Proc. Indian Acad. Sci. (Math. Sci.) (2018) 128:45
https://doi.org/10.1007/s12044-018-0427-y

Special properties of Hurwitz series rings

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MS received 16 September 2016; accepted 30 December 2017;
published online 28 July 2018

Abstract. In this paper, we study some properties of the Hurwitz series ring HR (resp. Hurwitz polynomial ring hR), such as the flatness or the faithful flatness of $HR/(f)$ (resp. $hR/(f)$), the strongly Hopfian property and the radical property of HR (resp. hR). We give some sufficient and necessary conditions for $HR/(f)$ (resp. $hR/(f)$) to be flat or faithful flat. We also prove that the strongly Hopfian property transfer between R and HR (resp. hR), and some radicals of HR can be determined in terms of those of R, in case R satisfies some additional conditions.

Keywords. Hurwitz series; strongly Hopfian ring; radical property.

2000 Mathemetics Subject Classification. 13F25.

1. Introduction

Throughout this paper, all rings R are associative with identity. For a ring R, let $\mathrm{Id}(R)$, $U(R)$, $\mathrm{nil}(R)$, $\mathrm{nil}_*(R)$ and $\mathrm{nil}^*(R)$ denote the set of idempotents of R, the set of units of R, the set of nilpotent elements of R, the lower nilradical (i.e. the intersection of all the prime ideals in R), and the upper nilradical (i.e. the sum of all the nil ideals), respectively. The symbol $\mathbb{Q}$ stands for the field of rational numbers. Recall that a ring R is right (resp. left) perfect if R satisfies the descending chain condition on the principal left (resp. right) ideals, and the ring R is coherent if every finitely generated ideal of R is finitely presented. Let R be a ring and $\mathbb{N}$ the set of all nonnegative integers. The Hurwitz series ring over R is denoted by HR and is defined as follows: the elements in HR are mappings $f : \mathbb{N} \longrightarrow R$, the operation of addition in HR is componentwise and the operation of multiplication for each $f, g \in HR$ is defined by

$$(fg)(n) = \sum_{k=0}^{n} C_n^k f(k)g(n-k) \quad \text{for all } n \in \mathbb{N},$$

where C_n^k is the binomial coefficient. For any $f \in HR$, let $\mathrm{supp}(f) = \{i \in \mathbb{N} \mid 0 \neq f(i) \in R\}$ denote the support of f, $\pi(f)$ the minimal number in $\mathrm{supp}(f)$ and $\Delta(f)$ the maximal number in $\mathrm{supp}(f)$ if $\Delta(f)$ exists. Clearly, the set $hR = \{f \in HR \mid \mathrm{supp}(f)$ is a finite subset of $\mathbb{N}\}$ is a subring of HR under the operations of addition and multiplication defined as above, and we call this subring the Hurwitz polynomial ring. If I is an ideal of R, then $HI = \{f \in HR \mid f(n) \in I$ for any $n \in \mathrm{supp}(f)\}$ (resp. $hI = \{f \in hR \mid f(n) \in I$ for any $n \in \mathrm{supp}(f)\}$) is an ideal of HR (resp. hR).

For any $n \in \mathbb{N}$, $r \in R$, define the mapping $e_r^n : \mathbb{N} \longrightarrow R$ via $e_r^n(n) = r$ and $e_r^n(k) = 0$ for each $k \in \mathbb{N}$ and $k \neq n$. So there are natural ring homomorphisms $\varepsilon_{HR} : HR \longrightarrow R$ and $\lambda_R : R \longrightarrow HR$ defined as follows: for $f \in HR$ and $x \in R$, $\varepsilon_{HR}(f) = f(0)$ and $\lambda_R(x) = e_x^0$. Clearly, λ_R is injective. So we can regard R as a subring of HR and for any $f \in HR$, $r \in R$, $fr = fe_r^0$. If f is an element in hR with $\Delta(f) = n$, then we can write f as $f = \sum_{i=0}^{n} f(i)e_1^i$.

The ring of Hurwitz series has been studied by many authors [1–4,8,9], in particular, by Keigher [8] and its importance was proved. In this paper we further extend the study of this type of ring. In section 2, we use the homological method which is different from those used by many authors to study the Hurwitz series ring (resp. Hurwitz polynomial ring) and obtain some information for the R-module $HR/(f)$ (resp. $hR/(f)$) to be flat or faithfully flat. In section 3, we study the Hurwitz series ring HR (resp. Hurwitz polynomial ring hR) over a strongly Hopfian ring and give some necessary and sufficient conditions for HR (resp. hR) to be strongly Hopfian. In section 4, the connections between the radicals of R and the corresponding radicals of HR are investigated. It is proved that some radicals of HR can be determined in terms of those of R in case R satisfies some additional conditions.

2. The flatness of Hurwitz series

DEFINITION 2.1

A ring R is a strongly torsion free ring if for every ideal P of R, every element $a \in R$ and every positive integer $n \in \mathbb{N}$, $na \in P$ implies $a \in P$.

Clearly, if the field of rational numbers $\mathbb{Q}$ is contained in the ring R, then the ring R is a strongly torsion free ring.

Lemma 2.2. *Let I be an ideal of R. If R is a strongly torsion free ring, then for any $a \in U(R)$, $b \in R$ and any positive integer $n \in \mathbb{N}$, $nab \in I$ implies $b \in I$.*

Proof. Since R is a strongly torsion free ring, $nab \in I$ implies $ab \in I$ and so $b = a^{-1}(ab) \in I$.

Lemma 2.3. *Let R be a left perfect and right coherent ring. Then HR is flat as a right R-module.*

Proof. As a right R-module, $HR \cong \Pi R$ is projective and so is flat.

Lemma 2.4 [10]. *Let F be flat and $0 \longrightarrow K \longrightarrow F \longrightarrow B \longrightarrow 0$ be an exact sequence of right R-modules. Then the following conditions are equivalent*:

(1) *B is flat*;
(2) $K \cap FI = KI$ *for every left ideal* I;
(3) $K \cap FI = KI$ *for every finitely generated left ideal* I.

PROPOSITION 2.5

Let R be a perfect coherent strongly torsion free commutative ring and $f \in HR$. If there exists some positive integer $n \in \mathbb{N}$ such that $f(k) \in \mathrm{Id}(R)$ for all $0 \leq k \leq n-1$ and $f(n) \in U(R)$, then $HR/(f)$ is a flat R-module.

Proof. Consider the exact sequence $0 \longrightarrow (f) \longrightarrow HR \longrightarrow HR/(f) \longrightarrow 0$. Then by Lemma 2.4, it suffices to show that $(f) \cap HR \cdot I = (f) \cdot I$ for every finitely generated ideal I of R. Clearly, $(f) \cdot I \subseteq (f) \cap HR \cdot I$. So we only need to show the reverse containment. For any $\sum h_i c_i \in HR \cdot I$, where $h_i \in HR$, and $c_i \in I$, if $\sum h_i c_i \in (f)$, then there exists some $g \in HR$ such that $\sum h_i c_i = fg$. Now we show that $\sum h_i c_i = fg \in (f) \cdot I$. Clearly, $fg = \sum h_i c_i \in HI$. So for each $s \in \mathbb{N}$, we have

$$(fg)(s) = \sum_{k=0}^{s} C_s^k f(k) g(s-k) = \sum h_i(s) c_i \in I.$$

For $s = 0$, we have $(fg)(0) = f(0)g(0) \in I$. For $s = 1$, we have

$$(fg)(1) = f(0)g(1) + f(1)g(0) \in I.$$

Since $f(0) \in \mathrm{Id}(R)$ and $f(0)g(0) \in I$, we obtain

$$f(0)g(1) = f(0)(fg)(1) - f(0)g(0)f(1) \in I,$$

and so $f(1)g(0) \in I$. Hence $f(i)g(j) \in I$ for each $i + j = 1$, and so

$$f(i)g(j) \in I \text{ for each } 0 \leq i + j \leq 1.$$

For $s = 2$, we have

$$(fg)(2) = f(0)g(2) + C_2^1 f(1)g(1) + f(2)g(0) \in I.$$

Since $f(0) \in \mathrm{Id}(R)$ $f(0)g(0) \in I$ and $f(0)g(1) \in I$, we have

$$f(0)g(2) = f(0)(fg)(2) - C_2^1 f(0)g(1)f(1) - f(0)g(0)f(2) \in I,$$

and so

$$\Delta_0 = C_2^1 f(1)g(1) + f(2)g(0) \in I.$$

Since $f(1) \in \mathrm{Id}(R)$ and $f(1)g(0) \in I$, we have

$$C_2^1 f(1)g(1) = f(1)\Delta_0 - f(1)g(0)f(2) \in I,$$

and so $f(1)g(1) \in I$ since R is a strongly torsion free ring. Then $f(2)g(0) \in I$. Hence $f(i)g(j) \in I$ for all $i + j = 2$, and so

$$f(i)g(j) \in I \quad \text{for all } 0 \leq i + j \leq 2.$$

Similarly, for each $0 \leq s \leq n$, we can show that $f(i)g(j) \in I$ for each $i + j = s$, and so

$$f(i)g(j) \in I \quad \text{for each } 0 \leq i + j \leq n,$$

and so $f(n)g(0) \in I$. Then by Lemma 2.2, we obtain $g(0) \in I$.

For $s = n + 1$, we have

$$(fg)(n+1) = \sum_{k=0}^{n+1} C_{n+1}^{k} f(k)g(n+1-k) \in I.$$

Since $g(0) \in I$, we obtain

$$\Delta_1 = f(0)g(n+1) + C_{n+1}^{1} f(1)g(n) + \cdots + C_{n+1}^{n} f(n)g(1) \in I. \tag{1}$$

Multiplying eq. (1) by $f(0)$, we obtain $f(0)g(n+1) \in I$, and so

$$\Delta_2 = C_{n+1}^{1} f(1)g(n) + C_{n+1}^{2} f(2)g(n-1) + \cdots + C_{n+1}^{n} f(n)g(1) \in I. \tag{2}$$

Multiplying eq. (2) by $f(1)$, we obtain $C_{n+1}^{1} f(1)g(n) \in I$ and so $f(1)g(n) \in I$ by Lemma 2.2. Similarly, we can show that $f(i)g(j) \in I$ for each $i + j = n + 1$, and so

$$f(i)g(j) \in I \quad \text{for each } 0 \leq i + j \leq n + 1.$$

Hence $f(n)g(1) \in I$ and so $g(1) \in I$. Continuing this procedure yields that $g(s) \in I$ for each $s \in \mathbb{N}$. Assume that the ideal I is generated by $b_1, b_2, \ldots, b_k$ and assume that for each $s \in \mathbb{N}$,

$$g(s) = r_1^s b_1 + r_2^s b_2 + \cdots + r_k^s b_k.$$

For each $1 \leq i \leq k$, define $d_i \in HR$ via

$$d_i(s) = r_i^s \quad \text{for each } s \in \mathbb{N}.$$

Then it is easy to see that

$$g = d_1 b_1 + d_2 b_2 + \cdots + d_k b_k.$$

So $\sum h_i c_i = fg = fd_1 b_1 + fd_2 b_2 + \cdots + fd_k b_k \in (f) \cdot I$. Therefore $HR/(f)$ is a flat R-module.

PROPOSITION 2.6

Let R be a perfect coherent strongly torsion free commutative ring and $f \in HR$. If $f(0) \in U(R)$, then $HR/(f)$ is a flat R-module.

Proof. If $f(0) \in U(R)$, then by [8], $f \in U(HR)$, and so $HR/(f) = 0$ is a flat R-module.

Lemma 2.7 [5]. *Let R be a subring of A, and A_R be a flat right R-module. Then A_R is faithfully flat if and only if for any proper ideal I of R, $A \cdot I \neq A$.*

PROPOSITION 2.8

Let R be a perfect coherent strongly torsion free commutative ring and $f \in HR$. If there exists some positive integer $n \in \mathbb{N}$ such that $f(0) = 0$, $f(k) \in \mathrm{Id}(R)$ for all $0 < k \leq n-1$ and $f(n) \in U(R)$, then $HR/(f)$ is a faithfully flat R-module.

Proof. By Proposition 2.5, $HR/(f)$ is flat. So by Lemma 2.7, it suffices to show that for any proper ideal I of R, $HR/(f) \cdot I \neq HR/(f)$. Now we show that $1 + (f)$ is not in $HR/(f) \cdot I$. Assume on the contrary that $1 + (f) \in HR/(f) \cdot I$. Then there exist some $h_i \in HR$ and $c_i \in I$ such that

$$1 + (f) = \sum (h_i + (f))c_i = \sum h_i c_i + (f).$$

So there exists $g \in HR$ such that $1 - fg = \sum h_i c_i \in HR \cdot I \subseteq HI$ and so $(1 - fg)(0) = 1 - f(0)g(0) = 1 \in I$, a contradiction. Hence $HR/(f) \cdot I \neq HR/(f)$. Therefore $HR/(f)$ is a faithfully flat R-module.

COROLLARY 2.9

Let R be a perfect coherent strongly torsion free commutative ring and $f \in HR$. If $\pi(f) \neq 0$, and $f(\pi(f)) \in U(R)$, then $HR/(f)$ is a faithfully flat R-module.

Proof. It is obvious from Proposition 2.8.

In the following, we investigate the flatness or faithful flatness of $hR/(f)$, where (f) is the ideal of hR generated by f.

PROPOSITION 2.10

Let R be a strongly torsion free commutative ring and $f \in hR$. If $f(0) \in U(R)$, then $hR/(f)$ is flat as an R-module.

Proof. Consider the following exact sequence:

$$0 \longrightarrow (f) \longrightarrow hR \longrightarrow hR/(f) \longrightarrow 0.$$

Since $hR \cong \oplus R$ is flat, by Lemma 2.4, it suffices to show that $(f) \cap hR \cdot I = (f) \cdot I$ for every ideal I of R. Clearly, $(f) \cdot I \subseteq (f) \cap hR \cdot I$. Now we see that $(f) \cap hR \cdot I \subseteq (f) \cdot I$. Let $\sum h_i c_i \in hR \cdot I$ where $h_i \in hR$ and $c_i \in I$. If $\sum h_i c_i \in (f)$, then there exists some $g \in hR$ with $\Delta(g) = m$ such that $\sum h_i c_i = fg$. Then for any $s \in \mathbb{N}$, we have

$$(fg)(s) = \sum_{v=0}^{s} C_s^v f(v) g(s-v) = \sum h_i(s) c_i \in I.$$

Now we see that $g(i) \in I$ for each $0 \leq i \leq m$. For $s = 0$, we have

$$(fg)(0) = f(0)g(0) \in I,$$

and so by Lemma 2.2, we obtain $g(0) \in I$.

For $s = 1$, we have

$$(fg)(1) = f(1)g(0) + f(0)g(1) \in I.$$

Since $g(0) \in I$, we obtain $f(1)g(0) \in I$, and so $f(0)g(1) \in I$. Then by Lemma 2.2, we obtain $g(1) \in I$. Applying the preceding method repeatedly, we obtain that $g(i) \in I$ for each $0 \leq i \leq m$. Then g can be written as

$$g = g(0)e_1^0 + g(1)e_1^1 + \cdots + g(m)e_1^m.$$

So

$$fg = fe_1^0 g(0) + fe_1^1 g(1) + \cdots + fe_1^m g(m) \in (f) \cdot I.$$

Hence for any ideal I of R, $hR \cdot I \cap (f) = (f) \cdot I$. Therefore $hR/(f)$ is flat as an R-module.

PROPOSITION 2.11

Let R be a strongly torsion free commutative ring and $f \in hR$. If $\pi(f) > 0$ and $f(\pi(f)) \in U(R)$, then $hR/(f)$ is faithfully flat as an R-module.

Proof. By analogy with the proof of Proposition 2.10, we can show that $hR/(f)$ is flat as an R-module. Then by using the same proof as in Proposition 2.8, we complete the proof.

Let $a_i \in R$, $1 \leq i \leq n$. We denote by $(a_1, a_2, \ldots, a_n)$ the ideal of R generated by $a_i \in R$, $1 \leq i \leq n$.

Lemma 2.12. *Let R be a strongly torsion free commutative ring, $f \in hR$ with $\Delta(f) = n$ and there exists some integer $0 < k < n$ such that the set $\{f(n), f(n-1), \ldots, f(n-k+1)\} \subseteq \mathrm{Id}(R)$ and $(f(n), f(n-1), \ldots, f(n-k+1), f(n-k)) = R$. Then we have the following*:

(1) *Let I be an ideal of R and $g \in hR$ with $\Delta(g) = m$. If for any $n-k \leq s \leq n+m$, $(fg)(s) \in I$, then $g(j) \in I$ for any $0 \leq j \leq m$;*
(2) *f is not a zero divisor in hR;*
(3) *f is not a unit in hR.*

Proof.

(1) Suppose that for any $n-k \leq s \leq n+m$, we have

$$(fg)(s) = \sum_{v=0}^{s} C_s^v f(v)g(s-v) \in I.$$

Now we show that $g(j) \in I$ for each $0 \leq j \leq m$. For convenience, we make the following chart:

$n+m$	$f(n)g(m)$				
$n+m-1$	$f(n-1)g(m)$	$f(n)g(m-1)$			
$\vdots$	$\vdots$	$\vdots$			
$n+m-k$	$f(n-k)g(m)$	$f(n-k+1)g(m-1)$	$\cdots$	$f(n)g(m-k)$	
$n+m-k-1$		$f(n-k)g(m-1)$	$\cdots$	$f(n-1)g(m-k)$	$\cdots$
$\vdots$				$\ddots$	
$n-k$				$f(n-k)g(0)$	

For $s=n+m$, we have $(fg)(n+m)=C^n_{n+m}f(n)g(m)\in I$, and so $f(n)g(m)\in I$ since R is a strongly torsion ring. Then we put $f(n)g(m)$ in the first line of the chart.

For $s=n+m-1$, we have

$$(fg)(n+m-1)=C^{n-1}_{n+m-1}f(n-1)g(m)+C^n_{n+m-1}f(n)g(m-1)\in I.$$

Note that $f(n)\in \mathrm{Id}(R)$ and $f(n)g(m)\in I$. Then we have $C^n_{n+m-1}f(n)g(m-1)=(fg)(n+m-1)f(n)-C^{n-1}_{n+m-1}f(n-1)g(m)f(n)\in I$, and so $f(n)g(m-1)\in I$. Then $C^{n-1}_{n+m-1}f(n-1)g(m)\in I$ and so $f(n-1)g(m)\in I$. We put $f(n-1)g(m)$ and $f(n)g(m-1)$ in the second line of the chart. Similarly, for each $0\le l\le k$, we can show that $f(n-l)g(m)\in I$, $f(n-l+1)g(m-1)\in I,\dots,f(n)g(m-l)\in I$, and we put $f(n-l)g(m)$, $f(n-l+1)g(m-1),\dots,f(n)g(m-l)$ in the $(l+1)$ line of the chart. Since $(f(n),f(n-1),\dots,f(n-k))=R$, there exist $r_0,r_1,\dots,r_k\in R$ such that $r_0f(n)+r_1f(n-1)+\cdots+r_kf(n-k)=1$. Then from the second column of the chart, we obtain

$$r_0f(n)g(m)+r_1f(n-1)g(m)+\cdots+r_kf(n-k)g(m)=g(m)\in I.$$

Then for $s=n+m-k-1$, we have

$$\begin{aligned}(fg)(n+m-k-1)&=\sum_{s=n-k}^{n}C^s_{n+m-k-1}f(s)g(n+m-k-1-s)\\&\quad+C^{n-k-1}_{n+m-k-1}f(n-k-1)g(m)\in I.\end{aligned}$$

Since $g(m)\in I$, we obtain $\sum_{s=n-k}^{n}C^s_{n+m-k-1}f(s)g(n+m-k-1-s)\in I$. Then by using the same way as above, we can show that $f(n-k)g(m-1)\in I$, $f(n-k+1)g(m-2)\in I,\dots,f(n)g(m-k-1)\in I$, and we put $f(n-k)g(m-1)$, $f(n-k+1)g(m-2),\dots,f(n)g(m-k-1)$ in the $(k+2)$ line of the chart. Then from the third column of the chart, we have

$$\begin{aligned}&r_0f(n)g(m-1)+r_1f(n-1)g(m-1)+\cdots+r_kf(n-k)g(m-1)\\&\quad=g(m-1)\in I.\end{aligned}$$

Continuing this procedure yields that $g(j)\in I$ for each $0\le j\le m$.

(2) Assume that there exists some $g\in hR$ such that $fg=0$. Then by (1), we have $g(j)=0$ for each $0\le j\le m$ and so $g=0$. Hence f is not a zero divisor in hR.

(3) Assume, on the contrary, that f is a unit in hR. Then there exists some $g\in hR$ such that $fg=1$, and so for each $s>0$, we have $(fg)(s)=0$. Then by (1), we have $g(j)=0$ for each $0\le j\le m$, and so $g=0$. This contradicts the fact that $fg=1$. Therefore f is not a unit in hR.

PROPOSITION 2.13

Let R be a strongly torsion free commutative ring and $f \in hR$ with $\Delta(f) = n$. If there exists some integer $0 < k < n$ such that the set $\{f(n), f(n-1), \ldots, f(n-k+1)\} \subseteq \mathrm{Id}(R)$ and $(f(n), f(n-1), \ldots, f(n-k+1), f(n-k)) = R$. Then $hR/(f)$ is faithfully flat as an R-module.

Proof. By Lemma 2.12, we know that f is not a unit in hR and so $hR/(f) \neq 0$. Let I be an ideal of R and $g \in hR$. If for any $s \in \mathbb{N}$, we have $(fg)(s) \in I$, then by Lemma 2.12, we obtain $g(i) \in I$ for each $0 \leq i \leq \Delta(g)$. Then by analogy with the proof of Proposition 2.10, we can show that $hR/(f)$ is flat as an R-module.

Now we show that $hR/(f)$ is faithfully flat. Let I be an ideal of R. By Lemma 2.7, it suffices to show that $hR/(f) \cdot I \neq hR/(f)$. Now we see that $1 + (f)$ is not in $hR/(f) \cdot I$. Assume, on the contrary, that $1 + (f) \in hR/(f) \cdot I$. Then there exist some $h_i \in hR$ and $d_i \in I$ such that $1 + (f) = \sum(h_i + (f))d_i = \sum h_i d_i + (f)$. So there exists $g \in hR$ such that $1 - fg \in hR \cdot I \subseteq hI$. Since for each $s > 0$, we have $(1 + fg)(s) = (fg)(s) \in I$. In particular, for each $s \geq n - k$, we have $(1 + fg)(s) = (fg)(s) \in I$. So by Lemma 2.12, we obtain $g(j) \in I$ for each $0 \leq j \leq \Delta(g)$. Hence $g(0) \in I$, Then from $(1 + fg)(0) = 1 + f(0)g(0) \in I$, we obtain $1 \in I$, a contradiction. Therefore $hR/(f)$ is faithfully flat as an R-module.

COROLLARY 2.14

Let R be a strongly torsion-free commutative ring and let $f \in hR$ with $\Delta(f) = n$. If $f(n) \in \mathrm{Id}(R)$ and $f(n-1) \in U(R)$, then $hR/(f)$ is faithfully flat as an R-module.

Proof. It is an immediate consequence of Proposition 2.13.

PROPOSITION 2.15

Let R be a strongly torsion-free commutative ring and $f \in hR$ with $\Delta(f) = n$. If $f(n) \in U(R)$, then we have the following:

(1) *$hR/(f)$ is a faithfully flat R-module*;
(2) *f is not a zero divisor in hR.*

Proof.

(1) Let I be an ideal of R and $g \in hR$ with $\Delta(g) = m$. From the proof of Proposition 2.10 we know that in order that $hR/(f)$ is flat, it suffices to show that $g(j) \in I$ for each $0 \leq j \leq m$ whenever $fg \in hR \cdot I = hI$. Note that for any $s \in \mathbb{N}$, we have

$$(fg)(s) = \sum_{v=0}^{s} C_s^v f(v)g(s-v) \in I.$$

For $s = n + m$, we have $(fg)(n+m) = C_{n+m}^n f(n)g(m) \in I$ and so $g(m) \in I$ by Lemma 2.2.

For $s = n + m - 1$, we obtain $(fg)(n+m-1) = C_{n+m-1}^{n-1} f(n-1)g(m) + C_{n+m-1}^n f(n)g(m-1) \in I$, and so $g(m-1) \in I$. Continuing this process until $s = n$, we

obtain $g(j) \in I$ for each $0 \leq j \leq m$. Hence the flatness of $hR/(f)$ is proved. Then by analogy with the proof of Proposition 2.13, we can show that $hR/(f)$ is faithfully flat.
(2) Assume that $fg = 0$ where $g \in hR$. Then by the proof of (1), we obtain that $g = 0$. Therefore f is not a zero divisor in hR.

In the following, we investigate the conditions that f should be satisfied whenever $hR/(f)$ is flat or faithfully flat.

Lemma 2.16. *Let e_1^i and $e_1^j \in hR$. Then $e_1^i e_1^j = C_{i+j}^i e_1^{i+j}$.*

Proof. It is trivial.

Let (R, M) be a local ring. For any $a \in R$, we denote by $\bar{a}$ the image of a in R/M, and for any $h = h(0)e_1^0 + h(1)e_1^1 + \cdots + h(p)e_1^p \in hR$, we use $\bar{h} = \overline{h(0)}e_1^0 + \overline{h(1)}e_1^1 + \cdots + \overline{h(p)}e_1^p$ to stand for the image of h in $hR/hM \cong h(R/M)$. Then we have the following lemma.

Lemma 2.17. *Let (R, M) be a commutative local ring which contains the field of rational numbers $\mathbb{Q}$, $f = f(0)e_1^0 + f(1)e_1^1 + \cdots + f(m)e_1^m \in hR$ with $\Delta(f) = m$, $A = hR/(f)$. Then $\dim_{R/M} A/MA = \Delta(\bar{f})$.*

Proof. Note that $A/MA \cong hR/(hM + (f))$. Suppose that $\Delta(\bar{f}) = n$. then $f(n) \in U(R)$ and $f(j) \in M$ for all $n+1 \leq j \leq m$. So for any $g = g(0)e_1^0 + g(1)e_1^1 + \cdots + g(k)e_1^k \in hR$, let $w = (C_k^n)^{-1}(f(n))^{-1}g(k)$. If $k \geq n$, then we have

$$\begin{aligned} g - wfe_1^{k-n} = & \sum_{i=0}^{k-n-1} g(i)e_1^i \\ & + \sum_{j=0}^{n-1} \left[g(k-n+j) - C_{k-n+j}^j wf(j)\right] e_1^{k-n+j} \\ & - w\left(\sum_{s=n+1}^{m} f(s)e_1^s\right) e_1^{k-n}. \end{aligned}$$

Let

$$g_1 = \sum_{i=0}^{k-n-1} g(i)e_1^i + \sum_{j=0}^{n-1} \left[g(k-n+j) - C_{k-n+j}^j wf(j)\right] e_1^{k-n+j}$$

and

$$\delta = -w\left(\sum_{s=n+1}^{m} f(s)e_1^s\right) e_1^{k-n}.$$

Then $g - wfe_1^{k-n} = g_1 + \delta$. Clearly, $\Delta(g_1) \leq k-1$ and $\delta \in hM$. By using the same way repeatedly, we see that g can be written as

$$g = b_0 e_1^0 + b_1 e_1^1 + \cdots + b_{n-1} e_1^{n-1} + fh + \Lambda,$$

where $b_i \in R$ for each $0 \leq i \leq n-1$, $h \in hR$ and $\Lambda \in hM$. Hence it is easy to see that $e_1^0 + MA, e_1^1 + MA, \ldots, e_1^{n-1} + MA$ form the basis of the vector space $A/MA \cong hR/hM + (f)$ over the field R/M. Hence $\dim_{R/M} A/MA = \Delta(\bar{f}) = n$.

COROLLARY 2.18

Let (R, M) be a commutative local ring which contains the field of rational numbers $\mathbb{Q}$, $f = f(0)e_1^0 + f(1)e_1^1 + \cdots + f(m)e_1^m \in hR$ with $\Delta(f) = m$, $A = hR/(f)$. Then $\dim_{R/M} A/MA = n$ if and only if $f(n) \in U(R)$ and $f(j) \in M$ for all $n+1 \leq j \leq m$.

Proof. Note that $\Delta(\bar{f}) = n$ if and only if $f(n) \in U(R)$ and $f(j) \in M$ for all $n+1 \leq j \leq m$. Then the result is an immediate consequence of Lemma 2.17.

Lemma 2.19 [3]. *The Hurwitz polynomial $f = f(0)e_1^0 + f(1)e_1^1 + \cdots + f(n)e_1^n$ is a unit if and only if $f(0)$ is a unit and for each $i \geq 1$, $f(i) \in \mathrm{nil}(R)$.*

PROPOSITION 2.20

Let (R, M) be a commutative local ring which contains the field of rational numbers $\mathbb{Q}$, $f \in hR$ and $A = hR/(f)$ be a flat R-module. Then the following conditions are equivalent:

(1) *A is a free R-module with $rank(A) = n$;*
(2) *There exists $g \in hR$ with $\Delta(g) = n$ and $g(n) = 1$ such that $(f) = (g)$;*
(3) *Let $f = f(0)e_1^0 + f(1)e_1^1 + \cdots + f(m)e_1^m$. Then $m \geq n$, $f(n) \in U(R)$ and $f(j) \in M$ for each $n+1 \leq j \leq m$.*

Proof.

(1) $\Rightarrow$ (2) Suppose that $A = hR/(f)$ is a free R-module with $\mathrm{rank}(A) = n$. Then $A \cong R^n$ and so $A/MA \cong R/M \bigotimes_R A \cong R/M \bigotimes_R R^n \cong (R/M)^n$. Hence $\dim_{R/M} A/MA = n$. Then by Corollary 2.18, we have $f(n) \in U(R)$ and $f(j) \in M$ for each $n+1 \leq j \leq m$ and so by the proof of Proposition 2.17, we know that $e_1^0 + MA, e_1^1 + MA, \ldots, e_1^{n-1} + MA$ form the basis of $A/MA \cong hR/hM + (f)$. Since R is a local ring, we have $e_1^0, e_1^1, \ldots, e_1^{n-1}$ form the basis of the free module $A = hR/(f)$. Hence there exist $r_0, r_1, \ldots, r_{n-1} \in R$ and $h \in hR$ such that

$$e_1^n - r_0 e_1^0 - r_1 e_1^1 - \cdots - r_{n-1} e_1^{n-1} = hf.$$

Set $g = e_1^n - r_0 e_1^0 - r_1 e_1^1 - \cdots - r_{n-1} e_1^{n-1}$. Then $\Delta(g) = n$ and $g(n) = 1$. By routine computations, we can show that $hR/(g)$ is a free R-module with rank $(hR/(g)) = n$. We can now define a mapping

$$\theta : hR/(g) \longrightarrow hR/(f)$$

via

$$v + (g) \longrightarrow v + (f)$$

for each $v + (g) \in hR/(g)$. Then θ is well-defined since $(g) \subseteq (f)$. Moreover, the surjectivity of θ is clear. Note that both $hR/(g)$ and $hR/(f)$ are free R-module and $\text{rank}(hR/(g)) = \text{rank}(hR/(f)) = n$. We obtain that θ is an isomorphism. Therefore $(g) = (f)$.
(2) $\Rightarrow$ (3) Assume that there exists $g \in hR$ with $\Delta(g) = n$ and $g(n) = 1$ such that $(f) = (g)$. Then by Proposition 2.15, g is not a zero divisor in hR and so there exists a unit $h \in U(hR)$ such that $f = hg$. Let $h = h(0)e_1^0 + h(1)e_1^1 + \cdots + h(k)e_1^k$ and $g = g(0)e_1^0 + g(1)e_1^1 + \cdots + g(n-1)e_1^{n-1} + e_1^n$. Then $m = n + k$. By Lemma 2.19, $h(0) \in U(R)$ and $h(i) \in \text{nil}(R)$ for each $0 < j \leq k$. From the following equations:

$$\begin{aligned}
&f(m) = (hg)(k+n) = C_{k+n}^k h(k);\\
&f(m-1) = (hg)(k+n-1) = C_{k+n-1}^{k-1} h(k-1) + C_{k+n-1}^{k} h(k) g(n-1);\\
&\vdots\\
&f(n+1) = (hg)(n+1) = C_{n+1}^1 h(1) + C_{n+1}^2 h(2) g(n-1) + \cdots;\\
&f(n) = (hg)(n) = h(0) + C_n^1 h(1) g(n-1) + \cdots,
\end{aligned}$$

we obtain $f(n) \in U(R)$ and $f(j) \in \text{nil}(R)$ for each $n < j \leq m$.
(3) $\Rightarrow$ (1) Assume that $f(n) \in U(R)$ and $f(j) \in \text{nil}(R)$ for each $n = 1 \leq j \leq m$. By Corollary 2.18, we obtain $\dim_{R/M} A/MA = n$. Since R is a local ring, we obtain that A is finitely generated. Since $A = hR/(f)$ is also a flat R-module, we obtain that $A = hR/(f)$ is a free R-module with $\text{rank}(A) = \text{rank}_{R/M}(A/MA) = n$.

Lemma 2.21 [12]. *Let R be a commutative ring and $a \in R$. Then $a \in R$ is nilpotent (resp. unit) if and only if for any maximal ideal M, $\frac{a}{1}$ is nilpotent (resp. unit) in R_M.*

PROPOSITION 2.22

Let R be a commutative ring which contains the rational field $\mathbb{Q}$, $f = f(0)e_1^0 + f(1)e_1^1 + \cdots + f(m)e_1^m \in hR$ and $A = hR/(f)$ be a flat R-module. Then the following conditions are equivalent:

(1) *A is a free R-module with $rank(A) = n$;*
(2) *$f(n) \in U(R)$ and $f(j) \in \text{nil}(R)$ for each $n + 1 \leq j \leq m$.*

Proof.

(1) $\Rightarrow$ (2) Suppose $A \cong R^n$. Then for any maximal ideal M, we have $A_M \cong R_M \bigotimes_R A \cong R_M \bigotimes_R R^n \cong (R_M)^n$, and so A_M is a free R_M-module with $\text{rank}(R_M) = n$. Then by Proposition 2.20, we obtain $\frac{f(n)}{1} \in U(R_M)$ and $\frac{f(j)}{1} \in \text{nil}(R_M)$ for each $n + 1 \leq j \leq m$, and so by Lemma 2.21, $f(n) \in U(R)$ and $f(j) \in \text{nil}(R)$ for each $n + 1 \leq j \leq m$.
(2) $\Rightarrow$ (1) Assume that $f(n) \in U(R)$ and $f(j) \in \text{nil}(R)$ for each $n + 1 \leq j \leq m$. Then by Proposition 2.20, A_M is a free module with $\text{rank}(A_M) = n$. Clearly, the images of $e_1^0, e_1^1, \ldots, e_1^{n-1}$ in A_M form the basis of A_M, and so $e_1^0, e_1^1, \ldots, e_1^{n-1}$ form the basis of A. Hence A is a free module with $\text{rank}(A) = n$.

3. The strongly Hopfian properties of Hurwitz series

Let R be a commutative ring and $a \in R$. We denote by $\mathrm{ann}(a)$ the annihilator of a in R. A ring R is a torsion free $\mathbb{Z}$-module if for any $a \in R$, any positive integer $n \in \mathbb{Z}$, $na = 0$ implies $a = 0$. Clearly, a strongly torsion free ring is a torsion free $\mathbb{Z}$-module. A commutative ring R is said to be strongly Hopfian if for each $a \in R$, the chain $\mathrm{ann}(a) \subseteq \mathrm{ann}(a^2) \subseteq \cdots$ stabilizes. It is well-known that any subring of a strongly Hopfian ring is strongly Hopfian. So if HR or hR is strongly Hopfian, then so does R, but the converse is not true, in general. Recall that in [4], the authors proved that if R is a ring with nonzero characteristic, then R is strongly Hopfian which implies that HR or hR is strongly Hopfian. But if R is a ring with zero characteristic, then HR or hR need not to be strongly Hopfian even if R is strongly Hopfian. In this section, we provide some sufficient conditions in order that HR or hR is strongly Hopfian in case R is a strongly Hopfian ring with zero characteristic.

The next lemma is known and very useful. We leave the proof for the reader.

Lemma 3.1. *Let $a \in R$. Then the chain* $\mathrm{ann}(a) \subseteq \mathrm{ann}(a^2) \subseteq \cdots$ *stabilizes if and only if there exists $n > m$ such that* $\mathrm{ann}(a^n) = \mathrm{ann}(a^m)$.

Let U be a subset of HR, we denote by $\wp(U)$ the set $\{f(\pi(f)) \mid f \in U\} \cup \{0\}$.

PROPOSITION 3.2

Let R be a torsion free $\mathbb{Z}$-module. For any $f \in HR$ (resp. hR), there exists some positive integer n such that $\wp(\mathrm{ann}(f^n)) = \wp(\mathrm{ann}(f^{n+1}))$. Then HR (resp. hR) is a strongly Hopfian ring.

Proof. Let $f \in HR$. Clearly $\mathrm{ann}(f^n) \subseteq \mathrm{ann}(f^{n+1})$. So by Lemma 3.1, it suffices to show that $\mathrm{ann}(f^{n+1}) \subseteq \mathrm{ann}(f^n)$. Let $g_0 \in \mathrm{ann}(f^{n+1})$. Now we show that $g_0 \in \mathrm{ann}(f^n)$. Clearly, $g_0(0) \in \wp(\mathrm{ann}(f^{n+1})) = \wp(\mathrm{ann}(f^n))$. So there exists $g_1 \in \mathrm{ann}(f^n)$ such that $g_0(0) = g_1(0)$ and $g_0 f^n = (g_0 - g_1) f^n$. Since $\pi(g_0 - g_1) \geq 1$, we have $\pi(g_0 f^n) \geq 1$. Since $(g_0 - g_1) f^{n+1} = 0$ and $\pi(g_0 - g_1) \geq 1$, we have $(g_0 - g_1)(1) \in \wp(\mathrm{ann}(f^{n+1}) = \wp(\mathrm{ann}(f^n))$. Hence there exists $g_2 \in \mathrm{ann}(f^n)$ such that $\pi(g_2) \geq 1$, $g_2(1) = (g_0 - g_1)(1)$, and $g_0 f^n = (g_0 - g_1 - g_2) f^n$. Since $\pi(g_0 - g_1 - g_2) \geq 2$, we obtain $\pi(g_0 f^n) \geq 2$. We proceed by induction on k that $\pi(g_0 f^n) \geq k$ for $k \in \mathbb{N}$. Then $g_0 f^n = 0$, as desired.

If R is a reduced ring, then $\mathrm{ann}(a) = \mathrm{ann}(a^2)$ for each $a \in R$. Hence any reduced ring is strongly Hopfian.

PROPOSITION 3.3

Let R be a torsion free $\mathbb{Z}$-module. If R is a reduced ring, then both HR and hR are strongly Hopfian rings.

Proof. By [3], HR and hR are reduced rings and so are strongly Hopfian rings.

By a routine computations, we have the following lemma.

Lemma 3.4. *Let* $f = \sum_{i=0}^{n} f(i)e_1^i \in hR$ *and* $g = \sum_{j=0}^{m} g(j)e_1^j \in hR$. *If* $f(i)g(j) = 0$ *for all* $1 \leq i \leq n$ *and* $1 \leq j \leq m$, *then* $fg = 0$.

By using the idea similar to the one used by Hmaimou *et al.* in [7], we prove the following proposition.

PROPOSITION 3.5

Let R *be a torsion free* $\mathbb{Z}$*-module. Then the following conditions are equivalent*:

(1) R *is a strongly Hopfian ring*;
(2) hR *is a strongly Hopfian ring.*

Proof. It suffices to show that (1) $\Longrightarrow$ (2). Let $f \in hR$ with $\Delta(f) = n$. Then f can be written as $f = \sum_{i=0}^{n} f(i)e_1^i$. Since R is a strongly Hopfian ring, there exists some positive integer k such that for all $l > k$ and all $0 \leq i \leq n$, $\mathrm{ann}((f(i))^l) = \mathrm{ann}((f(i))^k)$. Now we show that $\mathrm{ann}(f^{(n+1)k+1}) = \mathrm{ann}(f^{(n+1)k})$. Let $g \in \mathrm{ann}(f^{(n+1)k+1})$ with $\Delta(g) = m$. Write g as $g = \sum_{j=0}^{m} g(j)e_1^j$. Then $f^{(n+1)k+1}g = 0$ and so

$$(f^{(n+1)k+1}g)(((n+1)k+1)n+m) = c(f(n))^{(n+1)k+1}g(m) = 0,$$

where c is a product of some binomial coefficients. Since R is a torsion free $\mathbb{Z}$-module, we obtain $(f(n))^{(n+1)k+1}g(m) = 0$, and so

$$g(m) \in \mathrm{ann}((f(n))^{(n+1)k+1}) = \mathrm{ann}((f(n))^{(n+1)k}) = \cdots = \mathrm{ann}((f(n))^k).$$

From $f^{(n+1)k+1}g = 0$, we have

$$\begin{aligned} 0 &= (f(n))^k f^{(n+1)k+1} g \\ &= (f(n))^k f^{(n+1)k+1} \left(\textstyle\sum_{j=0}^{m-1} g(j)e_1^j + g(m)e_1^m\right) \\ &= (f(n))^k f^{(n+1)k+1} \left(\textstyle\sum_{j=0}^{m-1} g(j)e_1^j\right). \end{aligned}$$

Hence

$$\begin{aligned} 0 &= \left((f(n))^k f^{(n+1)k+1} \left(\textstyle\sum_{j=0}^{m-1} g(j)e_1^j\right)\right)(((n+1)k+1)n+m-1) \\ &= c_1(f(n))^k(f(n))^{(n+1)k+1}g(m-1), \end{aligned}$$

where c_1 is a product of some binomial coefficients, and so $(f(n))^{(n+2)k+1}g(m-1) = 0$ since R is a torsion free $\mathbb{Z}$-module. Hence

$$\begin{aligned} g(m-1) \in \mathrm{ann}((f(n))^{(n+2)k+1}) &= \cdots = \mathrm{ann}((f(n))^{(n+1)k+1}) = \cdots \\ &= \mathrm{ann}((f(n))^k). \end{aligned}$$

Repeating this process, we obtain that

$$\begin{aligned} g(j) \in \mathrm{ann}((f(n))^{(n+2)k+1}) &= \cdots = \mathrm{ann}((f(n))^{(n+1)k+1}) = \cdots \\ &= \mathrm{ann}((f(n))^k). \end{aligned}$$

for all $0 \leq j \leq m$. Let

$$f_1 = \sum_{i=0}^{n-1} f(i)e_1^i \quad \text{and} \quad N = (n+1)k + 1.$$

Then

$$\begin{aligned} 0 &= f^{(n+1)k+1} g = (f(n)e_1^n + f_1)^N g \\ &= \sum_{r=0}^{nk+1} C_N^r (f(n)e_1^n)^{N-r} f_1^r g + \sum_{r=nk+2}^{N} C_N^r (f(n)e_1^n)^{N-r} f_1^r g \\ &= \sum_{r=nk+2}^{N} C_N^r (f(n)e_1^n)^{N-r} f_1^r g \\ &= \sum_{i=1}^{k} C_N^{k-i} (f(n)e_1^n)^{k-i} f_1^{N-(k-i)} g. \end{aligned} \tag{3}$$

Multiplying eq. (3) by $(f(n)e_1^n)^{k-1}$ and then by Lemma 3.4, we obtain

$$(f(n)e_1^n)^{k-1} f_1^{(n+1)k+1} g = 0.$$

Multiplying eq. (3) by f_1^{k-1}, we obtain

$$\sum_{i=2}^{k} C_N^{k-i} (f(n)e_1^n)^{k-i} f_1^{N-(k-i)} g f_1^{k-1} = 0. \tag{4}$$

Multiplying eq. (4) by $(f(n)e_1^n)^{k-2}$, we obtain

$$(f(n)e_1^n)^{k-2} f_1^{(n+1)k+1} g f_1^{k-1} = 0.$$

Then multiplying eq. (4) by f_1^{k-2}, we obtain

$$\sum_{i=3}^{k} C_N^{k-i} (f(n)e_1^n)^{k-i} f_1^{N-(k-i)} g f_1^{k-1} f_1^{k-2} = 0.$$

Continuing this process, we obtain $f_1^{(n+1)k+1} g f_1^{\frac{k(k-1)}{2}} = 0$, and so $f_1^{(n+1)k+1+\frac{k(k-1)}{2}} g = 0$. By using the same way as above, we can show that for all $0 \leq j \leq m$,

$$g(j) \in \operatorname{ann}((f(n-1))^{(n+1)k+1+\frac{k(k-1)}{2}}) = \cdots = \operatorname{ann}((f(n-1))^k).$$

Using induction on n, we obtain

$$g(j) \in \operatorname{ann}((f(i))^{(n+1)k+1}) = \operatorname{ann}((f(i))^{(n+1)k}) = \cdots = \operatorname{ann}((f(i))^k),$$

for all $0 \leq j \leq m$ and $0 \leq i \leq n$. Then it is easy to see that $f^{(n+1)k} g = 0$. Hence $\operatorname{ann}(f^{(n+1)k+1}) = \operatorname{ann}(f^{(n+1)k})$. Therefore hR is a strongly Hopfian ring.

Remark. Although Benhissi stated in [4] that if R is a strongly Hopfian ring and a torsion free $\mathbb{Z}$-module, the method used in [7] shows that hR is strongly Hopfian. We are sure that the proof is somewhat interesting. So we also present it in this article.

Let R be a commutative ring. If, for every sequence $(a_n)_n$ of elements of R, the ascending chain $\text{ann}(a_1) \subseteq \text{ann}(a_1a_2) \subseteq \cdots$ stabilizes, we say that R satisfies *acc* on d-annihilators. Frohn [6] proved that if R is a reduced ring, then the polynomial ring $R[x]$ satisfies *acc* on d-annihilators if and only if R satisfies *acc* on d-annihilators. In the following, we give some necessary and sufficient conditions in order that the Hurwitz polynomial ring hR satisfies *acc* on d-annihilators.

The following lemma appears in [6].

Lemma 3.6. *Let R be a ring satisfying acc on d-annihilators. Then for every sequence $(A_n)_n$ of finitely generated ideals of R, the ascending chain* $\text{ann}(A_1) \subseteq \text{ann}(A_1A_2) \subseteq \cdots$ *stabilizes.*

PROPOSITION 3.7

Let R be a reduced commutative ring and a torsion free $\mathbb{Z}$-module. Then the following conditions are equivalent:

(1) *R satisfies acc on d-annihilators*;
(2) *hR satisfies acc on d-annihilators.*

Proof.

(1) $\Longrightarrow$ (2) For any $h \in hR$ with $\Delta(h) = q$, we denote by A_h the ideal of R generated by the set $\{h(i) \mid 0 \leq i \leq q\}$. Suppose that $f, g \in hR$ with $\Delta(f) = m$ and $\Delta(g) = n$. We first see that $\text{ann}(A_f \cdot A_g) = \text{ann}(A_{fg})$. Note that for any $s \in \mathbb{N}$, $(fg)(s) = \sum_{k=0}^{s} C_s^k f(k)g(s-k)$. If $a \in \text{ann}(A_f \cdot A_g)$, then $af(i)g(j) = 0$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$. Hence for any $s \in \mathbb{N}$, $a((fg)(s)) = \sum_{k=0}^{s} C_s^k af(k)g(s-k) = 0$, and so $a \in \text{ann}(A_{fg})$. So $\text{ann}(A_f \cdot A_g) \subseteq \text{ann}(A_{fg})$.

We now turn our attention to proving $\text{Ann}(A_{fg}) \subseteq \text{ann}(A_f \cdot A_g)$. Let $a \in \text{ann}(A_{fg})$. Then we have the following system of equations:

$$a((fg)(s)) = \sum_{k=0}^{s} C_s^k af(k)g(s-k) = 0, s = 0, 1, \ldots, m+n.$$

For $s = m+n$, we have $C_{m+n}^m af(m)g(n) = 0$, and so $af(m)g(n) = 0$ since R is a torsion free $\mathbb{Z}$-module.

For $s = m+n-1$, we have

$$C_{m+n-1}^{m-1} af(m-1)g(n) + C_{m+n-1}^{m} af(m)g(n-1) = 0. \tag{5}$$

Multiplying eq. (5) by $f(m)$ and using the condition that $f(m)g(n) = 0$, we obtain $C_{m+n-1}^{m} af^2(m)g(n-1) = 0$ and so $af^2(m)g(n-1) = 0$ since R is a torsion free $\mathbb{Z}$-module. Since R is a reduced ring, we obtain $af(m)g(n-1) = 0$. Then eq. (5) becomes $C_{m+n-1}^{m-1} af(m-1)g(n) = 0$ and so $af(m-1)g(n) = 0$. Continuing this procedure yields that

$$af(i)g(j) = 0 \quad \text{for all } 0 \leq i \leq m \text{ and } 0 \leq j \leq n.$$

Thus for each $\sum_{i=0}^{m} r_i f(i) \in A_f$ and $\sum_{j=0}^{n} s_j g(j) \in A_g$, it is easy to see that

$$a\left(\sum_{i=0}^{m} r_i f(i)\right)\left(\sum_{j=0}^{n} s_j g(j)\right) = 0.$$

Hence $a \in \operatorname{ann}(A_f \cdot A_g)$ and so $\operatorname{ann}(A_{fg}) \subseteq \operatorname{ann}(A_f \cdot A_g)$. Therefore $\operatorname{ann}(A_{fg}) = \operatorname{ann}(A_f \cdot A_g)$ is proved. Now we claim that hR satisfies *acc* on d-annihilators. To this, it suffices to see that $\operatorname{ann}(f) = \operatorname{ann}(fg)$ in hR whenever $\operatorname{ann}(A_f) = \operatorname{ann}(A_{fg})$ in R. Let $h \in \operatorname{ann}(fg)$, and assume that $\Delta(h) = r$, $\Delta(f) = m$ and $\Delta(g) = n$. Then for any $s \in \mathbb{N}$, we have

$$(h(fg))(s) = \sum_{k=0}^{s} C_s^k h(k)(fg)(s-k) = 0.$$

Thus we obtain a system of equations:

$$\textstyle\sum_{k=0}^{s} C_s^k h(k)(fg)(s-k) = 0, \quad s = 0, 1, \ldots, r+m+n.$$

By using the same way as above, we can show that $h(i)(fg)(j) = 0$ for all $0 \le i \le r$ and $0 \le j \le m+n$. Then $h(i) \in \operatorname{ann}(A_{fg}) = \operatorname{ann}(A_f)$ for each $0 \le i \le r$ and so $h(i)f(j) = 0$ for each $0 \le i \le r$ and $0 \le j \le m$. By Lemma 3.4, we obtain $fg = 0$ and so $h \in \operatorname{ann}(f)$. Hence $\operatorname{ann}(f) = \operatorname{ann}(fg)$. Therefore hR satisfies *acc* on d-annihilator.
(2) $\Longrightarrow$ (1) It is trivial.

4. The radical property of Hurwitz series

In this section, we determine some radicals of HR in terms of the corresponding radicals of R under some additional conditions. Throughout this section, all rings R are associative with identity (not necessary commutative). A ring R is an NI ring if $\operatorname{nil}(R)$ forms an ideal. The ring R is said to be prime if $AB \neq 0$ for any nonzero ideals A, B of R. An ideal P of R is prime if R/P is a prime ring. R is said to be strongly prime if R is prime with no nonzero nil ideals. An ideal P of R is strongly prime if R/P is a strongly prime ring. Observe that for a ring R, $\operatorname{nil}^*(R) = \{a \in R \mid (a) \text{ is a nil ideal of } R\} = \cap\{P \mid P \text{ is a strongly prime ideal of } R\}$ [11].

Lemma 4.1. *Let R be a strongly torsion free ring and P an ideal of R. If P is a prime ideal of R, then HP is a prime ideal of HR.*

Proof. Assume, on the contrary, that there exist some $f, g \in HR$ be such that $f \notin HP$, $g \notin HP$ and $f \cdot HR \cdot g \subseteq HP$. Then there exist some positive integers m and n such that for all $0 \le s < m$, $f(s) \in P$, $f(m) \notin P$ and for all $0 \le t < n$, $g(t) \in P$, $g(n) \notin P$. Then from equation

$$(fRg)(m+n) = \sum_{i=0}^{m+n} C_{m+n}^i f(i)Rg(m+n-i) \subseteq P,$$

we obtain $C^m_{m+n} f(m)Rg(n) \subseteq P$ and so $f(m)Rg(n) \subseteq P$ since R is a strongly torsion free ring. Since P is a prime ideal, we obtain $f(m) \in P$ or $g(n) \in P$, a contradiction. Therefore HP is a prime ideal of HR.

PROPOSITION 4.2

Let R be a strongly torsion free ring. Then the following conditions are equivalent:

(1) $\mathrm{nil}_*(R)$ *is nilpotent*;
(2) $\mathrm{nil}_*(HR)$ *is nilpotent and* $\mathrm{nil}_*(HR) = H(\mathrm{nil}_*(R))$.

Proof.

(1) $\Longrightarrow$ (2) Since $\mathrm{nil}_*(R)$ is nilpotent, $H(\mathrm{nil}_*(R))$ is also nilpotent, and so $H(\mathrm{nil}_*(R)) \subseteq \mathrm{nil}_*(HR)$. Let $f \in \mathrm{nil}_*(HR)$. Now we show that $f \in H(\mathrm{nil}_*(R))$. For any prime ideal P of R, by Lemma 4.1, HP is also a prime ideal of HR. Then for any prime ideal P of R, $f \in HP$. Thus for any $n \in \mathbb{N}$, we have $f(n) \in P$ and so for any $n \in \mathbb{N}$, $f(n) \in \mathrm{nil}_*(R)$. Hence $f \in H(\mathrm{nil}_*(R))$. Therefore $\mathrm{nil}_*(HR) = H(\mathrm{nil}_*(R))$.
(2) $\Longrightarrow$ (1) It is trivial.

Lemma 4.3. *Let R be a strongly torsion free ring and P an ideal of R. If P is a strongly prime ideal of R, then HP is a strongly prime ideal of HR.*

Proof. Suppose that P is a strongly prime ideal of R. Then by Lemma 4.1, HP is a prime ideal of HR. Now we show that HP is strongly prime. Assume on the contrary that there exists a nonzero nil ideal $\Omega \subseteq HR/HP \cong H(R/P)$ [2]. Then we can find a nonzero nilpotent element $f \in \Omega$ such that $H(R/P) \cdot f \cdot H(R/P) \subseteq H(R/P)$ is also a nonzero nil ideal. Then it is easy to see that $R/P \cdot f(\pi(f)) \cdot R/P$ is a nonzero nil ideal of R/P. This contradicts the fact that P is a strongly prime ideal of R. Therefore HP is a strongly ideal of HR.

PROPOSITION 4.4

Let R be a strongly torsion free ring. Then the following conditions are equivalent:

(1) $\mathrm{nil}^*(R)$ *is nilpotent*;
(2) $\mathrm{nil}^*(HR)$ *is nilpotent and* $\mathrm{nil}^*(HR) = H(\mathrm{nil}^*(R))$.

Proof. Note that for any ring R, $\mathrm{nil}^*(R) = \cap\{P \mid P$ is a strongly prime ideal of $R\}$. Then by using Lemma 4.3 and the same proof as in Proposition 4.2, we complete the proof.

COROLLARY 4.5

Let R be a strongly torsion free ring and $\mathrm{nil}^*(R)$ *is nilpotent. Then we have the following*:

(1) $\mathrm{nil}_*(R) = \mathrm{nil}^*(R)$;
(2) $\mathrm{nil}_*(HR) = \mathrm{nil}^*(HR) = H(\mathrm{nil}_*(R)) = H(\mathrm{nil}^*(R))$.

Proof.

(1) Since $\mathrm{nil}^*(R)$ is nilpotent, we have $\mathrm{nil}^*(R) \subseteq \mathrm{nil}_*(R)$ and so $\mathrm{nil}_*(R) = \mathrm{nil}^*(R)$.
(2) It follows from (1) that $\mathrm{nil}^*(R) = \mathrm{nil}_*(R)$. Then according to Propositions 4.2 and Proposition 4.4, we complete the proof.

Lemma 4.6. *Let R be a strongly torsion free NI ring with* $\mathrm{nil}(R)$ *nilpotent. Then for any $f \in HR$, $f \in \mathrm{nil}(HR)$ if and only if for all $i \in \mathbb{N}$, $f(i) \in \mathrm{nil}(R)$.*

Proof. Suppose that $f \in \mathrm{nil}(HR)$. Then there exists some positive integer k such that $f^k = 0$ and so $f(0) \in \mathrm{nil}(R)$. Now we claim that $f(i) \in \mathrm{nil}(R)$ for all $i \in \mathbb{N}$. We proceed by induction on i. Clearly, for $i = 0$, the claim is true. Now suppose that for all $0 \le i < n$, $f(i) \in \mathrm{nil}(R)$. We will see that $f(n) \in \mathrm{nil}(R)$. For convenience, we write the set

$$\{(u_1, u_2, \ldots, u_k) \mid u_1 + u_2 + \cdots + u_k = kn, u_v \in \mathbb{N}, 1 \le v \le k\}$$

as

$$\big\{(n, n, \ldots, n)\} \cup \{(u_{s1}, u_{s2}, \ldots, u_{sk}) \mid u_{s1} + \cdots + u_{sk} = kn, u_{sj} \in \mathbb{N},\\ 1 \le j \le k, s < +\infty\big\},$$

and for each $(u_{s1}, u_{s2}, \ldots, u_{sk}) \in \{(u_{s1}, u_{s2}, \ldots, u_{sk}) \mid u_{s1} + \cdots + u_{sk} = kn, u_{sj} \in \mathbb{N}, 1 \le j \le k, s < +\infty\}$, there exists some $1 \le p \le k$ such that $u_{sp} \ne n$. Now it is easy to see that for each $(u_{s1}, u_{s2}, \ldots, u_{sk}) \in \{(u_{s1}, u_{s2}, \ldots, u_{sk}) \mid u_{s1} + \cdots + u_{sk} = kn, u_{sj} \in \mathbb{N}, 1 \le j \le k, s < +\infty\}$, there exists some $1 \le q \le k$ such that $u_{sq} < n$. Hence by induction hypothesis, we obtain $f(u_{sq}) \in \mathrm{nil}(R)$ and so $f(u_{s1}) f(u_{s2}) \cdots f(u_{sq}) \cdots f(u_{sk}) \in \mathrm{nil}(R)$ for each $(u_{s1}, u_{s2}, \ldots, u_{sk}) \in \{(u_{s1}, u_{s2}, \ldots, u_{sk}) \mid u_{s1} + \cdots + u_{sk} = kn, u_{sj} \in \mathbb{N}, 1 \le j \le k, s < +\infty\}$. Then from $f^k = 0$, we obtain

$$0 = f^k(kn) = C^n_{kn} C^n_{(k-1)n} \cdots C^n_n f^k(n) + \Delta,$$

where $\Delta \in \mathrm{nil}(R)$, and so $f(n) \in \mathrm{nil}(R)$ since R is a strongly torsion free ring. Therefore by induction, $f(i) \in \mathrm{nil}(R)$ for all $i \in \mathbb{N}$.

Assume that for all $i \in \mathbb{N}$, $f(i) \in \mathrm{nil}(R)$. Since $\mathrm{nil}(R)$ is nilpotent, there exists some $p \in \mathbb{N}$ such that $(\mathrm{nil}(R))^p = 0$. Then it is easy to see that $f^p = 0$, and so $f \in \mathrm{nil}(HR)$.

PROPOSITION 4.7

Let R be a strongly torsion free NI ring. Then the following conditions are equivalent:

(1) $\mathrm{nil}(R)$ *is nilpotent*;
(2) $\mathrm{nil}(HR)$ *is nilpotent and* $\mathrm{nil}(HR) = H(\mathrm{nil}(R))$.

Proof. Since $\mathrm{nil}(R)$ is nilpotent, $H(\mathrm{nil}(R))$ is also nilpotent, and so $H(\mathrm{nil}(R)) \subseteq \mathrm{nil}(HR)$. By Lemma 4.5, we have $\mathrm{nil}(HR) \subseteq H(\mathrm{nil}(R))$. Hence $\mathrm{nil}(HR) = H(\mathrm{nil}(R))$.
(2) $\Longrightarrow$ (1) It is trivial.

COROLLARY 4.8

Let R be a strongly torsion free NI ring with $\mathrm{nil}(R)$ *nilpotent. Then we have the following*:

(1) $\mathrm{nil}_*(R) = \mathrm{nil}^*(R) = \mathrm{nil}(R)$;
(2) $\mathrm{nil}_*(HR) = \mathrm{nil}^*(HR) = \mathrm{nil}(HR)) = H(\mathrm{nil}_*(R)) = H(\mathrm{nil}^*(R)) = H(\mathrm{nil}(R))$.

Acknowledgements

This research is supported by the National Natural Science Foundation of China (11471108), the Natural Science Foundation of Hunan Province (2016JJ2050), the Scientific Research Foundation of Hunan Provincial Education Department (12B101) and the Teaching Reform Foundation of Hunan Province (G21316).

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COMMUNICATING EDITOR: B Sury