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Inhomogeneous diophantine approximation with prime constraints

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Abstract. We study the problem of inhomogeneous diophantine approximation under certain primality restrictions.

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1. Introduction

Minkowski started the subject of inhomogeneous diophantine approximation by proving that for any irrational number x and any real number θ, the system of inequalities

$$\|px-\theta\| \le \frac{1}{4}Q^{-1} \quad \text{and} \quad |p| \le Q \tag{1}$$

has an integer solution p for infinitely many integers Q. Here and henceforth $\| \ \|$ will denote distance to the nearest integer. In this paper, we study the problem of inhomogeneous diophantine approximation under certain primality considerations, namely we wish to explore certain analogues of (1) for prime p. In fact, we will consider the more general problem of inhomogeneous diophantine approximation with prime constraints on lines in $\mathbb{R}^d$ whose slopes satisfy an explicit diophantine condition. The subject of diophantine approximation on manifolds, in general, and affine subspaces, in particular, has received considerable attention recently. We refer the reader to the survey [7]. In [1,2], we have studied the problem of diophantine approximation on lines under primality constraints. In the present paper, we study an inhomogeneous version of the problem. Inhomogeneous diophantine approximation on manifolds has been extensively studied but to the best of our knowledge, the results in the present paper constitute the first work in this area under primality constraints. Indeed, the study of inhomogeneous diophantine approximation with constraints seems to be a nascent field, some of the only results that we are aware of are contained in the work of Baker *et al.* [3] who studied inhomogeneous diophantine approximation with square-free numbers.

A line in $\mathbb{R}^d$ cannot be expected to inherit generic diophantine properties of $\mathbb{R}^d$ unless some diophantine condition is imposed on its slope. Let $\| \|_\infty$ denote the supremum norm of a vector and for vectors $\mathbf{v}, \mathbf{c} \in \mathbb{R}^d$, denote by $\mathbf{v} \cdot \mathbf{c} = v_1 c_1 + \cdots + v_d c_d$ the inner product of $\mathbf{v}$ and $\mathbf{c}$. Recall that $\mathbf{c} \in \mathbb{R}^d$ is called k-diophantine ($\mathbf{c} \in D_k(\mathbb{R}^d)$) if there exists a constant $C > 0$ such that

$$||\mathbf{v} \cdot \mathbf{c}|| > \frac{C}{||\mathbf{v}||_\infty^k} \quad \text{for every } \mathbf{v} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}. \tag{2}$$

Our main result is as follows.

Theorem 1.1. *Let d be a positive integer and $k \geq d$ be a positive real number. Define*

$$\gamma_{d,k} := \frac{1}{d(3k+2)} \tag{3}$$

and suppose that $0 < \varepsilon < \gamma_{d,k}$. Let $c_1, \ldots, c_d$ be positive irrational numbers such that the vector $\mathbf{c} = (c_1, \ldots, c_d)$ is k-diophantine. Let $(\theta, \theta_1, \ldots, \theta_d)$ be a $(d+1)$-tuple of positive real numbers. Then for almost all positive real α, with respect to the Lebesgue measure, there are infinitely many $(d+2)$-tuples $(p, q_1, \ldots, q_d, r)$ with p and r prime and $q_1, \ldots, q_d$ positive integers such that simultaneously

$$\begin{aligned} &0 < p\alpha - r - \theta < p^{-\gamma_{d,k}+\varepsilon}, \\ &0 < pc_i\alpha - q_i - \theta_i < p^{-\gamma_{d,k}+\varepsilon} \quad \textit{for all } i \in \{1, \ldots, d\}. \end{aligned} \tag{4}$$

In [2], we proved the homogeneous analogue of Theorem 1.1 under the same diophantine condition. It is well known that $D_d(\mathbb{R}^d)$ is a set of zero Lebesgue measure and full Hausdorff dimension. These comprise the set of *badly approximable* vectors. Moreover, $D_k(\mathbb{R}^d)$ has full measure whenever $k > d$, see [5] for example. The proof of Theorem 1.1 is similar to our earlier work [2] with suitable modifications and follows the metrical approach of Harman. Indeed, one of the main messages in this paper is that the method we have developed for studying homogeneous diophantine approximation on lines with primality constraints is flexible enough to also deal with the inhomogeneous case. In a subsequent work, we will address the problem of homogeneous and inhomogeneous diophantine approximation on curves.

2. A metrical approach

For a positive number α, let $F_N(\alpha)$ be the number of solutions to (4) with $p < N$ and for $0 < A < B$, let

$$G_N(A, B) = \frac{A^2}{B^2} \cdot \frac{\min(c_1, \ldots, c_d, d)^{d-1}}{2^{d+1}} N^{1-(d+1)(\gamma_{d,k}-\varepsilon)} (\log N)^{-2}.$$

Of course both F_N and G_N depend on $(\theta, \theta_1, \ldots, \theta_d)$ and we suppress this in the interest of notational convenience. We will prove as follows.

Theorem 2.1. *There exists an infinite sequence $\mathcal{S}$ of natural numbers $\mathbb{N}$ such that the following hold*:

(*i*) *Let* $0 < A < B$. *Then for all* a, b *with* $A \le a < b \le B$, *we have*

$$\int_a^b F_N(\alpha)\mathrm{d}\alpha \ge (b-a)G_N(A,B)(1+o(1)) \tag{5}$$

if $N \in \mathcal{S}$ *and* $N \to \infty$.

(*ii*) *Let* $0 < A < B$ *and* $\varepsilon > 0$. *Then there exists a constant* $K = K(A, B, \varepsilon)$ *such that for* $\alpha \in [A, B]$, *we have*

$$F_N(\alpha) \le K G_N(A,B) + J_N(\alpha)$$

with

$$\int_A^B |J_N(\alpha)|\,\mathrm{d}\alpha = o\left(G_N(A,B)\right) \quad \text{as } N \to \infty$$

if $N \in \mathcal{S}$ *and* $N \to \infty$.

Theorem 2.1(i) corresponds to Lemma 2 and Theorem 2.1(ii) to Lemma 3 in [13]. Combining Theorem 2.1 with Lemma 1 in [13] will complete the proof of Theorem 1.1. The only difference is that $\mathcal{S} = \mathbb{N}$ in the said lemmas but as observed in [2], it is enough to prove the statement for an infinite sequence of natural numbers.

3. Proof of Theorem 2.1(i)

3.1 *Reduction to a counting problem*

To prove Theorem 2.1(i), we follow the approach we used in our earlier work [1,2]. Broadly, we follow the approach in section 3 of [13] but use exponential sums instead of zero density estimates. Below, $\mathbb{P}$ denotes the set of prime numbers.

Let

$$\mathcal{B}_p = \bigcup_{\substack{r \in \mathbb{P} \\ q_1,\ldots,q_d \in \mathbb{N}}} \left[\frac{r+\theta}{p}, \frac{r+\theta+\eta}{p}\right) \cap \left[\frac{1}{c_1}\cdot\frac{q_1+\theta_1}{p}, \frac{1}{c_1}\cdot\frac{q_1+\theta_1+\eta}{p}\right) \cap \cdots \cap \left[\frac{1}{c_d}\cdot\frac{q_d+\theta_d}{p}, \frac{1}{c_d}\cdot\frac{q_d+\theta_d+\eta}{p}\right) \cap [a,b],$$

where $\eta = p^{\varepsilon-\gamma_{d,k}}$. Then

$$\int_a^b F_N(\alpha)\mathrm{d}\alpha = \sum_{\substack{p \in \mathbb{P} \\ p \le N}} \lambda(\mathcal{B}_p), \tag{6}$$

where λ is the Lebesgue measure. Set

$$\mu := (a+b)/(2a). \tag{7}$$

Our strategy is to split the interval $[1, N]$ into subintervals $[P, P\mu]$ and sum up over the P's in the end. Accordingly, we restrict p to the interval $P \le p < P\mu$ with $P\mu \le N$. We then obtain a lower bound for (6) by replacing η with

$$\eta = (\mu P)^{\varepsilon - \gamma_{d,k}}. \tag{8}$$

We note that if

$$\frac{r+\theta}{p} \le \frac{1}{c_i} \cdot \frac{q_i+\theta_i}{p} \le \frac{r+\theta+\eta/2}{p} \quad \text{for } i = 1, \ldots, d,$$

then

$$\lambda\left(\left[\frac{r+\theta}{p}, \frac{r+\theta+\eta}{p}\right) \cap \left[\frac{1}{c_1} \cdot \frac{q_1+\theta_1}{p}, \frac{1}{c_1} \cdot \frac{q_1+\theta_1+\eta}{p}\right) \cap \cdots \cap \left[\frac{1}{c_d} \cdot \frac{q_d+\theta_d}{p}, \frac{1}{c_d} \cdot \frac{q_d+\theta_d+\eta}{p}\right)\right) \ge \nu,$$

where

$$\nu := \frac{\eta}{\mu P} \min\left(\frac{1}{2}, \frac{1}{c_1}, \ldots, \frac{1}{c_d}\right) = (\mu P)^{-1-\gamma_{d,k}+\varepsilon} \min\left(\frac{1}{2}, \frac{1}{c_1}, \ldots, \frac{1}{c_d}\right). \tag{9}$$

Also, for all $p \in [P, P\mu)$,

$$Pa\mu \le r \le bP \Longrightarrow a \le \frac{r}{p} \le b,$$

and r here runs over the primes in an interval of length $\frac{b-a}{2}P$. We thus have

$$\sum_{P \le p < \mu P} \lambda(\mathcal{B}_p) \ge \nu N(P), \tag{10}$$

where $N(P)$ counts the number of solutions $(p, q_1, \ldots, q_d, r) \in \mathbb{P} \times \mathbb{N}^d \times \mathbb{P}$ to

$$q_i \in [c_i r + \tilde{\theta}_i, c_i r + \tilde{\theta}_i + \delta) \quad \text{for } i = 1, \ldots, d,\ P \le p < P\mu,\ Pa\mu \le r \le bP,$$

where

$$\tilde{\theta}_i := c_i\theta - \theta_i$$

and

$$\delta := \frac{\min\{c_1, \ldots, c_d\}\eta}{2} = \frac{\min\{c_1, \ldots, c_d\}}{2(\mu P)^{\gamma_{d,k}-\varepsilon}}. \tag{11}$$

By the prime number theorem, the number $R(P)$ of prime solutions to

$$P \le p < P\mu$$

satisfies

$$R(P) \sim (\mu - 1)P(\log 2P)^{-1} \quad \text{as } P \to \infty. \tag{12}$$

It remains to count the number of solutions $(q_1, \ldots, q_d, r) \in \mathbb{N} \times \mathbb{P}$ to

$$q_i \in [c_i r + \tilde{\theta}_i, c_i r + \tilde{\theta}_i + \delta) \quad \text{for } i = 1, \ldots, d, \quad Pa\mu \le r \le bP,$$

which equals

$$S(P) := \sum_{\substack{Pa\mu \le r \le bP \\ r \text{ prime}}} \prod_{i=1}^{d} ([-(c_i r + \tilde{\theta}_i)] - [-(c_i r + \tilde{\theta}_i + \delta)]).$$

Let

$$T(P) := \sum_{Pa\mu \le n \le bP} \prod_{i=1}^{d} ([-(c_i n + \tilde{\theta}_i)] - [-(c_i n + \tilde{\theta}_i + \delta)])\Lambda(n).$$

We will show that

$$T(P) = \delta^d (b - a\mu)P(1 + o(1)) + O(N^{1-d\gamma_{d,k}+\varepsilon/2}) \quad \text{if } P\mu \le N. \tag{13}$$

As usual, from (13), it follows that

$$S(P) = \delta^d (b - a\mu)P(\log 2P)^{-1}(1 + o(1)) + O(N^{1-d\gamma_{d,k}+\varepsilon/2}) \quad \text{if } P\mu \le N,$$

which together with (12) gives

$$\begin{aligned} N(P) = R(P)S(P) &= \delta^d (b - a\mu)(\mu - 1)P^2(\log 2P)^{-2}(1 + o(1)) \\ &+ O(PN^{1-d\gamma_{d,k}+\varepsilon/2}) \quad \text{if } P\mu \le N. \end{aligned}$$

Combing this with (7), (8), (9), (10) and (11), we obtain

$$\begin{aligned} \sum_{P \le p < \mu P} \lambda(\mathcal{B}_p) \ge \frac{(b-a)^2}{4a} \cdot \frac{\min(2, c_1, \ldots, c_d)^{d-1}}{2^d} \\ \cdot (\mu P)^{-1-(d+1)(\gamma_{d,k}-\varepsilon)} P^2 (\log 2P)^{-2}(1 + o(1)) \\ + O(P^{-\gamma_{d,k}} N^{1-d\gamma_{d,k}+3\varepsilon/2}) \quad \text{if } P\mu \le N. \end{aligned} \tag{14}$$

By splitting the interval $[1, N)$ into intervals of the form $[P, \mu P)$ and summing up, it now follows from (6) and (14) that

$$\begin{aligned}\int_a^b F_N(\alpha)\mathrm{d}\alpha &\geq \frac{(b-a)^2}{4a}\cdot\frac{\min(2,c_1,\ldots,c_d)^{d-1}}{2^d}\\ &\quad\cdot\left(\sum_{k=0}^{\infty}\left(\frac{N}{\mu^k}\right)^{-1-(d+1)(\gamma_{d,k}-\varepsilon)}\left(\frac{N}{\mu^{k+1}}\right)^2(\log N)^{-2}\right)(1+o(1))\\ &= \frac{(b-a)^2}{4a}\cdot\frac{\min(2,c_1,\ldots,c_d)^{d-1}}{2^d}\\ &\quad\cdot\mu^{-2}\cdot\frac{1}{1-\mu^{-(1-(d+1)(\gamma_{d,k}-\varepsilon))}}\\ &\quad\cdot N^{1-(d+1)(\gamma_{d,k}-\varepsilon)}(\log N)^{-2}(1+o(1)).\end{aligned}$$

Further, since $\mu > 1$, we have

$$\begin{aligned}1-\mu^{-(1-(d+1)(\gamma_{d,k}-\varepsilon))} &\leq \big(1-(d+1)(\gamma_{d,k}-\varepsilon)\big)(\mu-1)\\ &= \big(1-(d+1)(\gamma_{d,k}-\varepsilon)\big)\cdot\frac{b-a}{2a}.\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}\int_a^b F_N(\alpha)\mathrm{d}\alpha &\geq (b-a)\cdot\frac{2a^2}{(a+b)^2}\\ &\quad\cdot\frac{1}{1-(d+1)(\gamma_{d,k}-\varepsilon)}\cdot\frac{\min(c_1,\ldots,c_d,2)^{d-1}}{2^d}\\ &\quad\cdot N^{1-(d+1)(\gamma_{d,k}-\varepsilon)}(\log N)^{-2}(1+o(1))\\ &\geq (b-a)\cdot\frac{A^2}{B^2}\cdot\frac{\min(c_1,\ldots,c_d,2)^{d-1}}{2^{d+1}}\\ &\quad\cdot N^{1-(d+1)(\gamma_{d,k}-\varepsilon)}(\log N)^{-2}(1+o(1)),\end{aligned} \tag{15}$$

establishing the claim of Theorem 2.1(i).

3.2 *Proof of* (13)

We proceed as in [2] by reducing to exponential sums. For $x \in \mathbb{R}$, let

$$\psi(x) := x - [x] - \frac{1}{2}.$$

Then we may write $T(P)$ in the form

$$\begin{aligned}T(P) &= \sum_{Pa\mu\leq n\leq bP}\Lambda(n)\prod_{i=1}^{d}(\delta-(\psi(-(c_in+\tilde{\theta}_i))-\psi(-(c_in+\tilde{\theta}_i+\delta))))\\ &= \sum_{\mathcal{A}\subseteq\{1,\ldots,d\}}\delta^{d-|\mathcal{A}|}T_{\mathcal{A}}(P),\end{aligned} \tag{16}$$

where

$$T_{\mathcal{A}}(P) := \sum_{Pa\mu\leq n\leq bP}\Lambda(n)\prod_{i\in\mathcal{A}}(\psi(-(c_in+\tilde{\theta}_i+\delta))-\psi(-(c_in+\tilde{\theta}_i)).$$

By the prime number theorem,

$$T_{\emptyset}(P) = \delta^d \sum_{Pa\mu \le n \le bP} \Lambda(n) \sim \delta^d (b - a\mu)P \quad \text{as } P \to \infty.$$

Hence, to establish (13), it suffices to prove that for any fixed $\varepsilon > 0$ a bound of the form

$$T_{\mathcal{A}}(P) = O(N^{1-d\gamma_{d,k}+\varepsilon/2}) \quad \text{if } P\mu \le N \tag{17}$$

for all non-empty subsets $\mathcal{A}$ of $\{1, \ldots, d\}$ holds. We reduce the left-hand side to exponential sums, using the following Fourier analytic tool developed by Vaaler [15].

Lemma 3.1 [15]. For $0 < |t| < 1$, let

$$W(t) = \pi t(1 - |t|) \cot \pi t + |t|.$$

Fix a positive integer J. For $x \in \mathbb{R}$, define

$$\psi^*(x) := - \sum_{1 \le |j| \le J} (2\pi i j)^{-1} W\left(\frac{j}{J+1}\right) e(jx)$$

and

$$\tau(x) := \frac{1}{2J+2} \sum_{|j| \le J} \left(1 - \frac{|j|}{J+1}\right) e(jx).$$

Then $\delta(x)$ is non-negative, and we have

$$|\psi^*(x) - \psi(x)| \le \tau(x)$$

for all real numbers x.

Proof. This is Theorem A6 in [8] and has its origin in [15]. □

We define

$$\psi(x) = \psi^*(x) + \tau^*(x).$$

Then

$$\begin{aligned} T_{\mathcal{A}}(P) = &\sum_{\mathcal{B} \subseteq \mathcal{A}} \sum_{Pa\mu \le n \le bP} \Lambda(n) \\ &\times \left(\prod_{i \in \mathcal{B}} (\psi^*(-(c_i n + \tilde{\theta}_i + \delta)) - \psi^*(-(c_i n + \tilde{\theta}_i)) \right) \\ &\times \left(\prod_{j \in \mathcal{A} \setminus \mathcal{B}} (\tau^*(-(c_j n + \tilde{\theta}_j + \delta)) - \tau^*(-(c_j n + \tilde{\theta}_j)) \right) \\ = &\, U_{\mathcal{A}}(P) + O\left(V_{\mathcal{A}}(P)\right), \end{aligned} \tag{18}$$

where

$$U_{\mathcal{A}}(P) := \sum_{Pa\mu \le n \le bP} \Lambda(n) \left(\prod_{i \in \mathcal{A}} (\psi^*(-(c_i n + \tilde{\theta}_i + \delta)) - \psi^*(-(c_i n + \tilde{\theta}_i)) \right)$$

and

$$V_{\mathcal{A}}(P) := (\log 2P) \sum_{i \in \mathcal{A}} \sum_{Pa\mu \le n \le bP} (\tau(-(c_i n + \tilde{\theta}_i + \delta)) + \tau(-(c_i n + \tilde{\theta}_i))),$$

where the O-term $V_{\mathcal{A}}(P)$ arrives by using $|\psi^*(x)| \le 2, |\tau^*(x)| \le \tau(x) \le 1, \Lambda(n) \le \log n$ and the triangle inequality.

The definition of the function $\tau(x)$ gives

$$V_{\mathcal{A}}(P) = \frac{\log 2P}{2J+2} \sum_{i \in \mathcal{A}} \sum_{|j| \le J} \left(1 - \frac{|j|}{J+1}\right) (1 + e(j\delta)) \cdot e(j\tilde{\theta}_i) \cdot \sum_{Pa\mu \le n \le bP} e(jc_i n),$$

and the definition of $\psi^\star(x)$ gives

$$U_{\mathcal{A}}(P) = - \sum_{1 \le |j_1| \le J} \cdots \sum_{1 \le |j_h| \le J} \times \left(\prod_{k=1}^{h} \left((2\pi i j_k)^{-1} W\left(\frac{j_k}{J+1}\right) (1 + e(j_k\delta)) \cdot e(j_k \tilde{\theta}_k) \right) \right) \times \sum_{Pa\mu \le n \le bP} \Lambda(n) e\left(n \sum_{k \in \mathcal{A}} j_k c_{l_k}\right),$$

where we suppose that J is a positive integer satisfying $J \le P$ and

$$h := |\mathcal{A}| \quad \text{and} \quad \mathcal{A} = \{l_1, \ldots, l_h\}.$$

Now $V_{\mathcal{A}}(P)$ can be estimated exactly as in [2] to get

$$V_{\mathcal{A}}(P) \ll \frac{P \log 2P}{J} + (\log 2P) \tilde{V}_d(P). \tag{19}$$

Then we estimate $U_{\mathcal{A}}(P)$ by

$$U_{\mathcal{A}}(P) \ll \tilde{U}_{\mathcal{A}}(P) := \sum_{1 \le |j_1| \le J} \cdots \sum_{1 \le |j_h| \le J} \frac{1}{|j_1 \cdots j_h|} \times \left| \sum_{Pa\mu \le n \le bP} \Lambda(n) e\left(n \sum_{k \in \mathcal{A}} j_k c_{l_k}\right) \right|. \tag{20}$$

It remains to estimate $\tilde{U}_{\mathcal{A}}(P)$ and $\tilde{V}_d(P)$. This analysis was carried out in §3 of [2] and uses Vaughan's identity and this is also where the diophantine properties of **c** come into play. It was proved there that

$$\tilde{U}_{\mathcal{A}}(P) \ll P^{\varepsilon}(P^{4/5} + (PJ)^{1-1/(k+1)} + P^{1-1/(2(k+1))} J^{1/2-1/(2(k+1))}) \tag{21}$$

and that

$$V_{\mathcal{A}}(P) \ll (\log 2P)(PJ^{-1} + J + P^{1-1/(k+1)}). \tag{22}$$

Putting together the estimates, we get

$$T_{\mathcal{A}}(P) \ll P^{\varepsilon}(PJ^{-1} + J + P^{4/5} + (PJ)^{1-1/(k+1)} + P^{1-1/(2(k+1))}J^{1/2-1/(2(k+1))}).$$

Now choosing

$$J := [P^{1/(3k+2)}], \tag{23}$$

we get

$$T_{\mathcal{A}}(P) \ll P^{1-1/(3k+2)+\varepsilon},$$

which proves (17) upon replacing ε by $\varepsilon/2$. This completes the proof of Theorem 2.1(i).

4. Proof of Theorem 2.1(ii)

This proof also follows [2], namely we use a sieve theoretic approach. This section also uses the diophantine properties of the vector $\mathbf{c}$. Let $\{\cdot\}$ represent the fractional part, and put

$$\mu := N^{\varepsilon-\gamma_{d,k}}.$$

Write

$$\mathcal{A} = \mathcal{A}(\alpha) = \{n[n\alpha - \theta] \ : \ 1 \le n \le N, \ \{n\alpha - \theta\} < \mu, \ \{nc_i\alpha - \theta_i\} < \mu \text{ for } i = 1, \dots, d\}.$$

We will show that $\mathcal{A}$ does not contain too many products of two primes using a two-dimensional upper bound sieve (see [9], Theorem 5.2). To do this, we need to obtain an asymptotic formula for the number of solutions to

$$n[n\alpha - \theta] \equiv 0 \bmod q, \ \ 1 \le n \le N,$$

with

$$\{n\alpha - \theta\} < \mu \quad \text{and} \quad \{nc_i\alpha - \theta_i\} < \mu \text{ for } i = 1, \dots, d, \tag{24}$$

where

$$q \le Q := N^{\varepsilon}.$$

For this, it suffices to establish a formula for the number of solutions to

$$n \equiv 0 \bmod t_1, \quad [n\alpha - \theta] \equiv 0 \bmod t_2$$

subject to (24). We can combine (24) with the congruence conditions to require

$$1 \le n \le \frac{N}{t_1}, \quad \left\{\frac{nt_1\alpha - \theta}{t_2}\right\} < \frac{\mu}{t_2}, \quad \{nt_1c_i\alpha - \theta_i\} < \mu \quad \text{for } i = 1, \dots, d, \tag{25}$$

and count the number $S(\alpha; t_1, t_2)$ of solutions to (25) using Fourier analysis. To do so, we proceed as in [2] and write

$$\begin{aligned} S(\alpha, t_1, t_2) &= \sum_{1\le n\le N/t_1} \left(\left[\tfrac{nt_1\alpha-\theta}{t_2}\right] - \left[\tfrac{nt_1\alpha-\theta}{t_2} - \tfrac{\mu}{t_2}\right]\right) \\ &\quad \cdot \prod_{i=1}^{d} ([nt_1c_i\alpha - \theta_i] - [nt_1c_i\alpha - \theta_i - \mu]) \end{aligned}$$

As in *loc. cit.*, we have the asymptotic estimate

$$S(\alpha; t_1, t_2) = \frac{N\mu^{d+1}}{t_1t_2} + O\left(\frac{N\mu^d}{L} + E(\alpha; t_1, t_2)\right), \tag{26}$$

where $L := Q^3\mu^{-1}$ and

$$E(\alpha; t_1, t_2) := \frac{\mu^{d+1}}{t_2} \sum_{\substack{|m_0|\le L,\dots,|m_d|\le L \\ (m_0,\dots,m_d)\ne(0,\dots,0)}} \left| \sum_{1\le n\le N/t_1} e\left(n\alpha t_1\left(\frac{m_0}{t_2} + \sum_{i=1}^{d} c_i m_i\right)\right)\right|.$$

Applying the upper bound sieve gives

$$F_N(\alpha) \le \frac{C(\varepsilon)N\mu^d}{\log^2 N} + O\left(J_N(\alpha)\right), \tag{27}$$

where

$$\begin{aligned} J_N(\alpha) &:= \sum_{t_1t_2\le Q} (t_1t_2)^{\varepsilon} \left(\frac{N\mu^d}{L} + E(\alpha; t_1, t_2)\right) \\ &= \sum_{t_1t_2\le Q} (t_1t_2)^{\varepsilon} E(\alpha; t_1, t_2) + o\left(\frac{N\mu^{d+1}}{\log^2 N}\right) \quad \text{as } N \to \infty. \end{aligned}$$

Hence Theorem 2.1(i) is proved once we have shown that

$$\sum_{t_1t_2\le Q} (t_1t_2)^{\varepsilon} \int_A^B E(\alpha; t_1, t_2)\mathrm{d}\alpha = o\left(\frac{N\mu^{d+1}}{\log^2 N}\right) \quad \text{as } N \to \infty,\ N \in \mathcal{S}, \tag{28}$$

which is proved in [2].

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