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A note on signs of Fourier coefficients of two cusp forms

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Abstract. Kohnen and Sengupta (*Proc. Am. Math. Soc.* **137(11)** (2009) 3563–3567) showed that if two Hecke eigencusp forms of weight k_1 and k_2 respectively, with $1 < k_1 < k_2$ over $\Gamma_0(N)$, have totally real algebraic Fourier coefficients $\{a(n)\}$ and $\{b(n)\}$ respectively for $n \geq 1$ with $a(1) = 1 = b(1)$, then there exists an element σ of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ such that $a(n)^{\sigma} b(n)^{\sigma} < 0$ for infinitely many n. Later Gun *et al.* (*Arch. Math. (Basel)* **105(5)** (2015) 413–424) extended their result by showing that if two Hecke eigen cusp forms, with $1 < k_1 < k_2$, have real Fourier-coefficients $\{a(n)\}$ and $\{b(n)\}$ for $n \geq 1$ and $a(1)b(1) \neq 0$, then there exists infinitely many n such that $a(n)b(n) > 0$ and infinitely many n such that $a(n)b(n) < 0$. When $k_1 = k_2$, the simultaneous sign changes of Fourier coefficients of two normalized Hecke eigen cusp forms follow from an earlier work of Ram Murty (*Math. Ann.* **262** (1983) 431–446). In this note, we compare the signs of the Fourier coefficients of two cusp forms simultaneously for the congruence subgroup $\Gamma_0(N)$ where the coefficients lie in an arithmetic progression. Next, we consider an analogous question for the particular sparse sequences of Fourier coefficients of normalized Hecke eigencusp forms for the full modular group.

Keywords. Sign changes; Fourier coefficients; modular forms; Rankin–Selberg L-function.

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1. Introduction

Throughout the paper, let $z \in \mathfrak{H}$ be an element of the Poincaré upper-half plane and $q = e^{2\pi i z}$. Let S_k denote the space of cusp forms of weight k for full modular group $SL_2(\mathbb{Z})$ and $S_k(N)$ denote the space of cusp forms of weight k for the congruence subgroup $\Gamma_0(N)$.

The sign changes of Fourier coefficients of cusp forms in one or in several variables have been studied in various aspects by many authors. It is known that, if the Fourier coefficients of a cusp form are real then they change signs infinitely often [6]. Further, many quantitative results for the number of sign changes for the sequence of the Fourier coefficients have been established. The sign changes of the subsequence of the Fourier coefficients at prime numbers was first studied by Ram Murty [14]. Later, Meher *et al.* [13] investigated that for a normalized Hecke eigencusp form with Fourier coefficients $\{a(n)\}$, the subsequences $\{a(n^j)\}_{n\geq 1}$ $(j = 2, 3, 4)$ of the Fourier coefficients change signs infinitely often. Kohnen and Martin [8] generalized their result by showing that the subsequence $\{a(p^{jn})\}_{n\geq 0}$ has infinitely many sign changes for a set of primes of density one and for $j \in \mathbb{N}$ with $4 \nmid j$.

Recently in 2009, the question of simultaneous sign change of Fourier coefficients of two cusp forms of different integral weights with totally real algebraic Fourier coefficients was considered by Kohnen and Sengupta [9]. They showed that for $1 < k_1 < k_2$ if f and g is in $S_{k_i}(\Gamma_0(\mathbb{N}))$ for $i = 1, 2$ with totally real algebraic Fourier coefficients $\{a(n)\}$ and $\{b(n)\}$ respectively for $n \geq 1$ with $a(1) = 1 = b(1)$, then there exist an element σ of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ such that $a(n)^\sigma b(n)^\sigma < 0$ for infinitely many n. Later Gun *et al.* [5] extended their result by showing that if f and g have real Fourier coefficients $\{a(n)\}$ and $\{b(n)\}$ respectively for $n \geq 1$ and $a(1)b(1) \neq 0$, then there exist infinitely many n such that $a(n)b(n) > 0$ and infinitely many n such that $a(n)b(n) < 0$. When $k_1 = k_2$, the simultaneous sign changes of Fourier coefficients of f and g follow from an earlier work of Ram Murty [14].

In this article, we have considered two normalized Hecke eigencusp forms f and g of weight $1 < k_2 < k_1$ for $\Gamma_0(\mathbb{N})$ with Fourier coefficients $\{a(n)\}$ and $\{b(n)\}$ respectively and have studied the simultaneous sign changes of Fourier coefficients where the coefficients are in arithmetic progression. We have also studied the simultaneous sign changes of the sparse sequences $\{a(n^j)\}$ and $\{b(n^j)\}$ for $j = 2, 3, 4$ respectively of f and g over full modular group $SL_2(\mathbb{Z})$.

2. Statement of the results

The following two theorems are the main results alluded off earlier.

Theorem 2.1. *Let $f, g \in S_{k_i}(N)$ (with $i = 1, 2$ respectively) having Fourier coefficients $a(n)$ and $b(n)$ respectively which are normalized, i.e. $a(1) = b(1) = 1$. Then for any $m \in \mathbb{N}$ and $l \in \mathbb{Z}$ with $(l, m) = 1$, the subsequences $(a(n))_{n\equiv l \pmod m}$ and $(b(n))_{n\equiv l \pmod m}$ have infinitely many terms of the same as well as of different signs simultaneously.*

Theorem 2.2. *Let $f, g \in S_{k_i}$ (with $i = 1, 2$ respectively) be normalized Hecke eigenform with Fourier coefficients $a(n)$ and $b(n)$ respectively. Then for $j = 2, 3, 4$ the sparse sequences $\{a(n^j)\}_{n\geq 1}$ and $\{b(n^j)\}_{n\geq 1}$ have infinitely many terms of the same as well as of different signs simultaneously.*

3. Proof of Theorem 2.1

We begin by stating and sometimes outlining a proof of a series of lemmas. Let us define the counting function

$$I_l(n) := \begin{cases} 1 & \text{if } n \equiv l (\text{mod } m) \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

with m and l as before.

Lemma 3.1. *Let χ be any Dirichlet character modulo N. Consider $f \in S_k(N, \chi)$ having Fourier coefficients $a(n) \in \mathbb{R}$ and l, m be co-prime positive integers. If $g(z) := \sum_{n=1}^{\infty} I_l(n)a(n)q^n$, then $g \in S_k(\Gamma_1(Nm^2))$.*

Proof. We refer Proposition 17, p. 127 of [7] for the proof. □

Lemma 3.2. *Suppose $f \in S_k(\Gamma_1(N))$. Then its associated L function $L(s, f)$ has an analytic continuation to the full s-plane.*

Proof. We refer Theorem 5.10.2 of [3] for the proof. □

Lemma 3.3 [10]. *Suppose that $d(n) \geq 0$ for all but finitely many n's and that the Dirichlet series*

$$\Psi(s) = \sum_{n \geq 1} \frac{d(n)}{n^s}$$

has finite abscissa of convergence σ_c. Then $\Psi(s)$ has a singularity on the real axis at the point $s = \sigma_c$.

3.1 *Proof of Theorem* 2.1

Consider,

$$f_1(z) = \sum_{n \geq 1} I_l(n)a(n)q^n$$

and

$$g_1(z) = \sum_{n \geq 1} I_l(n)b(n)q^n,$$

with $I_l(n)$ as in (3.1). Applying Lemma 3.1, we have $f_1 \in S_{k_1}(\Gamma_1(Nm^2))$ and $g_1 \in S_{k_2}(\Gamma_1(Nm^2))$. Let

$$R_{f_1,g_1}(s) = \sum_{n \geq 1} \frac{I_l(n)a(n)b(n)}{n^s} \tag{3.2}$$

be the Rankin–Selberg Dirichlet series attached to f_1 and g_1. Let us take $k_1 > k_2$ without any loss of generality and set

$$R^*_{f_1,g_1}(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s - k_2 + 1)\zeta_{Nm^2}(2s - (k_1 + k_2) + 2)R_{f_1,g_1}(s), \tag{3.3}$$

where

$$\zeta_{Nm^2}(s) = \prod_{p|Nm^2} (1 - p^{-s})\zeta(s).$$

It is well known that $R^*_{f_1,g_1}(s)$ extends to an entire function on $\mathbb{C}$ (cf. [1]). Since $\Gamma(z)$ has no zeros for all $z \in \mathbb{C}$, it follows that $\zeta_{Nm^2}(2s - (k_1 + k_2) + 2)R_{f_1,g_1}(s)$ extends to an entire function.

If possible, let us assume that the sequence $\{a(n)b(n)\}_{n\equiv l\,(\mathrm{mod}\ m)}$ has all but finitely many terms ≥ 0 for $(l,m)=1$. If we denote the coefficients of $\zeta_{Nm^2}(2s-(k_1+k_2)+2)R_{f_1,g_1}(s)$ by $e(n)$ for all $n\in\mathbb{N}$, then $e(1)\neq 0$ and, by assumption, we have that $\{I_l(n)a(n)b(n)\}_{n\geq 1}$ has all but finitely many terms ≥ 0. Now by following the similar argument as in the proof of the theorem in [9], one can show that $e(1)=0$, which is a contradiction. Thus we conclude that the subsequences $(a(n))_{n\equiv l\,(\mathrm{mod}\ m)}$ and $(b(n))_{n\equiv l\,(\mathrm{mod}\ m)}$ have infinitely many terms of different signs simultaneously for $(l,m)=1$.

Now we assume that the sequence $\{a(n)b(n)\}_{n\equiv l\,(\mathrm{mod}\ m)}$ has all but finitely many terms ≤ 0 for $(l,m)=1$. Now we can argue similar to the proof of theorem 1 in [5] to arrive at the contradiction. Thus the subsequences $(a(n))_{n\equiv l\,(\mathrm{mod}\ m)}$ and $(b(n))_{n\equiv l\,(\mathrm{mod}\ m)}$ have infinitely many terms of the same signs simultaneously for $(l,m)=1$. This completes the proof.

4. Proof of Theorem 2.2

Let us denote $\alpha_j = 1/2, 3/4, 7/9$ and $\beta_j = 2/11, 1/9, 2/27$ respectively for $j=2,3,4$. Let

$$f(z)=\sum_{n\geq 1} a(n)\mathrm{e}^{2\pi i n z}\in S_{k_1}$$

and

$$g(z)=\sum_{n\geq 1} b(n)\mathrm{e}^{2\pi i n z}\in S_{k_2}$$

be normalized Hecke eigenform. Here we want to compare the sparse sequences $\{a(n^j)\}_{n\geq 1}$ and $\{b(n^j)\}_{n\geq 1}$ simultaneously for $j=2,3,4$. The following two lemmas will be required to prove the theorem.

Lemma 4.1. *Let $f(z)=\sum_{n\geq 1} a(n)\mathrm{e}^{2\pi i n z}\in S_k$ be normalized Hecke eigenform. Then for any $\epsilon>0$ and $j=2,3,4$, we have*

$$\sum_{n\leq x} a(n^j) \ll_{f,\epsilon} x^{\alpha_j+\epsilon}$$

where α_j's are defined as above.

Proof. We refer [4] and [12] for the proof. □

Lemma 4.2. *Let $f\in S_{k_1}$ and $g\in S_{k_2}$ be normalized Hecke eigenform with Fourier coefficients $a(n)$ and $b(n)$ respectively. Then for any $\epsilon>0$ and $j=2,3,4$, we have*

$$\sum_{n\leq x} a(n^j)b(n^j) = C_j x + O_{f,\epsilon}(x^{1-\beta_j+\epsilon}),$$

where C_j's are absolute constants and β_j's are defined above.

Proof. The proof of the lemma is an immediate consequence of the proof of Theorems 1.1, 1.2, 1.3 of [11]. □

4.1 *Proof of Theorem* 2.2

Let $h = h(x) = x^{1-\beta_j+2\epsilon}$, where $\epsilon(>0)$ is sufficiently small. If possible let us assume that for $j = 2, 3, 4$, the sparse sequence $\{a(n^j)b(n^j)\}_{n\geq 1}$ are of constant sign say positive $\forall\, n \in (x, x+h]$.

Now from Lemma 4.2, we have

$$\sum_{x<n\leq x+h} a(n^j)b(n^j) = C_j h + O_{f,\epsilon}(x^{1-\beta_j+\epsilon}) \gg x^{1-\beta_j+2\epsilon} \tag{4.1}$$

On the other hand, using Lemma 4.1 and Deligne's bound (cf. [2]) we get

$$\sum_{x<n\leq x+h} a(n^j)b(n^j) \ll x^{2\epsilon} \sum_{x<n\leq x+h} b(n^j) \ll x^{2\epsilon}\big((x+h)^{\alpha_j+\epsilon} + x^{\alpha_j+\epsilon}\big) \ll x^{\alpha_j+3\epsilon}. \tag{4.2}$$

Now comparing $1-\beta_j$ and α_j in 4.1 and 4.2, we can see that the bounds contradict each other. Therefore, atleast one $a(n^j)b(n^j)$ for $n \in (x, x+h]$ are of negetive sign. Hence we can conclude that, for $j = 2, 3, 4$ the sparse sequences $\{a(n^j)\}_{n\geq 1}$ and $\{b(n^j)\}_{n\geq 1}$ have infinitely many terms of different signs simultaneously. By similar argument, one can also show that the sparse sequences have infinitely many terms of the same signs simultaneously. This completes the proof of the theorem.

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References

[1] Bump D, The Rankin–Selberg method: a survey, in: number theory, trace formula and discrete groups, edited by K E Aubert *et al.* (1989) (London Academic Press) pp. 49–109

[2] Deligne P, La Conjecture de Weil, *I. Inst. Hautes Etudes Sci. Publ. Math.*, **43** (1974) 273–307, MR0340258 (49:5013)

[3] Diamond F and Shurman J, A first course in modular forms, GTM 228 (2005) (Springer-Verlag)

[4] Fomenko O M, On the behaviour of the automorphic L-functions at the centre of the critical strip, *J. Math. Sci. (N.Y.)*, **118** (2003) 4910–4917

[5] Gun S, Kohnen W and Rath P, Simultaneous sign change of Fourier-coefficients of two cusp forms, *Arch. Math. (Basel)* **105(5)** (2015) 413–424

[6] Knopp M, Kohnen W and Pritbikin W, On the signs of Fourier coefficients of cusp forms, *The Ramanujan J.*, **7** (2003) 269–277

[7] Koblitz N, Introduction to elliptic curves and modular forms, GTM 97 (1993) (Springer-Verlag)

[8] Kohnen W and Martin Y, Sign changes of Fourier coefficients of cusp forms supported on prime power indices, *Int. J. Number Theory*, **10(8)** (2014) 1921–1927

[9] Kohnen W and Sengupta J, Signs of Fourier coefficients of two cusp forms of different weights, *Proc. Am. Math. Soc.* **137(11)** (2009) 3563–3567

[10] Landau E, Über einen Satz von Tschebyschef, *Math. Ann.*, **61** (1906) 527–550

[11] Lao H and Sankaranarayanan A, The average behaviour of Fourier coefficients of cusp forms over sparse sequences, *Proc. Amer. Math. Soc.*, **8** (2009) 2557–2565
[12] Lü G S, On an open problem of Sankaranarayanan, *Science China Math.*, **53** (2010) 1319–1324
[13] Meher J, Shankhadhar K and Viswanadham G K, A short note on sign changes, *Proc. Indian Acad. Sci. (Math. Sci.)*, **123** (2013) 315–320
[14] Ram Murty M, Oscillations of the Fourier coefficients of modular forms, *Math. Ann.*, **262** (1983) 431–446

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