Proc. Indian Acad. Sci. (Math. Sci.) (2018) 128:42
https://doi.org/10.1007/s12044-018-0422-3

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Certain Somos's $P-Q$ type Dedekind η-function identities

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MS received 29 July 2016; revised 1 February 2018; accepted 27 April 2018; published online 25 July 2018

Abstract. In this paper, we provide a new proof for the Dedekind η-function identities discovered by Somos. During this process, we found two new Dedekind η-function identities. Furthermore, we extract interesting partition identities from some of the η-function identities.

Keywords. Dedekind η-function; theta functions; modular equations; colored partitions.

2010 Mathematics Subject Classification. Primary: 11F20, 11B65, 11P83; Secondary: 14K25.

1. Introduction

Ramanujan [7,15] recorded 23 identities in his unorganized pages that involved the ratio of Dedekind η-function. They are proved by Berndt and Zhang [6] by utilizing Ramanujan's modular equations of various degree, mixed modular equations or through the hypothesis of modular forms. The other 14 identities of the comparative sort were found on page 55 of his lost notebook [16] and are demonstrated by Berndt [10] employing the above techniques. Berndt and Chan [9] and Berndt *et al.* [8] have employed some of the above mentioned $P-Q$ modular equations for the explicit evaluation of Rogers–Ramanujan continued fractions and Ramanujan–Weber class invariants. Influenced by their work, several new $P-Q$ η-function identities have been discovered and employed to find the explicit evaluation of continued fractions, class invariants and ratio of theta functions by numerous mathematicians. For the introduction, one can see [1–4,11,14,20–25]. Influenced by Ramanujan's work, Somos [17] used a computer to discover nearly 6200 Dedekind η-function identities. Somos formulated these identities after examining thousands of Dedekind η-function identities computationally. He ran the PARI/GP scripts to look at each identity to see if it is equivalent to an identity in $P-Q$ form without offering the proof. Recently, Yuttanan[27] proved some of these identities and established certain interesting partition identities for them. Furthermore, Vasuki and Veeresha [26] demonstrated Somos's η-function identities of level 14 and Srivatsa Kumar and Veeresha [18] have established certain partition identities for the same. Also, Srivatsa Kumar and Anuradha [19] have proved Somos's η-function identities of level 10 and established certain colored partition identities for them.

Motivated by the above work, in this paper we provide a simple proof of several Dedekind η-function identities discovered by Somos and other mathematicians. Section 2 records some preliminary results. In section 3, we prove $P-Q$ type Dedekind η-function identities. Theorems 3.1–3.10 are Somos's identities. Theorem 3.8 is also due to Vasuki and Kahtan [25]. Theorems 3.11–3.13 were due to Adiga *et al.* [1], Naika and Dharmendra [14] and, Vasuki and Srivatsa Kumar [22] respectively. In addition, two new identities, Theorem 3.14 and 3.15 are found. Furthermore, in section 4, by the notion of colored partitions, we are able to extract colored partitions for some of these identities.

2. Preliminaries

For $q = \exp(2\pi i\tau)$, the Dedekind η-function $\eta(\tau)$ is defined for $\mathrm{Im}(\tau) > 0$ by

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n).$$

Here and throughout this paper, we assume $|q| < 1$. As is customary, we define

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k) \qquad \text{and} \qquad (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

For $|ab| < 1$, Ramanujan's general theta-function $f(a,b)$ is given by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}.$$

Jacobi's triple product identity [5, p. 35] is given by

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.1}$$

The three most important special cases of $f(a,b)$ [5, p. 36] are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)^2_\infty(q^2;q^2)_\infty = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{2.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.3}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty := q^{-1/24}\eta(\tau). \tag{2.4}$$

Also after Ramanujan, define

$$\chi(q) := (-q; q^2)_\infty,$$

where the product representation in (2.2)–(2.4) follows from (2.1). For convenience, we denote $f(-q^n)$ by f_n for a positive integer n. It is easy to see that

$$\varphi(-q) = \frac{f_1^2}{f_2}, \qquad \psi(q) = \frac{f_2^2}{f_1}, \qquad \varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \qquad \psi(-q) = \frac{f_1 f_4}{f_2},$$
$$\chi(q) = \frac{f_2^2}{f_1 f_4}, \qquad \chi(-q) = \frac{f_1}{f_2} \qquad \text{and} \qquad f(q) = \frac{f_2^3}{f_1 f_4}. \tag{2.5}$$

Ramanujan recorded many modular equations in his notebooks [15,16] without offering the proof. We now define a modular equation as understood by Ramanujan. The complete elliptic integral of the first kind $K(k)$ is defined by

$$K := K(k) := \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{(n!)^2} k^{2n}$$
$$= \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right),$$

where $0 < k < 1$. The series representation is found by expanding the integrand in a binomial series and integrating termwise. Also ${}_2F_1$ is the basic hypergeometric function defined by

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n, \qquad 0 \le |x| < 1$$

where a, b and c are complex numbers such that c is not a nonpositive integer with $(a)_n := a(a+1)(a+2)\cdots(a+n-1)$ for $n \ge 1$. The number k is called the modulus of $K := K(k)$ and $k' := \sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with moduli k, k', l and l' respectively. Suppose the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{2.6}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is induced by (2.6). Ramanujan expressed his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. Then we say that β has degree n over α. The corresponding multiplier m is defined by $m = K/L$. If $q = \exp(-\pi K'/K)$, then one of the fundamental properties of elliptic function affirms that [5, p. 101]

$$\varphi^2(q) = \frac{2}{\pi} K(k) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right). \tag{2.7}$$

We conclude this section by recalling some formulae for φ, ψ at different arguments in terms of α, q and $z := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)$ by using (2.7) recorded by Ramanujan [5, pp. 122–124].

$$\varphi(q) := \sqrt{z}, \tag{2.8}$$
$$\varphi(-q) := \sqrt{z}(1-\alpha)^{1/4}, \tag{2.9}$$
$$\varphi(-q^2) := \sqrt{z}(1-\alpha)^{1/8}, \tag{2.10}$$
$$\varphi(q^2) := \sqrt{\frac{z}{2}}(1+\sqrt{1-\alpha})^{1/2}, \tag{2.11}$$
$$\psi(q) := \sqrt{\frac{z}{2}}\left(\alpha q^{-1}\right)^{1/8}, \tag{2.12}$$
$$\psi(-q) := \sqrt{\frac{z}{2}}\left(\alpha(1-\alpha)q^{-1}\right)^{1/8}, \tag{2.13}$$
$$\psi(q^2) := \frac{\sqrt{z}}{2}(\alpha q^{-1})^{1/4}. \tag{2.14}$$

Lemma 2.1. *We have*

$$\varphi(-q)\varphi(q) = \varphi^2(-q^2), \tag{2.15}$$
$$\varphi(q)\psi(q^2) = \psi^2(q), \tag{2.16}$$
$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2), \tag{2.17}$$
$$\varphi^2(q) - \varphi^2(-q) = 8q\psi^2(q^4). \tag{2.18}$$

The identities (2.15)−(2.18) are due to Ramanujan [15] and for proof, see [5].

Lemma 2.2. *We have*

$$\varphi^2(q) + \varphi^2(q^3) = 2\frac{\varphi(-q^2)\varphi(-q^3)\varphi(-q^6)}{\varphi(-q)}, \tag{2.19}$$
$$\varphi^2(q) - 3\varphi^2(q^3) = -2\frac{\varphi(q^3)\varphi(-q^2)}{\varphi(q)\varphi(-q^6)}, \tag{2.20}$$
$$\varphi^2(-q) - 5\varphi^2(-q^5) = -4f_2^2\frac{\chi(-q^5)}{\chi(-q)} \tag{2.21}$$

and

$$\psi^2(q) - 5q\psi^2(q^5) = f_1^2\frac{\chi(-q)}{\chi(-q^5)}. \tag{2.22}$$

The identities (2.19)−(2.22) are due to Vasuki *et al.* [24].

3. Somos's Dedekind η-function identities

Theorem 3.1. *If*

$$P := \frac{\varphi(q)}{\varphi(q^3)} \quad \textit{and} \quad Q := \frac{\psi(q^2)}{\psi(q^6)}$$

then

$$PQ - \frac{3q}{PQ} = \frac{Q}{P} + q\frac{P}{Q}.$$

Proof. It can be seen from (2.5) that

$$PQ = \frac{\psi^2(q)}{\psi^2(q^3)} \qquad \text{and} \qquad \frac{P}{Q} = \frac{\psi^2(q)\psi^2(q^6)}{\psi^2(q^2)\psi^2(q^3)}.$$

We need to show that

$$\psi^2(q^2)\left\{\frac{\psi^4(q)}{\psi^2(q^2)} - \frac{\psi^4(q^3)}{\psi^2(q^6)}\right\} - q\psi^2(q^6)\left\{3\frac{\psi^4(q^3)}{\psi^2(q^6)} + \frac{\psi^4(q)}{\psi^2(q^2)}\right\} = 0.$$

With the help of (2.16), the above can be rewritten as

$$\frac{\psi^2(q^2)}{q\psi^2(q^6)} - \frac{3\varphi^2(q^3)+\varphi^2(q)}{\varphi^2(q)-\varphi^2(q^3)} = 0. \tag{3.1}$$

Now from [5, p. 232, eq.(5.1)], we have

$$\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8} = \frac{m+1}{2}, \qquad \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8} = \frac{3-m}{2m},$$
$$\left(\frac{\alpha^3}{\beta}\right)^{1/8} = \frac{3+m}{2m} \qquad \text{and} \qquad \left(\frac{\beta^3}{\alpha}\right)^{1/8} = \frac{m-1}{2}. \tag{3.2}$$

where β has degree 3 over α and m is the multiplier. We deduce

$$\varphi^2(q) + 3\varphi^2(q^3) = 4\frac{\varphi(q^3)\psi^3(q)}{\varphi(q)\psi(q^3)} \tag{3.3}$$

and

$$\varphi^2(q) - \varphi^2(q^3) = 4q\frac{\psi^3(q^3)\varphi(q)}{\varphi(q^3)\psi(q)}. \tag{3.4}$$

by using (2.8) and (2.12) in the third and fourth terms of (3.2). Now (3.1) can be established easily by employing (3.3) and (3.4). □

Theorem 3.2. *If*

$$P := \frac{f^2(-q)}{f^2(q)} \qquad \textit{and} \qquad Q := \frac{f^2(q^2)}{q^{1/2}f^2(-q^8)},$$

then

$$PQ + \frac{4}{PQ} = \frac{Q}{P} - 4\frac{P}{Q}.$$

Proof. Employing (2.5), we find that

$$PQ = \frac{\varphi^4(-q^4)}{q^{1/2}\psi^4(q)} \qquad \text{and} \qquad \frac{P}{Q} = q^{1/2}\frac{\chi^4(-q)}{\chi^4(-q^4)}.$$

Substituting these in the desired equation, it suffices to prove that

$$\chi^4(-q^4)\varphi^4(-q^4)\{\psi^4(q)\chi^4(-q^4) - \chi^4(-q)\varphi^4(-q^4)\}$$
$$-4q\chi^4(-q)\psi^4(q)\{\psi^4(q)\chi^4(-q^4) + \chi^4(-q)\varphi^4(-q^4)\} = 0.$$

Also by using (2.5), one can easily see that

$$\chi^4(-q)\varphi^4(-q^4) = \psi^4(-q)\chi^4(-q^4).$$

On using this in the above, we deduce

$$\frac{\chi^4(-q^4)\varphi^4(-q^4)}{4q\chi^4(-q)\psi^4(q)} - \frac{\psi^4(q)+\psi^4(-q)}{\psi^4(q)-\psi^4(-q)} = 0. \tag{3.5}$$

Multiplying (2.17) and (2.18) by $\psi^2(q^2)$ and then using (2.16), we obtain

$$\psi^4(q) + \psi^4(-q) = 2\varphi^2(q^2)\psi^2(q^2) \tag{3.6}$$

and

$$\psi^4(q) - \psi^4(-q) = 8q\psi^2(q^2)\psi^2(q^4). \tag{3.7}$$

Employing (3.6) and (3.7) in (3.5) and further using (2.5), we readily arrive at (3.5). □

Theorem 3.3. *If*

$$P := \frac{\varphi(-q)}{\varphi(q^2)} \qquad \textit{and} \qquad Q := \frac{\varphi(q)}{q^{1/2}\psi(q^4)},$$

then

$$PQ + \frac{8}{PQ} = \frac{Q}{P} + 4\frac{P}{Q}.$$

Proof. From (2.15) and (2.16), we note that

$$PQ = \frac{\varphi^2(-q^2)}{q^{1/2}\psi^2(q^2)} \qquad \text{and} \qquad \frac{P}{Q} = q^{1/2}\frac{\varphi(-q)\psi(q^4)}{\varphi(q)\varphi(q^2)}.$$

Using these in the required equation and then employing (2.15) and (2.16), it suffices to prove that

$$\{\varphi^2(q) - 4q\psi^2(q^4)\}\{\varphi^2(-q) - \varphi^2(q^2)\} + 4q\psi^4(q^2) = 0. \tag{3.8}$$

Adding (2.17) and (2.18), we obtain

$$\varphi^2(q) - \varphi^2(q^2) = 4q\psi^2(q^4). \tag{3.9}$$

Changing q to $-q$ in the above, we get

$$\varphi^2(-q) - \varphi^2(q^2) = -4q\psi^2(q^4). \tag{3.10}$$

Employing (3.9) and (3.10) in (3.8) and also by using (2.16), we complete the proof. □

Theorem 3.4. *If*

$$P := \chi^3(q) \qquad \textit{and} \qquad Q := \chi^3(q^3),$$

then

$$PQ - \frac{8q}{PQ} = \left(\frac{Q}{P}\right)^2 + q\left(\frac{P}{Q}\right)^2.$$

Proof. Employing (2.5) in P and Q, we obtain

$$PQ = \frac{\varphi(q)\varphi(q^3)}{\psi(-q)\psi(-q^3)} \quad \text{and} \quad \frac{P}{Q} = \frac{\varphi(q)\psi(-q^3)}{\varphi(q^3)\psi(-q)}.$$

Using these in the desired equation, it suffices to prove that

$$\frac{\varphi^3(q^3)\psi(-q)}{q\varphi(q)\psi^3(-q^3)} - \frac{\varphi^3(q)\psi(-q^3) + 8\varphi(q^3)\psi^3(-q)}{\varphi^3(q)\psi(-q^3) - \varphi(q^3)\psi^3(-q)} = 0. \tag{3.11}$$

Now from (2.8) and (2.13), we have

$$\frac{\psi(-q)}{\varphi(q)} = \frac{1}{\sqrt{2}}\left(\frac{\alpha(1-\alpha)}{q}\right)^{1/8}$$

and

$$\frac{\psi^3(-q^3)}{\varphi^3(q^3)} = \frac{1}{2\sqrt{2}}\left(\frac{\beta(1-\beta)}{q^3}\right)^{3/8},$$

where α and β have degrees 1 and 3 respectively. Substituting the above two equations in (3.11) and then simplifying, we complete the proof. □

Theorem 3.5. *If*

$$P := \frac{f(q)}{f(-q)} \qquad \textit{and} \qquad Q := \frac{f(q^3)}{f(-q^3)},$$

then

$$2\left(PQ-\frac{1}{PQ}\right)=\left(\frac{P}{Q}\right)^2-\left(\frac{Q}{P}\right)^2.$$

Proof. From (2.5), one can see that

$$PQ=\frac{\varphi(-q^2)\varphi(-q^6)}{\varphi(-q)\varphi(-q^3)} \quad \text{and} \quad \frac{P}{Q}=\frac{\varphi(-q^2)\varphi(-q^3)}{\varphi(-q)\varphi(-q^6)}.$$

We need to show

$$\varphi^3(-q^2)\varphi(-q^3)+\varphi(-q^6)\varphi^3(-q)=0. \tag{3.12}$$

From (2.9) and (2.10), we have

$$\varphi(-q)=\sqrt{z_1}(1-\alpha)^{1/4}, \quad \varphi(-q^2)=\sqrt{z_1}(1-\alpha)^{1/8}$$

then

$$\varphi(-q^3)=\sqrt{z_3}(1-\beta)^{1/4}, \quad \varphi(-q^6)=\sqrt{z_3}(1-\beta)^{1/8}, \tag{3.13}$$

where β has degree 3 over α. Now using the above identities in (3.12), we deduce

$$z_1\sqrt{z_1z_3}\left\{\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8}+\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/4}\right\}=0.$$

On employing (3.2) in the above, we complete the proof. □

Theorem 3.6. *If*

$$P:=\frac{f(q)}{f(-q^4)} \quad \textit{and} \quad Q:=\frac{f(q^3)}{f(-q^{12})},$$

then

$$PQ-\frac{4q}{PQ}=\left(\frac{Q}{P}\right)^2-q\left(\frac{P}{Q}\right)^2.$$

Proof. It follows from (2.5) that

$$PQ=\frac{\psi(q)\psi(q^3)}{\psi(q^2)\psi(q^6)} \quad \text{and} \quad \frac{P}{Q}=\frac{\psi(q)\psi(q^6)}{\psi(q^2)\psi(q^3)}.$$

Now it suffices to prove that

$$\frac{\psi(q^2)\psi^3(q^3)}{q\psi(q)\psi^3(q^6)}-\frac{4\frac{\psi^3(q^2)}{\psi(q^6)}-\frac{\psi^3(q)}{\psi(q^3)}}{\frac{\psi^3(q)}{\psi(q^3)}-\frac{\psi^3(q^2)}{\psi(q^6)}}=0. \tag{3.14}$$

Employing (2.12) and (2.14) in (3.14) and then simplifying, we obtain

$$\frac{\psi(q^2)\psi^3(q^3)}{q\psi(q)\psi^3(q^6)} - \frac{4\frac{\psi^3(q^2)}{\psi(q^6)} - \frac{\psi^3(q)}{\psi(q^3)}}{\frac{\psi^3(q)}{\psi(q^3)} - \frac{\psi^3(q^2)}{\psi(q^6)}} = 2\left(\frac{\alpha}{\beta^3}\right)^{1/8} - \frac{2\left(\frac{\alpha^3}{\beta}\right)^{1/8} - 1}{1 - \frac{1}{2}\left(\frac{\alpha^3}{\beta}\right)^{1/8}}, \tag{3.15}$$

where β has degree 3 over α. Finally, using (3.2) in (3.15), we obtain (3.14). □

Theorem 3.7. *If*

$$P := \frac{\chi(-q^4)}{\chi(q)} \quad \textit{and} \quad Q := \frac{\chi(-q^{12})}{\chi(q^3)},$$

then

$$PQ + \frac{2q}{PQ} = \left(\frac{Q}{P}\right)^2 - q\left(\frac{P}{Q}\right)^2.$$

Proof. From (2.5), we note that

$$PQ = \frac{\varphi(-q^4)\varphi(-q^{12})}{\psi(q)\psi(q^3)} \quad \text{and} \quad \frac{P}{Q} = \frac{\varphi(-q^4)\psi(q^3)}{\varphi(-q^{12})\psi(q)}.$$

We need to prove that

$$\frac{\psi(q)}{q\psi^3(q^3)}\frac{\varphi(-q^{12})}{\varphi(-q^4)} - \frac{2\frac{\psi^3(q)}{\psi(q^3)} + \frac{\varphi^3(-q^4)}{\varphi(-q^{12})}}{\frac{\psi^3(q)}{\psi(q^3)} - \frac{\varphi^3(-q^4)}{\varphi(-q^{12})}} = 0. \tag{3.16}$$

Using (2.10) and (2.11) in (2.15), we have

$$\varphi(-q^4) = \sqrt{z}(1-\alpha)^{1/16}\left(\frac{1+\sqrt{1-\alpha}}{2}\right)^{1/4}.$$

Now employing (2.12) and the above in the left-hand side of (3.16), we obtain

$$\begin{aligned}
&\frac{\psi(q)}{q\psi^3(q^3)}\frac{\varphi(-q^{12})}{\varphi(-q^4)} - \frac{2\frac{\psi^3(q)}{\psi(q^3)} + \frac{\varphi^3(-q^4)}{\varphi(-q^{12})}}{\frac{\psi^3(q)}{\psi(q^3)} - \frac{\varphi^3(-q^4)}{\varphi(-q^{12})}} \\
&= \sqrt{2}\left(\frac{\alpha}{\beta^3}\right)^{1/8}\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/16}\left(\frac{(1+\sqrt{1-\beta})^3}{1+\sqrt{1-\beta}}\right)^{1/4} \\
&\quad - \frac{\sqrt{2}\left(\frac{\alpha^3}{\beta}\right)^{1/8}\left(\frac{1-\beta}{(1-\alpha)^3}\right)^{1/16} + \left(\frac{(1+\sqrt{1-\alpha})^3}{1+\sqrt{1-\beta}}\right)^{1/4}}{\frac{1}{\sqrt{2}}\left(\frac{\alpha^3}{\beta}\right)^{1/8}\left(\frac{1-\beta}{(1-\alpha)^3}\right)^{1/16} - \left(\frac{(1+\sqrt{1-\alpha})^3}{1+\sqrt{1-\beta}}\right)^{1/4}}.
\end{aligned} \tag{3.17}$$

Now, from [5, p. 233, eq. (5.2)], we have

$$\alpha = \frac{(m-1)(3+m)^3}{16m^3}, \quad \beta = \frac{(m-1)^3(m+3)}{16m}.$$

where β has degree 3 over α and m is the multiplier. Also it follows that

$$1+\sqrt{1-\alpha} = \frac{4m\sqrt{m}+\sqrt{16m^3-(m-1)(3+m)^3}}{4m\sqrt{m}} \tag{3.18}$$

and

$$1+\sqrt{1-\beta} = \frac{4\sqrt{m}+\sqrt{16m-(m-1)^3(3+m)}}{4\sqrt{m}}. \tag{3.19}$$

With the help of (3.18) and (3.19), (3.17) can be rewritten as

$$\begin{aligned}
&\frac{\psi(q)}{q\psi^3(q^3)}\frac{\varphi(-q^{12})}{\varphi(-q^4)} - \frac{2\frac{\psi^3(q)}{\psi(q^3)}+\frac{\varphi^3(-q^4)}{\varphi(-q^{12})}}{\frac{\psi^3(q)}{\psi(q^3)}-\frac{\varphi^3(-q^4)}{\varphi(-q^{12})}} \\
&= \frac{\sqrt{m+1}}{m-1}\left(\frac{\left(4\sqrt{m}+\sqrt{16m-(m-1)^3(3+m)}\right)^3}{4m\sqrt{m}+\sqrt{16m^3-(m-1)(3+m)^3}}\right)^{1/4} \\
&- \frac{3+m+\sqrt{3-m}\left(\frac{\left((4m\sqrt{m}+\sqrt{16m^3-(m-1)(3+m)^3}\right)^3}{16m^4(4m+\sqrt{16m-(m-1)^3(3+m)})}\right)^{1/4}}{\frac{3+m}{2}-\sqrt{3-m}\left(\frac{\left((4m\sqrt{m}+\sqrt{16m^3-(m-1)(3+m)^3})\right)^3}{16m^4(4m+\sqrt{16m-(m-1)^3(3+m)})}\right)^{1/4}}
\end{aligned}$$

which is valid only for $m = 3$. We thus complete the proof. □

Theorem 3.8 [25]. *If*

$$P := \frac{\varphi(-q)}{\varphi(-q^3)} \quad \textit{and} \quad Q := \frac{\varphi(q)}{\varphi(q^3)},$$

then

$$\frac{P}{Q}+\frac{Q}{P} = \frac{3}{PQ} - PQ.$$

Proof. The above identity is equivalent to

$$P^2(1+Q^2) = 3 - Q^2$$

Using (2.15), (2.19) and (2.20), we observe that

$$\begin{aligned}P^2(1+Q^2) &= \frac{\varphi^2(-q)\{\varphi^2(q)+\varphi^2(q^3)\}}{\varphi^2(-q^3)\varphi^2(q^3)}\\ &= \frac{2\varphi(-q)\varphi(-q^2)\varphi(-q^6)}{\varphi(-q^3)\varphi(q^3)\varphi(q^3)}\\ &= \frac{2\varphi(-q)\varphi(-q^2)}{\varphi(-q^6)\varphi(q^3)}\\ &= 3-Q^2.\end{aligned}$$

□

Theorem 3.9. *If*

$$P := \frac{\psi(q)}{\varphi(-q^8)} \quad \textit{and} \quad Q := \chi(-q)\chi(-q^8),$$

then

$$(PQ)^2 - \frac{4q}{(PQ)^2} = \frac{Q}{P} - 2q\left(\frac{P}{Q}\right).$$

Proof. From (2.5), we observe that

$$PQ = \frac{f_2}{f_8} \quad \text{and} \quad \frac{P}{Q} = \frac{f_2^3 f_{16}^2}{f_1^2 f_8^3}.$$

Using these in the desired equation, it suffices to prove that

$$\begin{aligned}A(q) &= \varphi(-q)\big\{f_2\psi^2(q) - f_8\varphi^2(-q^8)\big\}\\ &\quad -2q\psi(q^8)\big\{2f_8\varphi^2(-q^8) - f_2\psi^2(q)\big\} = 0.\end{aligned}$$

Employing (2.2), (2.3) and (2.4) in the above,

$$A(q) = q^4\big(2-6q-8q^2-6q^3-10q^4-18q^5+12q^6+\cdots\big).$$

By analytic continuation, $A(q)\to 0$ as $q\to 0$. Hence the proof. □

Theorem 3.10. *If*

$$P := \frac{f(-q)}{f(-q^2)} \quad \textit{and} \quad Q := \frac{f(-q^5)}{f(-q^{10})},$$

then

$$(PQ)^2 - \frac{4q}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 + q\left(\frac{P}{Q}\right)^3. \tag{3.20}$$

Proof. Employing (2.5) in (3.20) and after simplification, it is sufficient to prove that

$$\frac{\chi^5(-q)+4\chi(-q^5)}{\chi^5(-q)-\chi(-q^5)}+\frac{\chi^5(-q^5)}{q\chi(-q)}=0. \tag{3.21}$$

On employing (2.5) in (2.21) and (2.22), we deduce

$$\chi^5(-q)+4\chi(-q^5)=5\frac{f_5^2}{f_2^2}\chi^2(-q^5)\chi(-q) \tag{3.22}$$

and

$$\chi^5(-q)-\chi(-q^5)=-5q\frac{f_{10}^2}{f_1^2}\frac{\chi^4(-q)}{\chi(-q^5)}. \tag{3.23}$$

Using (3.22) and (3.23) in (3.21), we complete the proof. □

Theorem 3.11 **[1].** *If*

$$P:=\frac{\psi(-q)}{q^{1/4}\psi(-q^3)} \quad \text{and} \quad Q:=\frac{\varphi(q)}{\varphi(q^3)},$$

then

$$\left(\frac{P}{Q}\right)^2-\left(\frac{Q}{P}\right)^2=(PQ)^2-\frac{9}{(PQ)^2}.$$

Proof. The desired equation is equivalent to

$$P^4-Q^4=(PQ)^4-9.$$

Employing (2.5) in the above and after simplification, it is sufficient to prove that

$$\frac{\psi^4(-q)}{q\psi^4(-q^3)}-\frac{\varphi^4(q)-9\varphi^4(q^3)}{\varphi^4(q^3)-\varphi^4(q)}=0. \tag{3.24}$$

Now multiplying (2.19) and (3.4), we have

$$\varphi^4(q^3)-\varphi^4(q)=-8q\frac{\psi^3(q^3)\varphi(q)\varphi(-q^2)\varphi(-q^3)\varphi(-q^6)}{\psi(q)\varphi(-q)\varphi(q^3)}. \tag{3.25}$$

Also multiplying (2.20) and (3.3), we obtain

$$\varphi^4(q)-9\varphi^4(q^3)=-8\frac{\varphi^2(q^3)\varphi^3(-q^2)\psi^3(q)}{\varphi(-q^6)\varphi^2(q)\psi(q^3)}. \tag{3.26}$$

Employing (3.25) and (3.26) in (3.24) and then using the identity (2.16) repeatedly, we complete the proof. □

Theorem 3.12 **[14].** *If*

$$P := \frac{\varphi(-q)\varphi(-q^2)}{\varphi(q)\varphi(q^2)} \quad and \quad Q := \frac{\varphi(-q)\varphi(q^2)}{\varphi(q)\varphi(-q^2)},$$

then

$$\sqrt{PQ} + \frac{1}{\sqrt{PQ}} = 2\frac{Q}{P}.$$

Proof. It is easy to see that

$$PQ = \frac{\varphi^2(-q)}{\varphi^2(q)} \quad \text{and} \quad \frac{P}{Q} = \frac{\varphi^2(-q^2)}{\varphi^2(q^2)}.$$

Using (2.15) and (2.17), we observe that

$$\begin{aligned}\sqrt{PQ} + \frac{1}{\sqrt{PQ}} &= \frac{\varphi(-q)}{\varphi(q)} + \frac{\varphi(q)}{\varphi(-q)} \\ &= \frac{\varphi^2(-q) + \varphi^2(q)}{\varphi(q)\varphi(-q)} \\ &= 2\frac{Q}{P}.\end{aligned}$$

□

Theorem 3.13 **[22].** *If*

$$P := \frac{\varphi(-q)}{\varphi(q)} \quad and \quad Q := \frac{\varphi(-q^3)}{\varphi(q^3)},$$

then

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 = 4\left(PQ + \frac{1}{PQ}\right) - 6. \tag{3.27}$$

Proof. For $0 < t < 1$, set

$$(1-t)^{1/4} = \frac{\varphi(-q)}{\varphi(q)}. \tag{3.28}$$

Then

$$P = (1-\alpha)^{1/4} \quad \text{and} \quad Q = (1-\beta)^{1/4}.$$

or

$$\alpha = 1 - P^4 \quad \text{and} \quad \beta = 1 - Q^4. \tag{3.29}$$

From Entry 5(ii) of [5, p. 230], we have

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1, \tag{3.30}$$

where β has degrees 3 over α. Employing (3.29) in the above and then dividing throughout by $(PQ)^2$, we readily arrive at (3.27). □

Theorem 3.14. *If*

$$P := \frac{\varphi(-q)}{\varphi(q)} \quad \textit{and} \quad Q := \frac{\varphi(-q^7)}{\varphi(q^7)},$$

then

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 8\left((PQ)^{3/2} + \frac{1}{(PQ)^{3/2}}\right) + 28\left(PQ + \frac{1}{PQ}\right)$$
$$-56\left(\sqrt{PQ} + \frac{1}{\sqrt{PQ}}\right) + 70 = 0. \tag{3.31}$$

Proof. If β has degree 7 over α, then we have from Entry 19(i) of [5, p.314]

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1. \tag{3.32}$$

Now employing (3.28) and (3.29) in the above, we obtain

$$(1-P^4)(1-Q^4) = (1-\sqrt{PQ})^8.$$

On simplifying the above and then dividing throughout by $(PQ)^2$, we easily arrive at (3.31). □

Theorem 3.15. *If*

$$P := \frac{\psi(q)\psi(q^2)}{\psi(-q)\psi(-q^2)} \quad \textit{and} \quad Q := \frac{\psi(q)\psi(-q^2)}{\psi(-q)\psi(q^2)},$$

then

$$PQ + \frac{1}{PQ} = 2\,\frac{P^2}{Q^2}.$$

Proof. It is easy to see that

$$PQ = \frac{\psi^2(q)}{\psi^2(-q)} \quad \text{and} \quad \frac{P}{Q} = \frac{\psi^2(q^2)}{\psi^2(-q^2)}.$$

Using (3.6) and the identity $\psi(q)\psi(-q) = \psi(q^2)\varphi(-q^2)$, we have

$$PQ + \frac{1}{PQ} = \frac{\psi^4(q) + \psi^4(-q)}{\psi^2(q)\psi^2(-q)} = 2\,\frac{\varphi^2(q^2)}{\varphi^2(-q^2)} = 2\,\frac{\psi^4(q^2)}{\psi^4(-q^2)} = 2\,\frac{P^2}{Q^2}.$$

□

4. Application to the theory of partitions

The identities stated in section 3 have application to the theory of colored partitions. Haung [13] and Chen and Haung [12] introduced the concept of colored partitions to Göllnitz–Gordon functions. We demonstrate this by giving combinatorial interpretations for Theorem 3.11 and 3.15. For simplicity, we adopt the standard notation

$$(q_1, q_2, \ldots, q_n; q)_\infty := \prod_{j=1}^{n}(q_j; q)_\infty,$$

and define

$$(q^{r\pm}; q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty, \quad (r < s); r, s \in \mathbb{N}.$$

For example, $(q^{2\pm}; q^8)_\infty$ means $(q^2, q^6; q^8)_\infty$ which is $(q^2; q^8)_\infty\,(q^6; q^8)_\infty$.

A positive integer n has l colors if there are l copies of n available colors and all of them are viewed as distinct objects. Partitions of a positive integer into parts with colors are colored partitions.

For example, if 2 is allowed to have two colors, say b (black) and y (yellow), then the colored partitions of 4 are

$$4, 3+1, 2_y+2_b, 2_b+2_b, 2_y+2_y, 2_b+1+1, 2_y+1+1, 1+1+1+1.$$

An important fact is that, $(q^a; q^b)_\infty^{-k}$ is the generating function for the number of partitions of n, where all the parts are congruent to $a (\text{mod } b)$ having k colors.

Theorem 4.1. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to $\pm 2, \pm 4 (\text{mod } 12)$ with twelve and four colors respectively. Let $p_2(n)$ denote the number of partitions of n into parts congruent to $\pm 1, \pm 5 (\text{mod } 12)$ with twelve colors each, and $\pm 4 (\text{mod } 12)$ with four colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to $\pm 1, \pm 5 (\text{mod } 12)$ with eight colors each. Let $p_4(n)$ denote the number of partitions of n into parts congruent to $\pm 1, \pm 5 (\text{mod } 12)$ with four colors each and ± 2, $\pm 4 (\text{mod } 12)$ with twelve, eight colors respectively. Then for any positive integer $n \geq 1$, the following relation holds true*:

$$p_1(n) - p_2(n-1) = p_3(n) - 9p_4(n-1).$$

Proof. Rewriting the products of Theorem 3.11 subject to the common base q^{12}, we obtain

$$\frac{1}{(q_{12}^{2\pm}, q_4^{4\pm}; q^{12})_\infty} - \frac{q}{(q_{12}^{1\pm}, q_4^{4\pm}, q_{12}^{5\pm}; q^{12})_\infty} = \frac{1}{(q_8^{1\pm}, q_8^{5\pm}; q^{12})_\infty} - \frac{9q}{(q_4^{1\pm}, q_{12}^{2\pm}, q_8^{4\pm}, q_4^{5\pm}; q^{12})_\infty}.$$

The quotients of the above identity represent the generating functions for $p_1(n)$, $p_2(n)$, $p_3(n)$ and $p_4(n)$ respectively. Hence the above identity is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n - q\sum_{n=0}^{\infty} p_2(n)q^n = \sum_{n=0}^{\infty} p_3(n)q^n - 9q\sum_{n=0}^{\infty} p_4(n)q^n,$$

where we set $p_1(0) = p_2(0) = p_3(0) = p_4(0) = 1$. Now equating the coefficients of q^n on both sides, we lead to the desired result.
The following table verifies the case for $n = 2$ in the above theorem.

$p_1(2) = 12$:	$2_r, 2_w, 2_b, 2_g, 2_y, 2_o, 2_m, 2_p, 2_{gr}, 2_{bl}, 2_{br}, 2_i$
$p_2(1) = 12$:	$1_r, 1_w, 1_b, 1_g, 1_y, 1_o, 1_m, 1_p, 1_{gr}, 1_{bl}, 1_{br}, 1_i$
$p_3(2) = 36$:	$1_r + 1_r, 1_w + 1_w, 1_b + 1_b, 1_g + 1_g, 1_y + 1_y, 1_o + 1_o,$ $1_m + 1_m, 1_p + 1_p, 1_r + 1_w, 1_r + 1_b, 1_r + 1_g, 1_r + 1_y,$ $1_r + 1_o, 1_r + 1_m, 1_r + 1_p, 1_w + 1_b, 1_w + 1_g, 1_w + 1_y,$ $1_w + 1_o, 1_w + 1_m, 1_w + 1_p, 1_b + 1_g, 1_b + 1_y, 1_b + 1_o,$ $1_b + 1_m, 1_b + 1_p, 1_g + 1_y, 1_g + 1_o, 1_g + 1_m, 1_g + 1_p,$ $1_y + 1_o, 1_y + 1_m, 1_y + 1_p, 1_o + 1_m, 1_o + 1_p, 1_m + 1_p.$
$p_4(1) = 4$:	$1_r, 1_w, 1_b, 1_g$

□

Theorem 4.2. *Let* $p_1(n)$ *denote the number of partitions of n into parts congruent to* $\pm1, \pm3, +4 \pmod 8$ *with eight, eight and eighteen colors respectively. Let* $p_2(n)$ *denote the number of partitions of n into parts congruent to* $\pm2, +4 \pmod 8$ *with four and eighteen colors respectively. Let* $p_3(n)$ *denote the number of partitions of n into parts congruent to* $\pm1, \pm2, \pm3 \pmod 8$ *with four, ten and four colors respectively. Then for any positive integer* $n \geq 0$, *the following equality holds true*:

$$p_1(n) + p_2(n) = 2p_3(n).$$

Proof. On rewriting the products of Theorem 3.15 subject to the common base q^8, we obtain

$$\frac{1}{(q_8^{1\pm}, q_8^{3\pm}, q_{18}^{4+}; q^8)_\infty} + \frac{1}{(q_4^{2\pm}, q_{18}^{4+}; q^8)_\infty} = \frac{2}{(q_4^{1\pm}, q_{10}^{2\pm}, q_4^{3\pm}; q^8)_\infty}.$$

The quotients of the above identity represent the generating functions for $p_1(n)$, $p_2(n)$ and $p_3(n)$ respectively. Hence the above identity is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n + \sum_{n=0}^{\infty} p_2(n)q^n = 2\sum_{n=0}^{\infty} p_3(n)q^n,$$

where we set $p_1(0) = p_2(0) = p_3(0) = 1$. Equating the coefficients of q^n in the above, we lead to the desired result.

The following table verifies the case for $n = 2$ in the above theorem. □

$p_1(2) = 36$:	$1_r + 1_r, 1_w + 1_w, 1_b + 1_b, 1_g + 1_g, 1_y + 1_y, 1_o + 1_o,$ $1_m + 1_m, 1_p + 1_p, 1_r + 1_w, 1_r + 1_b, 1_r + 1_g, 1_r + 1_y,$ $1_r + 1_o, 1_r + 1_m, 1_r + 1_p, 1_w + 1_b, 1_w + 1_g, 1_w + 1_y,$ $1_w + 1_o, 1_w + 1_m, 1_w + 1_p, 1_b + 1_g, 1_b + 1_y, 1_b + 1_o,$ $1_b + 1_m, 1_b + 1_p, 1_g + 1_y, 1_g + 1_o, 1_g + 1_m, 1_g + 1_p,$ $1_y + 1_o, 1_y + 1_m, 1_y + 1_p, 1_o + 1_m, 1_o + 1_p, 1_m + 1_p.$
$p_2(2) = 4$:	$2_r, 2_w, 2_b, 2_g.$
$p_3(2) = 20$:	$1_r + 1_r, 1_w + 1_w, 1_b + 1_b, 1_g + 1_g, 1_r + 1_w, 1_r + 1_b,$ $1_r + 1_g, 1_w + 1_b, 1_w + 1_g, 1_b + 1_g, 2_r, 2_w, 2_g, 2_b, 2_y,$ $2_o, 2_m, 2_{bl}, 2_p, 2_i$

Acknowledgements

The authors are extremely grateful to the referee for numerous suggestions which have significantly improved the presentation of this work. The authors would also like to thank Dr. K R Vasuki, Department of Studies in Mathematics, Manasa Gangothri, University of Mysore, Mysore, India for his guidance during the preparation of this paper. The research of the first author is partially supported under Extra Mural Research Funding by SERB, a statutory body of DST, Government of India (File No. EMR/2016/001601).

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Communicating Editor: S D Adhikari