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On the convergence of a new iterative algorithm of three infinite families of generalized nonexpansive multi-valued mappings

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Abstract. In this paper, we establish some weak and strong convergence theorems for a new iterative algorithm under some suitable conditions to approximate the common fixed point of three infinite families of multi-valued generalized nonexpansive mappings in a uniformly convex Banach spaces. Our results generalize and improve several previously known results of the existing literature.

Keywords. Common fixed point; generalized nonexpansive map; three step iterative scheme; weak and strong convergence; condition (A').

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1. Introduction

In recent years, approximation of fixed point of multi-valued nonexpansive mappings and multi-valued generalized nonexpansive mappings by iteration has been studied by many authors (see [2,3,20,22,24,26,28,31,32]). The fixed point theory of multi-valued mappings is much more complicated and difficult than the corresponding theory of single-valued mappings and has many fruitful applications in various fields, for example, game theory and mathematical economics. Thus, it is natural to extend the known fixed point results for single-valued mappings to the setting of multi-valued mappings. However, some classical fixed point theorems for single-valued nonexpansive mappings have already been extended to multi-valued mappings. The first set of results in this direction were established by Markin [17] in Hilbert spaces and by Browder [4] for spaces having weakly continuous duality mapping. Dozo [10] generalized these results in a Banach space satisfying Opial's condition.

In 1969, Nadler [18] proved a fixed point theorem for multi-valued contraction mappings and convergence of a sequence and also extended theorems on the stability of fixed points of single-valued mappings. Markin [17] first used the Hausdorff metric to study the fixed points for multi-valued contractions and nonexpansive mappings. Later in 1997, Hu *et al.* [13] obtained common fixed point of two nonexpansive multi-valued mappings satisfying certain contractive conditions.

Sastry and Babu [23] studied the Mann and Ishikawa iterative schemes for multi-valued mappings and proved that these schemes for a multi-valued map T with a fixed point z converges to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from z.

Panyanak [20] generalized results of Sastry and Babu [23] to uniformly convex Banach spaces and proved a convergence theorem of Mann iterates for a mapping defined on a noncompact domain.

In 2008, Song and Wang [31] showed the strong convergence for Mann and Ishikawa iterates of multi-valued nonexpansive mapping T under some appropriate conditions. In 2009, Shahzad and Zegeye [29] proved strong convergence theorems of quasi-nonexpansive multi-valued mapping for the Ishikawa iteration. They also constructed an iterative scheme which removes the restriction of T with $Tz = \{z\}$ for any $z \in F(T)$ which relaxed compactness of the domain of T.

Further in 2011, Abbas *et al.* [1] established weak and strong convergence theorems of two multi-valued nonexpansive mappings in a real uniformly convex Banach space by the one-step iterative process to approximate common fixed points under some basic boundary conditions.

In 2012, Chang *et al.* [8] used the modified block iterative method for solving the convex feasibility problems for an infinite family of quasi-ϕ-asymptotically nonexpansive mappings and obtained some strong convergence theorems under suitable conditions in Banach space.

In the same year, Bunyawat and Suantai [5] introduced an iterative method for finding a common fixed point of a countable family of multi-valued quasi-nonexpansive mapping in a uniformly convex Banach space. They proved that the iterative sequence generated by their proposed method is an approximating fixed point sequence of the multi-valued quasi-nonexpansive mapping under certain control conditions, and established some strong convergence theorems of their proposed method.

Phuangphoo and Kumam [21] extended and improved the above results in 2012. They introduced a new iterative procedure which was constructed by the shrinking hybrid projection method for solving the common solution of fixed point problems for two total quasi-ϕ-asymptotically nonexpansive multi-valued mappings. Under suitable conditions, the strong convergence theorems were established in a uniformly smooth and strictly convex real Banach space with Kadec–Klee property.

In 2013, Zhang *et al.* [34] introduced a new iterative scheme for finding a common fixed point of two countable families of multi-valued quasi-nonexpansive mappings and proved a weak convergence theorem under the suitable control conditions in a uniformly convex Banach space.

Recently, Rashwan and Altwqi [22] introduced a new SP-iterative scheme to approximate the common fixed points of three multi-valued nonexpansive mappings in CAT(k) spaces.

In this paper, we generalize and modify the iterative scheme of Rashwan and Altwqi [22] from three finite families of multi-valued nonexpansive mappings to three infinite families of multi-valued generalized nonexpansive mappings in a uniformly convex Banach space.

Let $\{T_i\}$, $\{S_i\}$, $\{R_i\}$ be three infinite families of multi-valued generalized nonexpansive mappings from a bounded and closed convex subset K of a Banach space E into $P(K)$ with $\mathcal{F} := [\bigcap_{i=1}^{\infty} F(T_i)] \bigcap [\bigcap_{i=1}^{\infty} F(S_i)] \bigcap [\bigcap_{i=1}^{\infty} F(R_i)] \neq \emptyset$ and let $z \in \mathcal{F}$. For $x_1 \in K$, we define

$$\begin{cases} x_{n+1} = \alpha_{0,n} y_n + \sum\limits_{i=1}^{\infty} \alpha_{i,n} u_{i,n}, \\ y_n = \beta_{0,n} z_n + \sum\limits_{i=1}^{\infty} \beta_{i,n} v_{i,n}, \\ z_n = \gamma_{0,n} x_n + \sum\limits_{i=1}^{\infty} \gamma_{i,n} w_{i,n}, \end{cases} \tag{1.1}$$

where $u_{i,n} \in T_i y_n$, $v_{i,n} \in S_i z_n$ and $w_{i,n} \in R_i x_n$ and, $\{\alpha_{i,n}\}$, $\{\beta_{i,n}\}$ and $\{\gamma_{i,n}\}$ are sequences in [0, 1] satisfying $\sum_{i=0}^{\infty} \alpha_{i,n} = \sum_{i=0}^{\infty} \beta_{i,n} = \sum_{i=0}^{\infty} \gamma_{i,n} = 1$.

Remark 1.

(1) If $\beta_{0,n}, \gamma_{0,n} = 1$ and $\sum_{i=1}^{\infty} \beta_{i,n} = \sum_{i=1}^{\infty} \gamma_{i,n} = 0$, the iterative algorithm (1.1) reduces to the iterative scheme of Bunyawat and Suantai [5].
(2) If $\sum_{i=2}^{\infty} \alpha_{i,n} = \sum_{i=2}^{\infty} \beta_{i,n} = \sum_{i=2}^{\infty} \gamma_{i,n} = 0$, the iterative algorithm (1.1) reduces to the iterative scheme of Rashwan and Altwqi [22] given by

$$\begin{cases} x_{n+1} = \alpha_{0,n} y_n + \alpha_{1,n} u_{1,n}, \\ y_n = \beta_{0,n} z_n + \beta_{1,n} v_{1,n}, \\ z_n = \gamma_{0,n} x_n + \gamma_{1,n} w_{1,n}, \end{cases} \tag{1.2}$$

where $u_{1,n} \in T_1 y_n$, $v_{1,n} \in S_1 z_n$ and $w_{1,n} \in R_1 x_n$.
(3) If $\gamma_{0,n} = 1$ and $\sum_{i=1}^{\infty} \gamma_{i,n} = 0$, the iterative algorithm (1.1) reduces to the following iterative scheme: $x_1 \in K$, and

$$\begin{cases} x_{n+1} = \alpha_{0,n} y_n + \sum\limits_{i=1}^{\infty} \alpha_{i,n} u_{i,n}, \\ y_n = \beta_{0,n} x_n + \sum\limits_{i=1}^{\infty} \beta_{i,n} v_{i,n}, \end{cases}$$

where $u_{i,n} \in T_i y_n$, $v_{i,n} \in S_i z_n$ and $\{\alpha_{i,n}\}, \{\beta_{i,n}\}$ are sequences in [0, 1] satisfying $\sum_{i=0}^{\infty} \alpha_{i,n} = \sum_{i=0}^{\infty} \beta_{i,n} = 1$.

2. Preliminaries

Throughout the paper, E stands for a real Banach space with the norm $\| \, . \, \|$ and K a nonempty subset of E. Let $\mathbb{N}$ denote the set of all positive integers. Let $CB(K)$, $C(K)$ and $P(K)$ denote the families of nonempty closed and bounded subsets, nonempty compact subsets and nonempty proximinal bounded subsets of K respectively. Recall that the set K is said to be proximinal if for any $x \in E$, there exists an element $y \in K$ such that $d(x, y) = \text{dist}(x, K)$, where $\text{dist}(x, K) = \inf\{\|x - y\|; y \in K\}$.

Remark 2. It is well-known that weakly compact convex subsets of a Banach space and closed convex subsets of a uniformly convex Banach space are proximinal.

Let H be the Hausdorff metric on $CB(E)$ defined by

$$H(A, B) = \max\left\{ \sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A) \right\} \quad \text{for all } A, B \in CB(E).$$

A multi-valued mapping $T : K \to CB(E)$ is said to be nonexpansive if

$$H(Tx, Ty) \leq \|x - y\| \quad \text{for all } \; x, y \in K.$$

A point $z \in K$ is called a fixed point of T if $z \in Tz$. As usual, $F(T)$ stands for the set of fixed points of a multi-valued mapping T. A multi-valued mapping $T : K \to CB(K)$ is said to be quasi-nonexpansive [30] if $F(T) \neq \emptyset$ and

$$H(Tx, Tz) \leq \|x - z\| \quad \text{for all } \; x \in K \; \text{ and } \; z \in F(T).$$

The study of fixed points for multi-valued nonexpansive mappings using the Hausdorff metric was given by Markin [17] while the existence of fixed points for multi-valued nonexpansive mappings in uniformly convex Banach spaces can be found in Lim [16].

In 2008, Suzuki [33] defined a generalization of nonexpansive mapping and called it a mapping satisfying condition (C) as follows.

DEFINITION 1 [33]

A single-valued mapping $\mathcal{T} : K \to K$ is said to satisfy condition (C) if

$$\frac{1}{2}\|x - \mathcal{T}x\| \leq \|x - y\| \Rightarrow \|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|, \; \forall \; x, y \in K.$$

Further, García-Falset *et al.* [12] proposed two new generalizations of condition (C) and termed them as conditions (E) and (C_λ) and studied the existence of fixed points for these classes of mappings whose set-valued version was defined and studied in [2,3,14] which are as follows.

DEFINITION 2 [14]

Let $T : K \to CB(E)$ be a multi-valued mapping. Then T is said to satisfy condition (C_λ) if for some $\lambda \in (0, 1)$ and for each $x, y \in K$,

$$\lambda \operatorname{dist}(x, Tx) \leq \|x - y\| \Rightarrow H(Tx, Ty) \leq \|x - y\|.$$

For $\lambda = \frac{1}{2}$, we re-capture the class of mappings satisfying condition (C). It is easy to see that for $0 < \lambda_1 < \lambda_2 < 1$, condition (C_{λ_1}) implies condition (C_{λ_2}).

Lemma 1 [14]. *Let $T : K \to CB(E)$ be a multi-valued mapping.*

(i) If T is nonexpansive, then T satisfies condition (C).
(ii) If T satisfies condition (C) and $F(T) \neq \emptyset$, then T is quasi-nonexpansive.

Lemma 2 [11]. *Let K be a nonempty subset of a Banach space E and $T : K \to P(E)$ a multi-valued map satisfying condition (C). Then*

$$H(Tx, Ty) \leq 2 \operatorname{dist}(x, Tx) + \|x - y\| \quad \textit{for all } x, y \in K.$$

Very recently, Abkar and Eslamian [3] used a modified Suzuki condition for multi-valued mappings which runs as follows.

DEFINITION 3 [3]

A multi-valued mapping $T : K \to CB(E)$ is said to satisfy condition (E_μ) if for some $\mu \geq 1$ (for all $x, y \in K$),

$$\text{dist}(x, Ty) \leq \mu\, \text{dist}(x, Tx) + \|x - y\|.$$

We say that T satisfies condition (E) on K whenever T satisfies condition (E_μ) for some $\mu \geq 1$.

Lemma 3 [3]. *Let $T : K \to CB(E)$ be a multi-valued nonexpansive mapping. Then T satisfies condition (E_1).*

Although generalized nonexpansive mappings are perhaps one of the most important topics in the so-called metric fixed point theory, one can find in [25]–[27] considerable amount of research about more general classes of mappings than the generalized nonexpansive ones.

To make our article self-contained, we collect some more basic definitions and needed results which will be used frequently in the text later.

DEFINITION 4 [19]

A Banach space E is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad \forall\, y \in E,\ y \neq x.$$

Examples of Banach spaces satisfying Opial's condition are Hilbert spaces and all l_p spaces $(1 < p < \infty)$. On the other hand, $L_p[0, 2]$ with $1 < p \neq 2$ fail to satisfy Opial's condition.

DEFINITION 5 [15]

A multi-valued mapping T defined on a subset K of a Banach space E is said to satisfy condition (A) if there exists a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$, $g(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \geq g(\text{dist}(x, F(T))$ for all $x \in K$, where $\text{dist}(x, F(T)) = \inf\{\|x - z\| : z \in F(T)\}$.

The multi-valued version of condition (A) (for three infinite families) is as follows.

DEFINITION 6

Three infinite families of multi-valued mappings $T_i, S_i, R_i : K \to CB(E)$, $(i = 1, 2, \ldots)$ are said to satisfy condition (A') if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $\text{dist}(x, T_i x) \geq f(\text{dist}(x, \mathcal{F}))$ or $\text{dist}(x, S_i x) \geq f(\text{dist}(x, \mathcal{F}))$ or $\text{dist}(x, R_i x) \geq f(\text{dist}(x, \mathcal{F}))$ for all $x \in K$, where $\mathcal{F} := [\bigcap_{i=1}^{\infty} F(T_i)] \bigcap [\bigcap_{i=1}^{\infty} F(S_i)] \bigcap [\bigcap_{i=1}^{\infty} F(R_i)]$, the set of all common fixed points of mappings T_i, S_i and R_i.

Lemma 4 [8]. *Let E be a uniformly convex Banach space and $B_r(0) := \{x \in X : \|x\| \le r\}$ a closed ball with center 0 and radius $r > 0$. For any given sequence $\{x_1, x_2, \ldots, x_n, \ldots\} \subset B_r(0)$ and any given number sequence $\{\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots\}$ with $\lambda_i \ge 0$, $\sum_{i=1}^{\infty} \lambda_i = 1$. Then there exists a continuous strictly increasing and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that for any positive integers i, j with $i < j$ the following holds*:

$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|^2 \le \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|). \tag{2.1}$$

Lemma 5 [9]. *Let E be a strictly convex Banach space and K a nonempty closed and convex subset of E. Let $T : K \to CB(K)$ be a multi-valued mapping satisfying condition (C). If $F(T)$ is nonempty, then it is a closed and convex subset of K.*

Lemma 6 [9]. *Let E be a uniformly convex Banach space satisfying the Opial's condition and K a nonempty closed and convex subset of E. Let $T : K \to CB(K)$ be a multi-valued mapping with convex-values and satisfies condition (C). Let $\{x_n\}$ be a sequence in K such that $x_n \rightharpoonup z \in K$, and $\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0$. Then $z \in Tz$, that is, $(I - T)$ is demi-closed at zero.*

3. Weak convergence theorems

In this section, we prove some weak convergence theorems by approximating the fixed points of three infinite families of multi-valued generalized nonexpansive mappings with non-empty convex-values by using iterative scheme (1.1).

In the sequel, $\mathcal{F}$ denotes the set of common fixed points of multi-valued mappings T_i, S_i and R_i, that is, $\mathcal{F} = [\bigcap_{i=1}^{\infty} F(T_i)] \bigcap [\bigcap_{i=1}^{\infty} F(S_i)] \bigcap [\bigcap_{i=1}^{\infty} F(R_i)]$, $\mathcal{F}'$ denotes the set of common fixed points of multi-valued mappings T, S and R, that is, $\mathcal{F}' = F(T) \cap F(S) \cap F(R)$ and $\mathbf{F}$ denotes the set of common fixed points of single-valued mappings $\mathcal{T}_i$, $\mathcal{S}_i$ and $\mathcal{R}_i$.

Theorem 1. *Let E be a uniformly convex Banach space satisfying Opial's condition and K a nonempty closed and convex subset of E. Let $T_i, S_i, R_i : K \to CB(K)$, $(i = 1, 2, \ldots)$ be three infinite families of multi-valued mappings satisfying condition (C) with non-empty convex-values and $\{x_n\}$ a sequence as defined by (1.1). If*

(1) $\sum_{i=0}^{\infty} \alpha_{i,n} = \sum_{i=0}^{\infty} \beta_{i,n} = \sum_{i=0}^{\infty} \gamma_{i,n} = 1$, *for each* $n \ge 1$,
(2) *for each* $i \ge 1$, $\lim_{n\to\infty} \inf \alpha_{0,n}\alpha_{i,n} > 0$, $\lim_{n\to\infty} \inf \beta_{0,n}\beta_{i,n} > 0$, $\lim_{n\to\infty} \inf \gamma_{0,n}\gamma_{i,n} > 0$,
(3) $\mathcal{F} \ne \emptyset$ *and* $T_i z = S_i z = R_i z = \{z\}$ *for all* $i \ge 1$ *and* $z \in \mathcal{F}$,

then $\{x_n\}$ converges weakly to some fixed point $z^* \in \mathcal{F}$.

Proof. We first claim that $\lim_{n\to\infty} \|x_n - z\|$ exists for each $z \in \mathcal{F}$. Since $\mathcal{F} \ne \emptyset$, therefore from Lemma 1(ii), it follows that for each $i \ge 1$, T_i, S_i and R_i are multi-valued quasi-nonexpansive mappings. Since $T_i z = S_i z = R_i z = \{z\}$ for any $z \in \mathcal{F}$, therefore by using condition (3) we have for each $n \ge 1$,

$$\begin{cases} \|u_{i,n} - z\| = \mathrm{dist}(u_{i,n}, T_i z) \le H(T_i y_n, T_i z) \le \|y_n - z\| \\ \|v_{i,n} - z\| = \mathrm{dist}(v_{i,n}, S_i z) \le H(S_i z_n, S_i z) \le \|z_n - z\| \\ \|w_{i,n} - z\| = \mathrm{dist}(w_{i,n}, R_i z) \le H(R_i x_n, R_i z) \le \|x_n - z\|. \end{cases} \tag{3.1}$$

Now, from (1.1) and (3.1) we have

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_{0,n} y_n + \sum_{i=1}^{\infty} \alpha_{i,n} u_{i,n} - z\| \\ &\le \alpha_{0,n} \|y_n - z\| + \sum_{i=1}^{\infty} \alpha_{i,n} \|u_{i,n} - z\| \\ &\le \alpha_{0,n} \|y_n - z\| + \sum_{i=1}^{\infty} \alpha_{i,n} \|y_n - z\| \\ &= \|y_n - z\| \quad \forall\, n \ge 1 \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \|y_n - z\| &\le \beta_{0,n} \|z_n - z\| + \sum_{i=1}^{\infty} \beta_{i,n} \|v_{i,n} - z\| \\ &\le \beta_{0,n} \|z_n - z\| + \sum_{i=1}^{\infty} \beta_{i,n} \|z_n - z\| \\ &= \|z_n - z\| \quad \forall\, n \ge 1, \end{aligned} \tag{3.3}$$

and similarly,

$$\begin{aligned} \|z_n - z\| &\le \gamma_{0,n} \|x_n - z\| + \sum_{i=1}^{\infty} \gamma_{i,n} \|w_{i,n} - z\| \\ &\le \gamma_{0,n} \|x_n - z\| + \sum_{i=1}^{\infty} \gamma_{i,n} \|x_n - z\| \\ &= \|x_n - z\| \quad \forall\, n \ge 1. \end{aligned} \tag{3.4}$$

Now, from (3.2)–(3.4) we have

$$\|x_{n+1} - z\| \le \|x_n - z\| \quad \forall\, n \ge 1. \tag{3.5}$$

This shows that $\{\|x_n - z\|\}$ is decreasing and bounded below. Hence $\lim_{n\to\infty} \|x_n - z\|$ exists for each $z \in \mathcal{F}$. So $\{\|x_n - z\|\}$, $\{\|y_n - z\|\}$, $\{\|z_n - z\|\}$ and $\{\|u_{i,n} - z\|\}$, $\{\|v_{i,n} - z\|\}$, $\{\|w_{i,n} - z\|\}$ are bounded. Now, we prove that

$$\lim_{n\to\infty} \mathrm{dist}(x_n, T_j y_n) = \lim_{n\to\infty} \mathrm{dist}(x_n, S_j z_n) = \lim_{n\to\infty} \mathrm{dist}(x_n, R_j x_n) = 0 \text{ for } j \ge 1.$$

Since $\{\|y_n - z\|\}$ and $\{\|u_{i,n} - z\|\}$ both are bounded, therefore from Lemma 4 and (3.1), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|\alpha_{0,n}(y_n-z)+\sum_{i=1}^{\infty}\alpha_{i,n}(u_{i,n}-z)\|^2\\ &\leq \alpha_{0,n}\|y_n-z\|^2+\sum_{i=1}^{\infty}\alpha_{i,n}\|u_{i,n}-z\|^2-\alpha_{0,n}\alpha_{j,n}g(\|y_n-u_{j,n}\|)\\ &\leq \alpha_{0,n}\|y_n-z\|^2+\sum_{i=1}^{\infty}\alpha_{i,n}\|y_n-z\|^2-\alpha_{0,n}\alpha_{j,n}g(\|y_n-u_{j,n}\|)\\ &= \|y_n-z\|^2-\alpha_{0,n}\alpha_{j,n}g(\|y_n-u_{j,n}\|) \quad \forall\, n\geq 1 \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\|y_n-z\|^2 &= \|\beta_{0,n}(z_n-z)+\sum_{i=1}^{\infty}\beta_{i,n}(v_{i,n}-z)\|^2\\ &\leq \beta_{0,n}\|z_n-z\|^2+\sum_{i=1}^{\infty}\beta_{i,n}\|v_{i,n}-z\|^2-\beta_{0,n}\beta_{j,n}g(\|z_n-v_{j,n}\|)\\ &\leq \|z_n-z\|^2-\beta_{0,n}\beta_{j,n}g(\|z_n-v_{j,n}\|) \quad \forall\, n\geq 1, \end{aligned} \tag{3.7}$$

and similarly,

$$\|z_n-z\|^2 \leq \|x_n-z\|^2-\gamma_{0,n}\gamma_{j,n}g(\|x_n-w_{j,n}\|) \quad \forall\, n\geq 1. \tag{3.8}$$

Therefore, from (3.6)–(3.8) we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\leq \|z_n-z\|^2-\beta_{0,n}\beta_{j,n}g(\|z_n-v_{j,n}\|)-\alpha_{0,n}\alpha_{j,n}g(\|y_n-u_{j,n}\|)\\ &\leq \|x_n-z\|^2-\gamma_{0,n}\gamma_{j,n}g(\|x_n-w_{j,n}\|)-\beta_{0,n}\beta_{j,n}g(\|z_n-v_{j,n}\|)\\ &\quad -\alpha_{0,n}\alpha_{j,n}g(\|y_n-u_{j,n}\|), \end{aligned} \tag{3.9}$$

and hence

$$\begin{aligned}&\alpha_{0,n}\alpha_{j,n}g(\|y_n-u_{j,n}\|)+\beta_{0,n}\beta_{j,n}g(\|z_n-v_{j,n}\|)+\gamma_{0,n}\gamma_{j,n}g(\|x_n-w_{j,n}\|)\\ &\quad\leq \|x_n-z\|^2-\|x_{n+1}-z\|^2\end{aligned}$$

which tends to zero as $n\to\infty$. Therefore by condition (2), we have

$$\begin{aligned}&\lim_{n\to\infty} g(\|y_n-u_{j,n}\|)=0,\quad \lim_{n\to\infty} g(\|z_n-v_{j,n}\|)=0\\ &\quad\text{and } \lim_{n\to\infty} g(\|x_n-w_{j,n}\|)=0.\end{aligned}$$

Since g is continuous and strictly increasing with $g(0)=0$, it implies that

$$\begin{aligned}&\lim_{n\to\infty} \|y_n-u_{j,n}\|=0,\quad \lim_{n\to\infty} \|z_n-v_{j,n}\|=0\\ &\quad\text{and } \lim_{n\to\infty} \|x_n-w_{j,n}\|=0 \quad \forall\, j\geq 1.\end{aligned}$$

Thus, we have

$$\lim_{n\to\infty} \text{dist}(y_n, T_j y_n) \leq \lim_{n\to\infty} \|y_n - u_{j,n}\| = 0 \;\; \forall\; j \geq 1, \tag{3.10}$$

$$\lim_{n\to\infty} \text{dist}(z_n, S_j z_n) \leq \lim_{n\to\infty} \|z_n - v_{j,n}\| = 0 \;\; \forall\; j \geq 1, \tag{3.11}$$

$$\lim_{n\to\infty} \text{dist}(x_n, R_j x_n) \leq \lim_{n\to\infty} \|x_n - w_{j,n}\| = 0 \;\; \forall\; j \geq 1. \tag{3.12}$$

Now, we prove that $\{x_n\}$ converges weakly to some fixed point $z^* \in \mathcal{F}$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup z^* \in K$. By Lemma 6, $(I - T_j)$ is demi-closed at zero. Hence from (3.10), we have $z^* \in F(T_j)$. Similarly, we can say that $z^* \in F(S_j)$ and $z^* \in F(R_j)$ (since by Lemma 6, $(I - S_j)$ as well as $(I - R_j)$ both are demi-closed at zero). Hence by the arbitrariness of $j \geq 1$, we have $z^* \in \mathcal{F}$.

Now, we prove that $\{x_n\}$ has a unique weak subsequential limit in $\mathcal{F}$. To show this, let z_1, z_2 and z_3 be weak limits of the subsequences $\{x_{n_m}\}$, $\{x_{n_k}\}$ and $\{x_{n_l}\}$ of $\{x_n\}$ respectively, with $z_1 \neq z_2 \neq z_3$ and $z_1, z_2, z_3 \in \mathcal{F}$ (same as above). Then $\lim_{n\to\infty} \|x_n - z_1\|$, $\lim_{n\to\infty} \|x_n - z_2\|$ and $\lim_{n\to\infty} \|x_n - z_3\|$ exist. Hence by Opial's condition

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{m\to\infty} \|x_{n_m} - z_1\| < \lim_{m\to\infty} \|x_{n_m} - z_2\| \\
&= \lim_{n\to\infty} \|x_n - z_2\| = \lim_{k\to\infty} \|x_{n_k} - z_2\| \\
&\quad < \lim_{k\to\infty} \|x_{n_k} - z_3\| = \lim_{n\to\infty} \|x_n - z_3\| \\
&= \lim_{l\to\infty} \|x_{n_l} - z_3\| < \lim_{l\to\infty} \|x_{n_l} - z_1\| \\
&= \lim_{n\to\infty} \|x_n - z_1\|,
\end{aligned}$$

which is a contradiction. Hence $z_1 = z_2 = z_3$. Therefore $x_n \rightharpoonup z^* \in \mathcal{F}$. □

COROLLARY 1

Let E be a uniformly convex Banach space satisfying Opial's condition and K a nonempty closed and convex subset of E. Let $T, S, R : K \to CB(K)$ be three multi-valued mappings satisfying condition (C). If $\mathcal{F}' = \emptyset$ and $\{x_n\}$ is a sequence as defined by (1.2), then $\{x_n\}$ converges weakly to a common fixed point of T, S and R.

Now, we obtain an immediate consequence of Theorem 1 for single-valued mapping.

Theorem 2. *Let E be a uniformly convex Banach space satisfying Opial's condition and K a nonempty closed and convex subset of E. Let $\mathcal{T}_i, \mathcal{S}_i, \mathcal{R}_i : K \to K$, $(i = 1, 2, \ldots)$ be three infinite families of single-valued mappings satisfying condition (C). For a given $\{x_n\} \in K$, let $\{x_n\}$ be a sequence as defined by*

$$\begin{cases} x_{n+1} = \alpha_{0,n} y_n + \sum_{i=1}^{\infty} \alpha_{i,n} \mathcal{T}_i y_n, \\ y_n = \beta_{0,n} z_n + \sum_{i=1}^{\infty} \beta_{i,n} \mathcal{S}_i z_n, \\ z_n = \gamma_{0,n} x_n + \sum_{i=1}^{\infty} \gamma_{i,n} \mathcal{R}_i x_n, \quad n \geq 1, \end{cases} \tag{3.13}$$

where $\{\alpha_{i,n}\}$, $\{\beta_{i,n}\}$ *and* $\{\gamma_{i,n}\}$ *are sequences in* $[0, 1]$ *as given in Theorem* 1. *If* $\mathbf{F} \neq \emptyset$, *then* $\{x_n\}$ *converges weakly to some fixed point* $z^* \in \mathbf{F}$.

4. Strong convergence theorems

The following result gives a necessary and sufficient condition for the strong convergence of the sequence (1.1) to a common fixed point of three infinite families of multi-valued mappings in a Banach space.

Theorem 3. *Let* E *be a uniformly convex Banach space and* K *a nonempty closed and convex subset of* E. *Let* $T_i, S_i, R_i : K \to CB(K)$, $(i = 1, 2, \ldots)$ *be three infinite families of multi-valued mappings satisfying condition* (*C*). *For a given* $x_1 \in K$, *let* $\{x_n\}$ *be a sequence as defined by* (1.1).

If conditions (1), (2) *and* (3) *of Theorem* 1 *are satisfied*, *then* $\{x_n\}$ *converges strongly to some fixed point* $z^* \in \mathcal{F}$ *if and only if the following condition is satisfied*:

$$\lim_{n\to\infty} \inf \operatorname{dist}(x_n, \mathcal{F}) = 0. \tag{4.1}$$

Proof. The necessity is obvious. Now, we prove the sufficiency. In fact, from (3.10)–(3.12) for each $i \geq 1$, we have

$$\lim_{n\to\infty} \operatorname{dist}(y_n, T_i y_n) = \lim_{n\to\infty} \operatorname{dist}(z_n, S_i z_n) = \lim_{n\to\infty} \operatorname{dist}(x_n, R_i x_n) = 0,$$

and for each $z \in \mathcal{F}$, $\lim_{n\to\infty} \|x_n - z\|$ exists. Hence by condition (4.1), we have

$$\lim_{n\to\infty} \operatorname{dist}(x_n, \mathcal{F}) = 0. \tag{4.2}$$

Therefore, we can choose a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and a subsequence $\{z_k\} \subset \mathcal{F}$ such that for all positive integers $k \geq 1$,

$$\|x_{n_k} - z_k\| < \frac{1}{2^k}.$$

Since the sequence $\{\|x_n - z\|\}$, $z \in \mathcal{F}$ is decreasing, we obtain

$$\|x_{n_{k+1}} - z_k\| \leq \|x_{n_k} - z_k\| < \frac{1}{2^k}.$$

Hence,

$$\|z_{k+1} - z_k\| \leq \|x_{n_{k+1}} - z_{k+1}\| + \|x_{n_{k+1}} - z_k\| < \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}.$$

This implies that $\{z_k\}$ is a Cauchy sequence in K. Without loss of generality, we can assume that $z_k \to z^*$. For each $i \geq 1$,

$$\begin{aligned}\operatorname{dist}(z^*, T_i z^*) &= \lim_{k\to\infty} \operatorname{dist}(z_k, T_i z^*) \leq \lim_{i\to\infty} H(T_i z_k, T_i z^*)\\ &\leq \lim_{k\to\infty} \|z_k - z^*\| = 0,\end{aligned}$$

which implies that $z^* \in T_i z^*$ for all $i \geq 1$. Similarly,

$$\begin{aligned}
\mathrm{dist}(z^*, S_i z^*) &= \lim_{k\to\infty} \mathrm{dist}(z_k, S_i z^*) \leq \lim_{i\to\infty} H(R_i z_k, R_i z^*) \\
&\leq \lim_{k\to\infty} \|z_k - z^*\| = 0, \\
\mathrm{dist}(z^*, R_i z^*) &= \lim_{k\to\infty} \mathrm{dist}(z_k, R_i z^*) \leq \lim_{i\to\infty} H(S_i z_k, S_i z^*) \\
&\leq \lim_{k\to\infty} \|z_k - z^*\| = 0,
\end{aligned}$$

which implies that $z^* \in S_i z^*$ and $z^* \in R_i z^*$ respectively, for all $i \geq 1$. Consequently, $z^* \in \mathcal{F}$ and $x_n \to z^*$. This completes the proof of the theorem. □

COROLLARY 2

Let E be a uniformly convex Banach space and K a nonempty closed and convex subset of E. Let $T, S, R : K \to CB(K)$ be three multi-valued mappings satisfying condition (C) and $\{x_n\}$ a sequence as defined by (1.2). If $\mathcal{F}' \neq \emptyset$ and $Tz = Sz = Rz = \{z\}$, for any $z \in \mathcal{F}'$, then $\{x_n\}$ converges strongly to a common fixed point of T, S and R if and only if $\lim_{n\to\infty} \inf \mathrm{dist}(x_n, \mathcal{F}') = 0$.

Theorem 4. *Let E be a uniformly convex Banach space and K a nonempty closed and convex subset of E. Let $T_i, S_i, R_i : K \to CB(K)$, $(i = 1, 2, \ldots)$ be three infinite families of multi-valued mappings satisfying condition (C) and condition (A'). For a given $x_1 \in K$, let $\{x_n\}$ be a sequence as defined by (1.1). If conditions (1), (2) and (3) of Theorem 1 are satisfied, then $\{x_n\}$ converges strongly to some fixed point $z^* \in \mathcal{F}$.*

Proof. From Theorem 1, for each $i \geq 1$, $\lim_{n\to\infty} \mathrm{dist}(y_n, T_i y_n) = 0$, $\lim_{n\to\infty} \mathrm{dist}(z_n, S_i z_n) = 0$ and $\lim_{n\to\infty} \mathrm{dist}(x_n, R_i x_n) = 0$. Since T_i, S_i and R_i $(i = 1, 2, \ldots)$ satisfy condition (A'), we have $\lim_{n\to\infty} f(\mathrm{dist}(x_n, \mathcal{F})) = 0$ which implies that $\lim_{n\to\infty} \mathrm{dist}(x_n, \mathcal{F}) = 0$. Hence the conclusion of Theorem 4 can be obtained from Theorem 1 immediately. □

COROLLARY 3

Let E be a uniformly convex Banach space and K a nonempty closed and convex subset of E. Let $T, S, R : K \to CB(K)$ be three multi-valued mappings satisfying condition (C) and condition (A'). Let $\{x_n\}$ be a sequence as defined by (1.2). If $\mathcal{F}' \neq \emptyset$ and $Tz = Sz = Rz = \{z\}$ for any $z \in \mathcal{F}'$, then $\{x_n\}$ converges strongly to a common fixed point of T, S and R.

Now, we intend to remove the condition $T_i(z) = S_i(z) = R_i(z) = \{z\}$ for each $z \in \mathcal{F}$ and for each $i \geq 1$. Let E be a uniformly convex Banach space and K a nonempty closed and convex subset of E. Let $T_i, S_i, R_i : K \to CB(K)$, $(i = 1, 2, \ldots)$ be three infinite families of multi-valued mappings with convex values. Then for each $i \geq 1$ and $y, z', x \in K$, $T_i y$, $S_i z'$ and $R_i x$ are non-empty closed and convex subsets in K respectively. Hence by Remark 2, they are proximinal. Now we define multi-valued mappings $P_{T_i}, P_{S_i}, P_{R_i} : K \to CB(K)$ by

$$\begin{cases} P_{T_i}(y) = \{t_1 \in T_i(y) : \|y - t_1\| = \operatorname{dist}(y, T_i(y))\}, \\ P_{S_i}(z') = \{t_2 \in S_i(z') : \|z' - t_2\| = \operatorname{dist}(z', S_i(z'))\}, \\ P_{R_i}(x) = \{t_3 \in R_i(x) : \|x - t_3\| = \operatorname{dist}(x, R_i(x))\}. \end{cases} \tag{4.3}$$

For given $x_1 \in K$, define a sequence $\{x_n\}$ by

$$\begin{cases} x_{n+1} = \alpha_{0,n} y_n + \sum_{i=1}^{\infty} \alpha_{i,n} u_{i,n}, \quad u_{i,n} \in P_{T_i}(y_n), \\ y_n = \beta_{0,n} z_n + \sum_{i=1}^{\infty} \beta_{i,n} v_{i,n}, \quad v_{i,n} \in P_{R_i}(z_n), \\ z_n = \gamma_{0,n} x_n + \sum_{i=1}^{\infty} \gamma_{i,n} w_{i,n}, \quad w_{i,n} \in P_{S_i}(x_n), \ n \geq 1. \end{cases} \tag{4.4}$$

$\{\alpha_{i,n}\}$, $\{\beta_{i,n}\}$ and $\{\gamma_{i,n}\}$ are sequences in [0, 1].

Theorem 5. *Let E, K, T_i, S_i, R_i and, $P_{T_i}, P_{S_i}, P_{R_i}$ and $\{x_n\}$ be the same as mentioned above. Let T_i, S_i, R_i satisfy condition (A') and $P_{T_i}, P_{S_i}, P_{R_i}$ satisfy condition (C). If conditions* (1), (2) *and* (3) *of Theorem* 1 *are satisfied, then $\{x_n\}$ converges strongly to some fixed point $z^* \in \mathcal{F}$.*

Proof. Let $z \in \mathcal{F}$. Then from (4.3), we have

$$\begin{cases} P_{T_i}(z) = \{t_1 \in T_i(z) : \|z - t_1\| = \operatorname{dist}(z, T_i(z)) = 0\} = \{z\} \\ P_{S_i}(z) = \{t_2 \in S_i(z) : \|z - t_2\| = \operatorname{dist}(z, S_i(z)) = 0\} = \{z\} \\ P_{R_i}(z) = \{t_3 \in R_i(x) : \|z - t_3\| = \operatorname{dist}(z, R_i(z)) = 0\} = \{z\}. \end{cases} \tag{4.5}$$

Moreover, on the same lines of proof of Theorem 1, we can also prove that $\lim_{n\to\infty} \|x_n - z\|$ exists for each $z \in \mathcal{F}$, and $\lim_{n\to\infty} \operatorname{dist}(x_n, \mathcal{F}) = 0$. Therefore, we can choose a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and a subsequence $\{z_k\} \subset \mathcal{F}$ such that for all positive integers $k \geq 1$,

$$\|x_{n_k} - z_k\| < \frac{1}{2^k}.$$

As in the proof of Theorem 1, $\{z_k\}$ is a Cauchy sequence in K and hence $z_k \to q$. By the definition of P_{T_i}, we have $P_{T_i}(q) \subset T_i(q), \ i \geq 1$. Hence from (4.5) we have

$$\operatorname{dist}(z_k, T_i(q)) \leq \operatorname{dist}(z_k, P_{T_i(q)}) \leq H(P_{T_i}(z_k), P_{T_i}(q)) \leq \|q - z_k\|.$$

Since $z_k \to q$ (as $k \to \infty$), it follows that $\operatorname{dist}(q, T_i(q)) = 0$ for $i \geq 1$. Hence $q \in \mathcal{F}$ and $\{x_{n_k}\}$ converges strongly to q. This completes the proof. □

COROLLARY 4

Let E be a uniformly convex Banach space and K a nonempty closed and convex subset of E. Let $T, S, R : K \to CB(K)$ be three multi-valued mappings satisfying condition (A') and $P_T, P_S, P_R : K \to CB(K)$ be three multi-valued mappings satisfying condition (C). Let $\{x_n\}$ be a sequence as defined by

$$\begin{cases} x_{n+1} = \alpha_{0,n} y_n + \alpha_{1,n} u_{1,n}, & u_{1,n} \in P_{T_1}(y_n) \\ y_n = \beta_{0,n} z_n + \beta_{1,n} v_{1,n}, & v_{1,n} \in P_{R_1}(z_n) \\ z_n = \gamma_{0,n} x_n + \gamma_{1,n} w_{1,n}, & w_{1,n} \in P_{S_1}(x_n), \ n \geq 1. \end{cases}$$

If $\mathcal{F}' \neq \emptyset$ and $Tz = Sz = Rz = \{z\}$ for any $z \in \mathcal{F}'$, then $\{x_n\}$ converges strongly to a common fixed point of T, S and R.

5. Applications

Recently, it has been found that the convex feasibility problem can also be used in various disciplines such as image restoration, computer tomograph and radiation therapy treatment planning. Censor and Elfving [7] were the first mathematician who introduced the convex feasibility problem for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [6].

Here we utilize Theorem 2 to study the convex feasibility problem for three infinite families of single-valued generalized nonexpansive mappings. Let K be a nonempty closed and convex subset of a Banach space E and $\{K_i\}$ a countable family of subsets of K. Then the convex feasibility problem for the family of subsets $\{K_i\}$ is to find a point $x^* \in \bigcap_{i=1}^{\infty} K_i$. The result is as follows:

Theorem 6. *Let E be a uniformly convex Banach space with Opial's condition and K a nonempty closed and convex subset of E. Let $\mathcal{T}_i, \mathcal{S}_i, \mathcal{R}_i : K \to K$, $(i = 1, 2, \ldots)$ be three infinite families of single-valued mappings satisfying condition (C). Let $\{K_i = F(\mathcal{T}_i) \cap F(\mathcal{S}_i) \cap F(\mathcal{R}_i),\ i = 1, 2, \ldots\}$. For a given $x_1 \in K$, let $\{x_n\}$ be a sequence as defined by* (3.13).

If $\mathbf{F} \neq \emptyset$, then there exists a point $x^* \in \mathbf{F}$ which is a solution of the convex feasibility problem for the family of subsets $\{K_i\}$ and the sequence $\{x_n\}$ defined by (3.13) *converges weakly to x^*.*

Proof. The proof of this theorem runs on the same lines as that of proof of Theorem 2 which is actually a consequence of Theorem 1 for single-valued mappings. □

6. Conclusion

Overall we establish here some weak as well as strong convergence theorems for a new iterative algorithm of three infinite families of multi-valued generalized nonexpansive mappings in a uniformly convex Banach space under some suitable conditions. Our results generalize and improve the corresponding results of [9,22] and several others of the existing literature.

Our work actually is a generalization of Rashwan and Altwqi [22] from three finite families of multi-valued nonexpansive mappings to three infinite families of multi-valued generalized nonexpansive mappings in a uniformly convex Banach space. As an application, we utilize Theorem 2 to study the convex feasibility problem for three infinite families of single-valued mappings satisfying condition (C).

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