$

If $\alpha \notin V(\pi(\gamma_2) \cup \pi(\gamma_3))$, then $\alpha \in \{\pi(x), \pi(y)\}$ and α is a midpoint in $G_1 \circ \{w\}$. Without loss of generality, we can assume that $\alpha = \pi(x)$ and, consequently, x is a midpoint in $G_1 \circ G_2$. Let V_x be the closest vertex to x in $\gamma_2 \cup \gamma_3$ and v_x the closest vertex to $\pi(x)$ in $\pi(\gamma_1)$. Hence, $[V_x, v_x] \in E(G_1 \circ G_2)$, $d_{G_1 \circ \{w\}}(V_p, v_x) \leq \delta(G_1) - 1/2$ and

$$d_{G_1 \circ G_2}(p, \gamma_2 \cup \gamma_3) \leq d_{G_1 \circ G_2}(p, V_p) + d_{G_1 \circ \{w\}}(V_p, v_x) + d_{G_1 \circ G_2}(v_x, V_x) \leq \delta(G_1) + 1.$$

If $\pi(\gamma_2)$ and $\pi(\gamma_3)$ are not geodesics in $G_1 \circ \{w\}$, then there is a point $\alpha \in [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]$ such that $d_{G_1 \circ \{w\}}(V_p, \alpha) \leq \delta(G_1)$. Notice that if α is not a vertex in $G_1 \circ \{w\}$, then we repeat the previous argument and obtain the result. Assume now that $\alpha \in V([\pi(x)\pi(z)] \cup [\pi(z)\pi(y)])$; by symmetry, we can assume that $\alpha \in V([\pi(x)\pi(z)])$. If $\alpha \in \pi(\gamma_2) \cup \pi(\gamma_3)$, then the previous argument gives $d_{G_1 \circ G_2}(p, \gamma_2 \cup \gamma_3) \leq \delta(G_1) + 3/2$.

Assume now that $\alpha \notin \pi(\gamma_2) \cup \pi(\gamma_3)$. Seeking for a contradiction, assume that there is no vertex $(v_\alpha, w) \in V(\pi(\gamma_2) \cup \pi(\gamma_3))$ such that $[\alpha, (v_\alpha, w)] \in E(G_1 \circ \{w\})$. Then $d_{G_1 \circ \{w\}}(\alpha, V(\pi(\gamma_2) \cup \pi(\gamma_3))) \geq 2$; hence $d_{G_1 \circ \{w\}}(\alpha, \pi(x)) \geq 3/2$ and $d_{G_1 \circ \{w\}}(\alpha, \pi(z)) \geq 3/2$. However, by Corollary 2.13, we have $d_{G_1 \circ \{w\}}(\pi(x), \pi(z)) = d_{G_1 \circ \{w\}}(\pi(x), \alpha) + d_{G_1 \circ \{w\}}(\alpha, \pi(z)) < 3$, which is a contradiction. Therefore, there exists a vertex $(v_\alpha, w) \in V(\pi(\gamma_2) \cup \pi(\gamma_3))$ such that $[\alpha, (v_\alpha, w)] \in E(G_1 \circ \{w\})$. Let V_α be a vertex in $(\{v_\alpha\} \circ G_2) \cap (\gamma_2 \cup \gamma_3)$. Then $[\alpha, V_\alpha] \in E(G_1 \circ G_2)$ and

$$d_{G_1 \circ G_2}(p, \gamma_2 \cup \gamma_3) \leq d_{G_1 \circ G_2}(p, V_p) + d_{G_1 \circ \{w\}}(V_p, \alpha) + d_{G_1 \circ G_2}(\alpha, V_\alpha) \leq \delta(G_1) + 3/2.$$

In both cases, $\pi(\gamma_2)$ is a geodesic in $G_1 \circ \{w\}$ but $\pi(\gamma_3)$ is not a geodesic in $G_1 \circ \{w\}$, and $\pi(\gamma_3)$ is a geodesic in $G_1 \circ \{w\}$ but $\pi(\gamma_2)$ is not a geodesic in $G_1 \circ \{w\}$. A similar argument gives the inequality. \square

Remark 3.15. Let G_1 be any hyperbolic graph which is not a tree and let G_2 be any graph. The argument in the proof of Theorem 3.14 gives that if $\delta(G_1 \circ G_2) = \delta(G_1) + 3/2$, then there is a geodesic triangle $T = \{x, y, z\}$ with $x, y, z \in J(G_1 \circ G_2)$ and a midpoint $p \in [xy]$ such that $d_{G_1 \circ G_2}(p, [xz] \cup [zy]) = \delta(G_1) + 3/2$. Besides, $d_{G_1 \circ \{w\}}(V_p, [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) = \delta(G_1)$ and the distance is attained in a vertex $\alpha \in [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]$.

Example 3.3 and Theorem 3.24 show that the equality in Theorem 3.14 is attained. We obtain the following consequence of Theorem 3.2 and Theorem 3.14.

Theorem 3.16. *Let G_1 be any non-trivial graph and G_2 be any graph. Then*

$$\delta(G_1) \leq \delta(G_1 \circ G_2) \leq \delta(G_1) + 3/2.$$

Theorems 3.12 and 3.14 have the following consequence.

COROLLARY 3.17

If G_1 is any infinite tree and G_2 is any non-trivial graph, then $\delta(G_1 \circ G_2) = 3/2$.

Theorem 3.18. *Let G_1 be any non-trivial graph and G_2 be any graph. The lexicographic product $G_1 \circ G_2$ is hyperbolic if and only if G_1 is hyperbolic.*

Remark 3.19. For any graph G and the trivial graph E_1 , the lexicographic product graph $E_1 \circ G$ is hyperbolic if and only if G is hyperbolic, since $\delta(E_1 \circ G) = \delta(G)$. This trivial result completes the characterization of hyperbolic lexicographic products.

The following results allow to characterize the graphs for which the bound in Theorem 3.14 is attained.

Theorem 3.20. *Let G_1 be any hyperbolic graph and let G_2 be any graph. If $\delta(G_1 \circ G_2) = \delta(G_1) + 3/2$, then G_1 is a tree, G_2 is a non-trivial graph and $\delta(G_1 \circ G_2) = 3/2$.*

Proof. Seeking for a contradiction, assume that G_1 is not a tree (i.e., $\delta(G_1) > 0$). By hypothesis, $G_1 \circ G_2$ is hyperbolic, thus, Theorem 3.10 and Remark 3.15 give that there is a geodesic triangle $T = \{x, y, z\}$ in $G_1 \circ G_2$ that is a cycle with $x, y, z \in J(G_1 \circ G_2)$ and a midpoint $p \in [xy]$ such that $d_{G_1 \circ G_2}(p, [xz] \cup [zy]) = \delta(G_1) + 3/2$. Let $V_p := (v, w)$ be a closest vertex to p in $[xy] \cap V(G_1 \circ G_2)$ as in the proof of Theorem 3.14, i.e., $d_{G_1 \circ \{w\}}(V_p, [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) = \delta(G_1)$ with π the canonical projection on $G_1 \circ \{w\}$; besides, this equality is attained in a vertex $\alpha \in [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]$. Note that $\delta(G_1)$ is an integer number since it is the distance between two vertices. Since $\delta(G_1) > 0$, we have $\delta(G_1) \geq 1$. Let V'_p be the vertex in $T \cap V(G_1 \circ G_2)$ such that $[V_p, V'_p]$ is the edge in $G_1 \circ G_2$ with $p \in [V_p, V'_p]$. Since $d_{G_1 \circ G_2}(p, \{x, y\}) \geq d_{G_1 \circ G_2}(p, [xz] \cup [zy]) = \delta(G_1) + 3/2$, there exist $a, b \in [xy] \cap V(G_1 \circ G_2)$ with $d_{G_1 \circ G_2}(a, p) = d_{G_1 \circ G_2}(b, p) = 3/2$ and $d_{G_1 \circ G_2}(a, b) = 3$. If $\pi(V_p) = \pi(V'_p)$, then $d_{G_1 \circ \{w\}}(\pi(a), \pi(b)) = 2$. This contradicts Lemma 2.4, and so, we have $\pi(V_p) \neq \pi(V'_p)$ and $\pi(V_p) \neq \pi(p) \neq \pi(V'_p)$. If $d_{G_1 \circ \{w\}}(\pi(p), [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) = d_{G_1 \circ \{w\}}(V_p, [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) = \delta(G_1) \geq 1$, then since $\pi(V_p) \neq \pi(p)$, we obtain that $d_{G_1 \circ \{w\}}(\xi, [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) = \delta(G_1) + 1/4$, where ξ is the midpoint of $[\pi(p)V_p]$. But this is a contradiction, since $d_{G_1 \circ \{w\}}(\xi, [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) \leq \delta(G_1)$. Then we have $d_{G_1 \circ \{w\}}(\pi(p), [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) < d_{G_1 \circ \{w\}}(V_p, [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) = \delta(G_1)$; hence, $d_{G_1 \circ \{w\}}(\pi(p), [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) = \delta(G_1) - 1/2$ and $d_{G_1 \circ \{w\}}(\pi(V'_p), [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) = \delta(G_1) - 1$. We can repeat the same argument in the proof of Theorem 3.14 for V'_p instead of V_p , and we obtain $d_{G_1 \circ G_2}(p, [xz] \cup [zy]) \leq \delta(G_1) + 1/2$. This is the contradiction we were looking for and G_1 is a tree.

Hence $\delta(G_1 \circ G_2) = 3/2$. If G_2 is a trivial graph, then $3/2 = \delta(G_1 \circ G_2) = \delta(G_1) = 0$, which is a contradiction. Therefore, G_2 is a non-trivial graph. \square

Theorem 3.24 below is a converse of Theorem 3.20; furthermore, it provides the exact value of the hyperbolicity constant of the lexicographic product of many trees and graphs. We need some lemmas.

Lemma 3.21. *Let G_1 be any tree with $1 \leq \text{diam } G_1 \leq 2$ and G_2 be any graph. Then $\delta(G_1 \circ G_2) = 3/2$ if and only if there is a geodesic triangle $T = \{x, y, z\}$ in $G_1 \circ G_2$ that is a cycle contained in $\{v_0\} \circ G_2$ for some $v_0 \in V(G_1)$ with $x, y, z \in J(\{v_0\} \circ G_2)$ and a vertex $p \in [xy]$ such that $d_{G_1 \circ G_2}(p, [xz] \cup [zy]) = d_{G_1 \circ G_2}(p, x) = d_{G_1 \circ G_2}(p, y) = 3/2$.*

Proof. Assume first that $\delta(G_1 \circ G_2) = 3/2$. By Theorem 3.10, there exists a geodesic triangle $T = \{x, y, z\}$ in $G_1 \circ G_2$ that is a cycle with $x, y, z \in J(G_1 \circ G_2)$ and a point $p \in [xy]$ such that $\delta(T) = d_{G_1 \circ G_2}(p, [yz] \cup [zx]) = 3/2$. Thus, $d_{G_1 \circ G_2}(p, \{x, y\}) \geq d_{G_1 \circ G_2}(p, [xz] \cup [zy]) = 3/2$ and $L([xy]) \geq 3$.

Assume that $\text{diam } G_1 = 2$ (the case $\text{diam } G_1 = 1$ is similar and simpler). We show now that $\text{diam } G_1 \circ G_2 = 3$. Note that $\text{diam } G_1 \circ G_2 \geq L([xy]) \geq 3$. Let $A, B \in V(G_1 \circ G_2)$. If $\pi(A) = \pi(B)$, then by Lemma 2.4, we have $d_{G_1 \circ G_2}(A, B) \leq 2$. If $\pi(A) \neq \pi(B)$, then by Lemma 2.4, we have $d_{G_1 \circ G_2}(A, B) \leq 2$ since $\text{diam } G_1 = 2$. Therefore, $\text{diam } V(G_1 \circ G_2) = 2$ and $\text{diam } G_1 \circ G_2 \leq 3$. Consequently, $\text{diam } G_1 \circ G_2 = 3$, $L([xy]) = 3$ and $d_{G_1 \circ G_2}(p, x) = d_{G_1 \circ G_2}(p, y) = 3/2$. Notice that x, y are midpoints of $G_1 \circ G_2$ and p is a vertex of $G_1 \circ G_2$.

Assume now that $x \in \{v_0\} \circ G_2$ for some $v_0 \in V(G_1)$ and $y \notin \{v_0\} \circ G_2$, where $x \in [A_1, A_2]$ and $y \in [B_1, B_2]$ with $A_1, B_1 \in [xy]$; then $d_{G_1 \circ G_2}(A_1, B_1) = 2$ since $L([xy]) = 3$. Note that $A_1 \in \{v_0\} \times V(G_2)$ and $B_1 \in \{w_0\} \times V(G_2)$ with $d_{G_1}(v_0, w_0) = 2$.

We have that $[xy] \cap ([yz] \cup [zx]) = \{x, y\}$ since T is a cycle. Hence, $A_2, B_2 \in V([yz] \cup [zx])$ and $d_{G_1 \circ G_2}(p, [yz] \cup [zx]) = d_{G_1 \circ G_2}(p, \{A_2, B_2\}) = 1$ since p is a vertex, and this is a contradiction. If $y \in \{v_0\} \circ G_2$ for some $v_0 \in V(G_1)$ and $x \notin \{v_0\} \circ G_2$, then the same argument gives a contradiction. If $x, y \notin \cup_{v_0 \in V(G_1)} \{v_0\} \circ G_2$, then one can check that $d_{G_1 \circ G_2}(x, y) \leq 2$, which is a contradiction. Hence, we conclude that $x, y \in \{v_0\} \circ G_2$ for some $v_0 \in V(G_1)$. We also have $p \in \{v_0\} \circ G_2$ and we conclude that $[xy]$ is contained in $\{v_0\} \circ G_2$. If $[yz] \cup [zx]$ is not contained in $\{v_0\} \circ G_2$, then there is a vertex $W \in [yz] \cup [zx]$ such that $W \in \{w_0\} \circ G_2$ and $d_{G_1}(v_0, w_0) = 1$. Hence, $d_{G_1 \circ G_2}(p, W) = 1$, which is a contradiction. Then T is contained in $\{v_0\} \circ G_2$.

It is easy to check that if there exists such a geodesic triangle T , then $\delta(G_1 \circ G_2) \geq \delta(T) \geq 3/2$. Theorem 3.14 concludes that $\delta(G_1 \circ G_2) = 3/2$. \square

Now we define some families of graphs which will be useful. Denote by C_n the cycle graph with $n \geq 3$ vertices and by $V(C_n) := \{v_1^{(n)}, \dots, v_n^{(n)}\}$ the set of their vertices such that $[v_n^{(n)}, v_1^{(n)}] \in E(C_n)$ and $[v_i^{(n)}, v_{i+1}^{(n)}] \in E(C_n)$ for $1 \leq i \leq n-1$. Let $\mathcal{C}_6^{(1)}$ be the set of graphs obtained from C_6 by adding a (proper or not) subset of the set of edges $\{[v_2^{(6)}, v_6^{(6)}], [v_4^{(6)}, v_6^{(6)}]\}$. Let us define the set of graphs

$$\mathcal{F}_6 := \{\text{graphs containing, as induced subgraph,} \\ \text{an isomorphic graph to some element of } \mathcal{C}_6^{(1)}\}.$$

Let $\mathcal{C}_7^{(1)}$ be the set of graphs obtained from C_7 by adding a (proper or not) subset of the set of edges $\{[v_2^{(7)}, v_6^{(7)}], [v_2^{(7)}, v_7^{(7)}], [v_4^{(7)}, v_6^{(7)}], [v_4^{(7)}, v_7^{(7)}]\}$. Define

$$\mathcal{F}_7 := \{\text{graphs containing, as induced subgraph,} \\ \text{an isomorphic graph to some element of } \mathcal{C}_7^{(1)}\}.$$

Let $\mathcal{C}_8^{(1)}$ be the set of graphs obtained from C_8 by adding a (proper or not) subset of the set $\{[v_2^{(8)}, v_6^{(8)}], [v_2^{(8)}, v_8^{(8)}], [v_4^{(8)}, v_6^{(8)}], [v_4^{(8)}, v_8^{(8)}]\}$. Also, let $\mathcal{C}_8^{(2)}$ be the set of graphs obtained from C_8 by adding a (proper or not) subset of $\{[v_2^{(8)}, v_8^{(8)}], [v_4^{(8)}, v_6^{(8)}], [v_4^{(8)}, v_7^{(8)}], [v_4^{(8)}, v_8^{(8)}]\}$. Define

$$\mathcal{F}_8 := \{\text{graphs containing, as induced subgraph,} \\ \text{an isomorphic graph to some element of } \mathcal{C}_8^{(1)} \cup \mathcal{C}_8^{(2)}\}.$$

Let $\mathcal{C}_9^{(1)}$ be the set of graphs obtained from C_9 by adding a (proper or not) subset of the set of edges $\{[v_2^{(9)}, v_6^{(9)}], [v_2^{(9)}, v_9^{(9)}], [v_4^{(9)}, v_6^{(9)}], [v_4^{(9)}, v_9^{(9)}]\}$. Define

$$\mathcal{F}_9 := \{\text{graphs containing, as induced subgraph,} \\ \text{an isomorphic graph to some element of } \mathcal{C}_9^{(1)}\}.$$

Finally, we define the set \mathcal{F} by

$$\mathcal{F} := \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{F}_8 \cup \mathcal{F}_9.$$

Note that $\mathcal{F}_6, \mathcal{F}_7, \mathcal{F}_8$ and \mathcal{F}_9 are not disjoint sets of graphs.

Remark 3.22. Note that to check if a graph G belongs to \mathcal{F} is a tractable problem from a computational viewpoint, fix a graph G with m edges and maximum degree Δ . By choosing

edges $[v_1, v_2] \in E(G)$, $[v_2, v_3] \in E(G)$, \dots and $[v_{a-1}, v_a] \in E(G)$, we can obtain the set of all paths of length $a - 1$ in time $O(m\Delta^{a-2})$; hence, we can compute all cycles with length a in time $O(m\Delta^{a-2})$. Therefore, it is possible to find a subgraph isomorphic to a fixed graph in \mathcal{C}_a (or in \mathcal{C}'_a) in time $O(m\Delta^{a-2})$. Note that there are 4, 16, 16, 16 and 16 graphs in $\mathcal{C}_6, \mathcal{C}_7, \mathcal{C}_8, \mathcal{C}'_8$ and \mathcal{C}_9 , respectively. Hence, we can find that every subgraph of G isomorphic to a graph in $\mathcal{C}_6 \cup \mathcal{C}_7 \cup \mathcal{C}_8 \cup \mathcal{C}'_8 \cup \mathcal{C}_9$ in time $O(m\Delta^7)$. Hence, we can check if the hypothesis $G \in \mathcal{F}$ is a tractable problem from a computational viewpoint, by using the algorithm sketched before.

For any non-empty set $S \subset V(G)$, the induced subgraph of S will be denoted by $\langle S \rangle$.

Lemma 3.23. *Let G be any graph. Then $G \in \mathcal{F}$ if and only if there is a geodesic triangle $T = \{x, y, z\}$ in G that is a cycle with $x, y, z \in J(G)$, $L([xy]), L([yz]), L([zx]) \leq 3$ and $\delta(T) = 3/2 = d_G(p, [yz] \cup [zx])$ where $p \in [xy] \cap V(G)$.*

Proof. Assume first that there is a geodesic triangle $T = \{x, y, z\}$ in G that is a cycle with $x, y, z \in J(G)$, $L([xy]), L([yz]), L([zx]) \leq 3$ and $\delta(T) = 3/2 = d_G(p, [yz] \cup [zx])$ for some $p \in [xy]$. Since $d_G(p, \{x, y\}) \geq d_G(p, [yz] \cup [zx]) = 3/2$, we have $L([xy]) = 3$ and p is the midpoint of $[xy]$. Since $L([yz]) \leq 3$, $L([zx]) \leq 3$ and $L([yz]) + L([zx]) \geq L([xy])$, we have $6 \leq L(T) \leq 9$.

Assume now that $L(T) = 6$. Denote by $\{v_1, \dots, v_6\}$ the vertices in T such that $T = \bigcup_{i=1}^6 [v_i, v_{i+1}]$ with $v_7 := v_1$. Without loss of generality, we can assume that $x \in [v_1, v_2]$, $y \in [v_4, v_5]$ and $p = v_3$. Since $d_G(x, y) = 3$, we have that $\langle \{v_1, \dots, v_6\} \rangle$ contains neither $[v_1, v_4], [v_1, v_5], [v_2, v_4]$ nor $[v_2, v_5]$; besides, since $d_G(p, [yz] \cup [zx]) > 1$, we have that $\langle \{v_1, \dots, v_6\} \rangle$ contains neither $[v_3, v_1], [v_3, v_5]$ nor $[v_3, v_6]$. Note that $[v_2, v_6], [v_4, v_6]$ may be contained in $\langle \{v_1, \dots, v_6\} \rangle$. Therefore, $G \in \mathcal{F}_6$.

Assume that $L(T) = 7$ and $G \notin \mathcal{F}_6$. Denote by $\{v_1, \dots, v_7\}$ the vertices in T such that $T = \bigcup_{i=1}^7 [v_i, v_{i+1}]$ with $v_8 := v_1$. Without loss of generality, we can assume that $x \in [v_1, v_2]$, $y \in [v_4, v_5]$ and $p = v_3$. Since $d_G(x, y) = 3$, we have that $\langle \{v_1, \dots, v_7\} \rangle$ contains neither $[v_1, v_4], [v_1, v_5], [v_2, v_4]$ nor $[v_2, v_5]$; besides, since $d_G(p, [yz] \cup [zx]) > 1$, we have that $\langle \{v_1, \dots, v_7\} \rangle$ contains neither $[v_3, v_1], [v_3, v_5], [v_3, v_6]$ nor $[v_3, v_7]$. Since $G \notin \mathcal{F}_6$, $[v_1, v_6]$ and $[v_5, v_7]$ are not contained in $\langle \{v_1, \dots, v_7\} \rangle$. Note that $[v_2, v_6], [v_2, v_7], [v_4, v_6], [v_4, v_7]$ may be contained in $\langle \{v_1, \dots, v_7\} \rangle$. Hence, $G \in \mathcal{F}_7$.

Assume that $L(T) = 8$ and $G \notin \mathcal{F}_6 \cup \mathcal{F}_7$. Denote by $\{v_1, \dots, v_8\}$ the vertices in T such that $T = \bigcup_{i=1}^8 [v_i, v_{i+1}]$ with $v_9 := v_1$. Without loss of generality, we can assume that $x \in [v_1, v_2]$, $y \in [v_4, v_5]$ and $p = v_3$. Since $d_G(x, y) = 3$, we have that $\langle \{v_1, \dots, v_8\} \rangle$ contains neither $[v_1, v_4], [v_1, v_5], [v_2, v_4]$ nor $[v_2, v_5]$; besides, since $d_G(p, [yz] \cup [zx]) > 1$, we have that $\langle \{v_1, \dots, v_8\} \rangle$ contains neither $[v_3, v_1], [v_3, v_5], [v_3, v_6], [v_3, v_7]$ nor $[v_3, v_8]$. Since $G \notin \mathcal{F}_6 \cup \mathcal{F}_7$, $[v_1, v_6], [v_1, v_7], [v_5, v_7], [v_5, v_8]$ and $[v_6, v_8]$ are not contained in $\langle \{v_1, \dots, v_8\} \rangle$. Since T is a geodesic triangle, we have that $z \in \{v_{6,7}, v_7, v_{7,8}\}$ with $v_{6,7}$ and $v_{7,8}$ the midpoints of $[v_6, v_7]$ and $[v_7, v_8]$, respectively. If $z = v_7$, then $\langle \{v_1, \dots, v_8\} \rangle$ contains neither $[v_2, v_7]$ nor $[v_4, v_7]$. Note that $[v_2, v_6], [v_2, v_8], [v_4, v_6], [v_4, v_8]$ may be contained in $\langle \{v_1, \dots, v_8\} \rangle$. If $z = v_{6,7}$, then $\langle \{v_1, \dots, v_8\} \rangle$ contains neither $[v_2, v_6]$ nor $[v_2, v_7]$. Note that $[v_2, v_8], [v_4, v_6], [v_4, v_7], [v_4, v_8]$ may be contained in $\langle \{v_1, \dots, v_8\} \rangle$. By symmetry, we obtain an equivalent result for $z = v_{7,8}$. Therefore, $G \in \mathcal{F}_8$.

Assume that $L(T) = 9$ and $G \notin \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{F}_8$. Denote by $\{v_1, \dots, v_9\}$ the vertices in T such that $T = \bigcup_{i=1}^9 [v_i, v_{i+1}]$ with $v_{10} := v_1$. Without loss of generality, we can assume that $x \in [v_1, v_2]$, $y \in [v_4, v_5]$ and $p = v_3$. Since $d_G(x, y) = 3$, we have that $\langle \{v_1, \dots, v_9\} \rangle$

contains neither $[v_1, v_4], [v_1, v_5], [v_2, v_4]$ nor $[v_2, v_5]$; besides, since $d_G(p, [yz] \cup [zx]) > 1$, we have that $\langle \{v_1, \dots, v_9\} \rangle$ contains neither $[v_3, v_1], [v_3, v_5], [v_3, v_6], [v_3, v_7], [v_3, v_8]$ nor $[v_3, v_9]$. Since T is a geodesic triangle, we have that z is the midpoint of $[v_7, v_8]$. Since $d_G(y, z) = d_G(z, x) = 3$, we have that $\langle \{v_1, \dots, v_9\} \rangle$ contains neither $[v_1, v_7], [v_1, v_8], [v_2, v_7], [v_2, v_8], [v_4, v_7], [v_4, v_8], [v_5, v_7]$ nor $[v_5, v_8]$. Since $G \notin \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{F}_8$, $[v_1, v_6], [v_5, v_9], [v_6, v_8], [v_6, v_9]$ and $[v_7, v_9]$ are not contained in $\langle \{v_1, \dots, v_9\} \rangle$. Note that $[v_2, v_6], [v_2, v_9], [v_4, v_6], [v_4, v_9]$ may be contained in $\langle \{v_1, \dots, v_9\} \rangle$. Hence, $G \in \mathcal{F}_9$.

Therefore, in any case $G \in \mathcal{F}$.

The previous argument also shows that if $G \in \mathcal{F}$, then there is a geodesic triangle with the required properties. □

Theorem 3.20 and the following result characterize the graphs for which the bound in Theorem 3.14 is attained.

Theorem 3.24. *Let G_1 be any tree and G_2 be any non-trivial graph.*

- (1) *If $\text{diam } G_1 \geq 3$, then $\delta(G_1 \circ G_2) = 3/2$.*
- (2) *If $1 \leq \text{diam } G_1 \leq 2$, then $\delta(G_1 \circ G_2) = 3/2$ if and only if $G_2 \in \mathcal{F}$.*
- (3) *If G_1 is trivial, then $\delta(G_1 \circ G_2) = 3/2$ if and only if $\delta(G_2) = 3/2$.*

Proof. If $\text{diam } G_1 \geq 3$, then Theorems 3.12 and 3.14 give the result since $\delta(G_1) = 0$.

In order to prove (2), by Lemma 3.21, we have that $\delta(G_1 \circ G_2) = 3/2$ if and only if there is a geodesic triangle $T = \{x, y, z\}$ in $G_1 \circ G_2$ that is a cycle contained in $\{v\} \circ G_2$ for some $v \in V(G_1)$ with $x, y, z \in J(\{v\} \circ G_2)$ and a vertex $p \in [xy]$ such that $d_{G_1 \circ G_2}(p, [xz] \cup [zy]) = d_{G_1 \circ G_2}(p, x) = d_{G_1 \circ G_2}(p, y) = 3/2$. By Lemma 2.4, $\text{diam } V(G_1 \circ G_2) = 2$, hence, $L([yz]), L([zx]) \leq 3$ and x, y are the midpoints with $L([xy]) = 3$. Hence, by Lemma 3.23, we have that $\delta(G_1 \circ G_2) = 3/2$ if and only if $\{v\} \circ G_2 \in \mathcal{F}$ and so, Remark 2.3 gives that this is equivalent to $G_2 \in \mathcal{F}$.

Finally, if G_1 is trivial, then Remark 2.3 gives the result. □

The following result allows to compute, in a simple way, the hyperbolicity constant of the lexicographic product of any tree and any graph.

Theorem 3.25. *Let G_1 be any tree and G_2 be any graph. Then*

$$\delta(G_1 \circ G_2) = \begin{cases} \delta(G_2), & \text{if } G_1 \simeq E_1, \\ 0, & \text{if } G_2 \simeq E_1, \\ 1, & \text{if } \text{diam } G_1 = 1 \text{ and } 1 \leq \text{diam } G_2 \leq 2, \\ 5/4, & \text{if } \text{diam } G_1 = 1 \text{ and } \text{diam } G_2 > 2 \text{ and } G_2 \notin \mathcal{F}, \\ 5/4, & \text{if } \text{diam } G_1 = 2 \text{ and } \text{diam } G_2 \geq 1 \text{ and } G_2 \notin \mathcal{F}, \\ 3/2, & \text{if } 1 \leq \text{diam } G_1 \leq 2 \text{ and } G_2 \in \mathcal{F}, \\ 3/2, & \text{if } \text{diam } G_1 \geq 3 \text{ and } \text{diam } G_2 \geq 1. \end{cases}$$

Proof. If $G_1 \simeq E_1$ or $G_2 \simeq E_1$, then we have the result by Remark 2.3.

If $\text{diam } G_1 = 1$ and $1 \leq \text{diam } G_2 \leq 2$, then Theorems 3.9, 3.11, 3.14 and 3.24 give $\delta(G_1 \circ G_2) \in \{1, 5/4\}$ since $G_2 \notin \mathcal{F}$. Seeking for a contradiction, we can assume that $\delta(G_1 \circ G_2) = 5/4$. Then by Theorem 3.10, there is a geodesic triangle $T = \{x, y, z\}$ in

$G_1 \circ G_2$ that is a cycle with $x, y, z \in J(G_1 \circ G_2)$ and a point $p \in [xy]$ such that $\delta(T) = d_{G_1 \circ G_2}(p, [yz] \cup [zx]) = 5/4$. Thus, $d_{G_1 \circ G_2}(p, \{x, y\}) \geq d_{G_1 \circ G_2}(p, [xz] \cup [zy]) = 5/4$, $L([xy]) \geq 5/2$ and $x, y \in \{v\} \circ G_2$ for some $v \in V(G_1)$, since $\text{diam } G_1 = 1$. This is a contradiction since $\text{diam } G_2 \leq 2$ and we conclude that $\delta(G_1 \circ G_2) = 1$.

If $\text{diam } G_1 = 1$ and $\text{diam } G_2 > 2$ or $\text{diam } G_1 = 2$ and $\text{diam } G_2 \geq 1$, then Theorems 3.9, 3.12, 3.13 and 3.14 give $\delta(G_1 \circ G_2) \in \{5/4, 3/2\}$. Finally, since $G_2 \notin \mathcal{F}$, Theorem 3.24 gives $\delta(G_1 \circ G_2) \neq 3/2$ and we have $\delta(G_1 \circ G_2) = 5/4$.

If $1 \leq \text{diam } G_1 \leq 2$ and $G_2 \in \mathcal{F}$ or $\text{diam } G_1 \geq 3$ and $\text{diam } G_2 \geq 1$, then we have the result by Theorem 3.24. \square

COROLLARY 3.26

Let G_1, G_2 be any trees. Then

$$\delta(G_1 \circ G_2) = \begin{cases} 0, & \text{if } G_1 \simeq E_1 \text{ or } G_2 \simeq E_1, \\ 1, & \text{if } \text{diam } G_1 = 1 \text{ and } 1 \leq \text{diam } G_2 \leq 2, \\ 5/4, & \text{if } \text{diam } G_1 = 1 \text{ and } \text{diam } G_2 \geq 3, \\ 5/4, & \text{if } \text{diam } G_1 = 2 \text{ and } \text{diam } G_2 \geq 1, \\ 3/2, & \text{if } \text{diam } G_1 \geq 3 \text{ and } \text{diam } G_2 \geq 1. \end{cases}$$

COROLLARY 3.27

Let P_n, P_m be two path graphs. Then

$$\delta(P_n \circ P_m) = \begin{cases} 0, & \text{if } n = 1 \text{ or } m = 1, \\ 1, & \text{if } n = 2 \text{ and } m = 2, 3, \\ 5/4, & \text{if } n = 2 \text{ and } m \geq 4 \text{ or } n = 3 \text{ and } m \geq 2, \\ 3/2, & \text{if } n \geq 4 \text{ and } m \geq 2. \end{cases}$$

Acknowledgements

The authors would like to thank the referee for his/her valuable comments which has improved the presentation of the paper considerably. This work was supported in part by two Grants from Ministerio de Economía y Competitividad (MTM2013-46374-P and MTM2015-69323-REDT), Spain.

References

- [1] Alonso J, Brady T, Cooper D, Delzant T, Ferlini V, Lustig M, Mihalik M, Shapiro M and Short H, Notes on word hyperbolic groups, in: Group Theory from a Geometrical Viewpoint edited by E Ghys, A Haefliger and A Verjovsky (1992) (Singapore: World Scientific)
- [2] Alvarez V, Portilla A, Rodríguez J M and Tourís E, Gromov hyperbolicity of Denjoy domains, *Geom. Dedicata* **121** (2006) 221–245
- [3] Balogh Z M and Bonk M, Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains, *Comm. Math. Helv.* **75** (2000) 504–533

- [4] Balogh Z M and Buckley S M, Geometric characterizations of Gromov hyperbolicity, *Invent. Math.* **153** (2003) 261–301
- [5] Benoist Y, Convexes hyperboliques et fonctions quasisymétriques, *Publ. Math. Inst. Hautes Études Sci.* **97** (2003) 181–237
- [6] Bermudo S, Rodríguez J M, Rosario O and Sigarreta J M, Small values of the hyperbolicity constant in graphs, *Discrete Math.* **339** (2016) 3073–3084
- [7] Bermudo S, Rodríguez J M and Sigarreta J M, Computing the hyperbolicity constant, *Comput. Math. Appl.* **62** (2011) 4592–4595
- [8] Bermudo S, Rodríguez J M, Sigarreta J M and Vilaire J-M, Gromov hyperbolic graphs, *Discrete Math.* **313** (2013) 1575–1585
- [9] Bermudo S, Rodríguez J M, Sigarreta J M and Tourís E, Hyperbolicity and complement of graphs, *Appl. Math. Lett.* **24** (2011) 1882–1887
- [10] Bonk M, Heinonen J and Koskela P, Uniformizing Gromov hyperbolic spaces, *Astérisque* (2001) vol. 270
- [11] Brinkmann G, Koolen J and Moulton V, On the hyperbolicity of chordal graphs, *Ann. Comb.* **5** (2001) 61–69
- [12] Carballosa W, Casablanca R M, de la Cruz A and Rodríguez J M, Gromov hyperbolicity in strong product graphs, *Electr. J. Comb.* **20(3)** (2013) P2
- [13] Carballosa W, Pestana D, Rodríguez J M and Sigarreta J M, Distortion of the hyperbolicity constant of a graph, *Electr. J. Comb.* **19** (2012) P67
- [14] Carballosa W, Rodríguez J M and Sigarreta J M, New inequalities on the hyperbolicity constant of line graphs, *Ars Combin.* **129** (2016) 367–386
- [15] Carballosa W, Rodríguez J M and Sigarreta J M, Hyperbolicity in the corona and join of graphs, *Aequat. Math.* **89** (2015) 1311–1327
- [16] Carballosa W, Rodríguez J M, Sigarreta J M and Villeta M, Gromov hyperbolicity of line graphs, *Electr. J. Comb.* **18** (2011) P210
- [17] Charney R, Artin groups of finite type are biautomatic, *Math. Ann.* **292** (1992) 671–683
- [18] Chen B, Yau S-T and Yeh Y-N, Graph homotopy and Graham homotopy, *Discrete Math.* **241** (2001) 153–170
- [19] Chepoi V and Estellon B, Packing and covering δ -hyperbolic spaces by balls, *APPROX-RANDOM* (2007) pp. 59–73
- [20] Eppstein D, Squarepants in a tree: sum of subtree clustering and hyperbolic pants decomposition, *SODA '2007, Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms* (2007) pp. 29–38
- [21] Gavoille C and Ly O, Distance Labeling in Hyperbolic Graphs, in: *Lecture Notes in Computer Science*, vol. 3827 (2005) pp. 1071–1079
- [22] Ghys E and de la Harpe P, *Sur les Groupes Hyperboliques d'après Mikhael Gromov*, *Progress in Mathematics* 83, (1990) (Boston, MA: Birkhäuser Boston Inc.)
- [23] Gromov M, Hyperbolic groups, in: *Essays in group theory* edited by S M Gersten, M.S.R.I. Publ. 8 (1987) (Springer) pp. 75–263
- [24] Hästö P A, Gromov hyperbolicity of the j_G and \tilde{j}_G metrics, *Proc. Amer. Math. Soc.* **134** (2006) 1137–1142
- [25] Hästö P A, Lindén H, Portilla A, Rodríguez J M and Tourís E, Gromov hyperbolicity of Denjoy domains with hyperbolic and quasihyperbolic metrics, *J. Math. Soc. Jpn.* **64** (2012) 247–261
- [26] Hästö P A, Portilla A, Rodríguez J M and Tourís E, Gromov hyperbolic equivalence of the hyperbolic and quasihyperbolic metrics in Denjoy domains, *Bull. Lond. Math. Soc.* **42** (2010) 282–294
- [27] Imrich W and Klavžar S, *Product Graphs: Structure and Recognition*, *Wiley Series in Discrete Mathematics and Optimization* (2000)
- [28] Jonckheere E and Lohsoonthorn P, A hyperbolic geometry approach to multipath routing, *Proceedings of the 10th Mediterranean Conference on Control and Automation (MED 2002)*, Lisbon, Portugal, July 2002, FA5-1 (2002)

- [29] Jonckheere E A, Contrôle du trafic sur les réseaux à géométrie hyperbolique-Vers une théorie géométrique de la sécurité l'acheminement de l'information, *J. Europ. Syst. Autom.* **8** (2002) 45–60
- [30] Jonckheere E A and Lohsoonthorn P, Geometry of network security, *Amer. Control Conf. ACC* (2004) 111–151
- [31] Karlsson A and Noskov G A, The Hilbert metric and Gromov hyperbolicity, *Enseign. Math.* **48** (2002) 73–89
- [32] Kraner-Šumenjaka T, Pavlic P and Tepeh A, On the Roman domination in the lexicographic product of graphs, *Discrete Appl. Math.* **160(13–14)** (2012) 2030–2036
- [33] Krauthgamer R and Lee J R, Algorithms on negatively curved spaces, FOCS 2006 (2006)
- [34] Krioukov D, Papadopoulos F, Kitsak M, Vahdat A and Boguñá M, Hyperbolic geometry of complex networks, *Phys. Rev. E* **82** (2010) 036106
- [35] Lindén H, Gromov hyperbolicity of certain conformal invariant metrics, *Ann. Acad. Sci. Fenn. Math.* **32(1)** (2007) 279–288
- [36] Michel J, Rodríguez J M, Sigarreta J M and Villeta M, Gromov hyperbolicity in cartesian product graphs, *Proc. Indian Acad. Sci. (Math. Sci.)* **120** (2010) 1–17
- [37] Michel J, Rodríguez J M, Sigarreta J M and Villeta M, Hyperbolicity and parameters of graphs, *Ars Comb.* **100** (2011) 43–63
- [38] Narayan O and Saniee I, Large-scale curvature of networks, *Phys. Rev. E* **84** (2011) 066108
- [39] Oshika K, Discrete groups (2002) (AMS Bookstore)
- [40] Pestana D, Rodríguez J M, Sigarreta J M and Villeta M, Gromov hyperbolic cubic graphs, *Central Eur. J. Math.* **10(3)** (2012) 1141–1151
- [41] Portilla A, Rodríguez J M, Sigarreta J M and Vilaire J-M, Gromov hyperbolic tessellation graphs, *Utilitas Math.* **97** (2015) 193–212
- [42] Portilla A, Rodríguez J M and Tourís E, Gromov hyperbolicity through decomposition of metric spaces II, *J. Geom. Anal.* **14** (2004) 123–149
- [43] Portilla A, Rodríguez J M and Tourís E, The topology of balls and Gromov hyperbolicity of Riemann surfaces, *Diff. Geom. Appl.* **21** (2004) 317–335
- [44] Portilla A, Rodríguez J M and Tourís E, Stability of Gromov hyperbolicity, *J. Advan. Math Stud.* **2** (2009) 1–20
- [45] Power S C, Infinite lexicographic products of triangular algebras, *Bull. Lond. Math. Soc.* **27** (1995) 273–277
- [46] Rodríguez J M, Characterization of Gromov hyperbolic short graphs, *Act. Math. Sin.* **30** (2014) 197–212
- [47] Rodríguez J M, Sigarreta J M, Vilaire J-M and Villeta M, On the hyperbolicity constant in graphs, *Discrete Math.* **311** (2011) 211–219
- [48] Rodríguez J M and Tourís E, Gromov hyperbolicity through decomposition of metric spaces, *Acta Math. Hung.* **103** (2004) 53–84
- [49] Rodríguez J M and Tourís E, A new characterization of Gromov hyperbolicity for negatively curved surfaces, *Publ. Mat.* **50** (2006) 249–278
- [50] Rodríguez J M and Tourís E, Gromov hyperbolicity of Riemann surfaces, *Acta Math. Sin.* **23** (2007) 209–228
- [51] Saputro S W, Simanjuntak R, Uttunggadewa S, Assiyatun H, Baskoro E T, Salman A N M and Bača M, The metric dimension of the lexicographic product of graphs, *Discrete Math.* **313(9)** (2013) 1045–1051
- [52] Shavitt Y, Tankel T, On internet embedding in hyperbolic spaces for overlay construction and distance estimation, INFOCOM 2004 (2004)
- [53] Sigarreta J M, Hyperbolicity in median graphs, *Proc. Indian Acad. Sci. (Math. Sci.)* **123** (2013) 455–467
- [54] Tourís E, Graphs and Gromov hyperbolicity of non-constant negatively curved surfaces, *J. Math. Anal. Appl.* **380** (2011) 865–881
- [55] Verbeek K and Suri S, Metric embeddings, hyperbolic space and social networks, in: Proceedings of the 30th Annual Symposium on Computational Geometry (2014) pp. 501–510

- [56] Wu Y, Zhang C, Chordality and hyperbolicity of a graph, *Electr. J. Comb.* **18** (2011) P43
- [57] Yang C and Xu J, Connectivity of lexicographic product and direct product of graphs, *Ars Comb.* **111** (2013) 3–12
- [58] Zhang X, Liu J and Meng J, Domination in lexicographic product graphs, *Ars Comb.* **101** (2011) 251–256

COMMUNICATING EDITOR: B V Rajarama Bhat