Proc. Indian Acad. Sci. (Math. Sci.) (2018) 128:41
https://doi.org/10.1007/s12044-018-0416-1

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On the diophantine equation $y^2 = \prod_{i \leq 8}(x + k_i)$

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MS received 11 October 2016; revised 24 November 2017; accepted 29 January 2018; published online 24 July 2018

Abstract. This paper improves the result of Tengely (*Periodica Math. Hung.*, **72(1)** (2016) 23–28).

Keyword. Diophantine equation.

2010 Mathematics Subject Classification. 11D41, 11D45.

1. Introduction

Consider the diophantine equation

$$Y^2 = \prod_{0 \leq i \leq 3} (X + i)(X + k + i), \tag{1}$$

where $k \in \mathbb{N}$, the set of all integers ≥ 1. In 2016, Tengely [4] showed that if $(x, y) \in \mathbb{N} \times \mathbb{N}$ is a solution of (1), then $x \leq 1.08k$. In this paper, we prove the following theorem. Define

$$x_i = (2 + i)\big(2k^4 + 5ik^2 + 2i^2\big)$$

and

$$\alpha_{i,\delta}(k) = \big(- (3ik + 2i - 2k^2) + \delta\sqrt{x_i}\big)/2i,$$

for any $i \in \mathbb{N}$ and $\delta \in \{1, -1\}$.

Theorem 1. *Let $r \geq 2$ be an integer and let $(x, y) \in \mathbb{N} \times \mathbb{N}$ be any solution of (1). If*

$$x \neq \big(- (k + 3) + ((k + 3)^2 + 4|\alpha_{i,\delta}(k)|)^{0.5}\big)/2,$$

for all $\delta \in \{1, -1\}$ and $i = 1, 2, \ldots, r - 1$, then

$$x \leq \big((2 + \sqrt{2r + 4})/2r + (2.7/k) + (1 + \sqrt{r + 2})/k^2\big)^{0.5} k.$$

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COROLLARY 1

Assume the hypothesis of Theorem 1 with $k^{0.8} \leq r < k^{0.8} + 1$. *Then we have* $x \leq 8k^{0.8}$. *Moreover, this bound improves the speed of the algorithm to have Theorem* 2 *of* [4].

In 2012, Srikanth and Subburam [2] provided an algorithm to find all integral solutions of a general diophantine equation of the type

$$y^p = f(x),$$

where x and y are unknown integers, p is prime and $f(x)$ is an integer polynomial. This algorithm can be restated as follows:

Theorem 2. *Let* p *be a prime number*, $B(x)$ *and* $C(x)$ *are non-zero rational polynomials with* $\deg(C(x)) < \deg(B(x))(p-1)$, l *a positive integer such that* $lB(x)$ *and* $l^pC(x)$ *have integer coefficients*, *for any non-negative integer* i *and* $\delta \in \{1, -1\}$

$$P_{i,\delta}(x) = \delta(lB(x) + \delta i)^p - \delta(lB(x))^p - \delta l^p C(x),$$

r *a positive integer such that* $P_{i,\delta}(x) = 0$ *has no integer roots for all* $\delta = 1, -1$ *and* $i = 0, 1, 2, \ldots, r-1$ *and*

$$H = \left\{\alpha \in \mathbb{R} : P_{r,1}(\alpha) = 0 \text{ or } P_{r,-1}(\alpha) = 0\right\},$$

where $\mathbb{R}$ *is the set of all real numbers. If* H *is empty, then the diophantine equation*

$$y^p = B(x)^p + C(x) \tag{2}$$

has no integer solution. Otherwise all integer solutions (x, y) *of the equation satisfy*

$$\min\ H \leq x \leq \max\ H.$$

A proof of this theorem has been presented in [2]. The method is: Assume that there is an integer solution (x, y) of equation (2) such that x is not in $[\min\ H, \max\ H]$. If $P_{r,1}(x) > 0$ and $P_{r,-1}(x) > 0$, then we have

$$(lB(x) - r)^p < (lB(x))^p + l^p C(x) < (lB(x) + r)^p.$$

So, $P_{i,\delta}(x) = 0$, for some $i < r$ and some $\delta = 1, -1$, since $(ly)^p = (lB(x))^p + l^p C(x)$. This is a contradiction. Similarly, we can deal with the case $P_{r,1}(x) < 0$ and $P_{r,-1}(x) < 0$. Also, the other two cases are impossible. This proves the theorem. The same theorem can be proved by the method of Srikanth and Subburam [1] also.

2. Proof of Theorem 1

Let (X, Y) be a positive integral solution of (1). Write (1) as

$$y^2 = B(x)^2 + C(x),$$

where $x = X^2 + (k+3)X$, $y = Y$, $B(x) = x^2 + (3k+2)x + (k^2 + 3k)$ and $C(x) = -4k^2x - (k^4 + 6k^3 + 9k^2)$. Take $l = 1$ in Theorem 2. So, $P_{i,\delta}(x) = 2ix^2 + (6ik + 4i + \delta 4k^2)x + \delta(k^4 + 6k^3 + (9 + \delta 2i)k^2 + \delta 6ik + i^2)$.

Since $\alpha_{i,1}(k)$ and $\alpha_{i,-1}(k)$ are the roots of $P_{i,-1}(x)$ and all the roots of $P_{i,1}(x)$ are negative, by Theorem 2, we have that if $X \in \mathbb{N}$ with

$$X^2 + (k+3)X \neq \alpha_{i,\delta}(k) \quad \text{and} \quad (k^2+6k+9)/4$$

for $\delta \in \{1, -1\}$ and $i = 1, 2, \ldots, r-1$, then

$$x \leq \max\{z \in \mathbb{R} : P_{r,1}(z) = 0 \text{ or } P_{r,-1}(z) = 0\} \leq \max\{\alpha_{r,1}(k), \alpha_{r,-1}(k)\}.$$

Therefore,

$$X^2 + (k+3)X \leq \max\{\alpha_{r,1}(k), \alpha_{r,-1}(k)\}.$$

It is clear that $X^2 + (k+3)X \neq (k^2+6k+9)/4$. This proves the theorem.

Acknowledgements

The first author (RS) gratefully acknowledges the TATA Realty and Infrastructure Limited for its financial support.

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Communicating Editor: B Sury