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Universal formulas for the number of partitions

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Abstract. In this paper, a formula that generalizes the total number of partitions of a natural number and the number of all possible decompositions of a certain number of parts can be united in the same formula. An advantage of this formula compared to similar ones is that it is given as a finite sum. Another advantage is that this amount may be expressed as a polynomial whose coefficients can be computed explicitly in an elementary form. The most important advantage of this approach is the fact that it is possible to express the results obtained in the general form, which so far in all similar attempts was not the case. From the general form we will prove as follows: (a) Partition functions, can be written with one fractal polynomial. (b) In partition functions, $p(n)$ is the first half of its coefficients with the highest degree which are in common with all unified polynomials that form it. (c) The remaining coefficients vary. The first variable coefficient can have two values; the next coefficient can have up to six values, etc. The variability of coefficients increases as the degree of polynomials decreases up to a free member whose variability is up to LCM $(2, 3, \ldots, n)$.

Keywords. Partitions function; the number of partitions; a partition class.

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1. Introduction

For determination of the number of partitions of any natural number, mathematicians most frequently use recurrent formulas, starting from the number one, with relatively quick access to each other. This paper gives a single formula determining both the number of partitions and the number of possible decompositions for any fixed number of parts, within partitioning of the arbitrary integer.

Number 5, for example, can be written as the sum of integers in the following ways:

$$5 = \begin{cases} 5 \\ 4+1 \\ 3+2 \\ 3+1+1 \\ 2+2+1 \\ 2+1+1+1 \\ 1+1+1+1+1 \end{cases}. \tag{1}$$

Among expressions on the right-hand side, sums with one, two, ..., five parts can be seen.

Let us denote $p(n)$ as the number of all possible partitions of the natural number n as the sum of other integers, sorted, for instance, in non-increasing order. Any such decomposition is called the partition of the number n.

The problem of finding the number of partitions of natural numbers is about 300 years old and the merit for the development of the term belongs to Leibniz. Euler gave the first valuable results in that field and one of them is a recurrent formula [4]:

$$p(n) = \sum_{k=1}^{n} (-1)^{k+1} \left(p\left(n - \frac{1}{2}k(3k-1)\right) + p\left(n - \frac{1}{2}k(3k+1)\right)\right),$$

which can compute values of all partitions.

Andrews [1] in his book, after providing this formula says: "No one has ever found a more efficient algorithm for computing $p(n)$. It computes a full table of values of $p(n)$ for $n > 5$, in time $O(n^{\frac{3}{2}})$."

Formulas given here can converge to $p(n)$ at an arbitrary rate. Formulas are given as finite multiplicative sums. The amount of necessary computations can be gradually shortened by increasing the speed of convergence.

One possible classification of partitions is their distinction by the number of parts they contain. In this sense, let us denote with $p(n, m)$ the number of all possible partitions of the number n having exactly m parts, $(1 \leq m \leq n)$, namely the number of partitions of the number n of class m, such that

$$n = n_1 + n_2 + \cdots + n_m, \text{ where } n_1 \geq n_2 \geq \cdots \geq n_m \geq 1.$$

It is clear that $p(n) = \sum_{k=1}^{n} p(n, k)$, i.e., if the number of partitions is determined for all of the possible number of classes, the total number of partitions is then the sum of all these values.

Note that all partitions can be arranged in a series of alphanumeric order – commutative property of addition is not considered here. Greater addends are added first, calculating the sum from left to right. The class with fewer addends is placed before the one with more addends. For instance, $3 + 2$ partition is placed before $3 + 1 + 1$, and so on. In this way, the resulting formula is not only a unique formula for the decomposition of some classes, but also the formula deriving the total number of partitions of any integer. Later, we will show that the number of partitions of a general formula class can also determine the total number of partitions.

From the theorem of vertical k-collecting, it is possible to prove all other statements in this work. In addition, some other assertions are already known, but the evidence for all these assertions is original. In addition to the evidence, the statements of the application in this work are also original.

The functions which compute the restriction of the partitions (classes of the partitions) appear in other places, such as in [8] in the form

$$p(n, 1) = 1; \quad p(n, 2) = \frac{1}{4}\left(2n - 1 + (-1)^n\right);$$

$$p(n, 3) = \frac{1}{72}\left(6n^2 - 7 - 9(-1)^n + 16\cos\left(\frac{2}{3}n\pi\right)\right);$$

$$p(n, 4) = \frac{1}{864}\left(3(n+1)\left(2n(n+2) - 13 + 9(-1)^n\right) - 96\cos\left(\frac{2}{3}n\pi\right) + 108(-1)^{\frac{n}{2}} \bmod (n+1, 2) + 32\sqrt{3}\sin\left(\frac{2}{3}n\pi\right)\right).$$

The biggest problem with this one and all similar formulas is the fact that it is not clear how the string will continue.

2. Determining the number of partitions of natural numbers *n*

Let us determine the values $p(n, 1)$, $p(n, 2), \ldots, p(n, n)$. Some of these are trivially determined, such as

$$p\,(n, 1) = 1, \quad p\,(n, n) = 1, \quad n \in \mathbb{N},$$
$$p\,(n, n-1) = 1, \quad n > 1, \ldots.$$

Now, let us determine the number of partitions of natural numbers n of class 2. We need to find a number of pairs: $(n-1)+1$, $(n-2)+2$, $(n-3)+3, \ldots$. Those numbers can be written as the whole half of n. Hence, the total number of partitions for the second class equals $p\,(n, 2) = \left[\frac{n}{2}\right]$.

In order to calculate $p\,(n, 3)$, $(n > 2)$, let us perform the systematization of all possible decompositions as follows. Start with partitions of the 3rd class where the last part is 1; they are of the form $(n-l-1)+l+1$, where $1 \le l \le \left[\frac{n-1}{2}\right]$ and there are $p\,(n-1, 2) = \left[\frac{n-1}{2}\right]$ of these.

The partitions of the third class, where the last part is 2 have the form $(n-l-2)+l+2$, where $2 \le l \le \left[\frac{n-4}{2}\right]$ have partition numbers of class 2 exactly of the form $p\,(n-4, 2) = \left[\frac{n-4}{2}\right]$. Continuing in this way, the generalization of all possible decompositions $p(n, 3)$ can be derived as follows:

$$\begin{aligned} p\,(n, 3) &= p\,(n-1, 2) + p\,(n-4, 2) + \cdots \\ &= \sum\nolimits_{i \ge 0} p\,(n-1-3i, 2)\,, \quad n > 2. \end{aligned}$$

Summing is performed while the difference $(n-1-3i)$ is positive. Therefore, we will not explicitly specify the upper limit index from this point onward, although this is possible. Thus, in $p(n, 3)$ it is sufficient to perform the addition with i from $i = 0$ to $i = \left[\frac{n-1}{3}\right]$. (In general, addition would be performed with i from $i = 0$ to $i = \left[\frac{n-1}{m}\right]$, for determining the number of partitions that begin with m.)

Using the previous procedure, we obtained the sum

$$\begin{aligned} p\,(n, 3) &= \left[\frac{n-1}{2}\right] + \left[\frac{n-4}{2}\right] + \cdots + \left[\frac{n-1-3m}{m}\right] \\ &\quad + \cdots = \sum\nolimits_{m \ge 0} \left[\frac{n-1-3m}{2}\right]. \end{aligned} \tag{2}$$

Similarly, differentiating the partitions ending with 1, 2,. . ., we make a conclusion for $p(n, 4)$. The symbol $p_{\ldots+f}\,(n, k)$ represents the number of partitions of $p\,(n, k)$ ending with f. With 1 ending as many partitions of the fourth class of the number n, as there are partitions of the third class of the number $(n-1)$. Therefore,

$$p_{\ldots+1}\,(n, 4) = p\,(n-1, 3) = \sum\nolimits_{i \ge 0} \left[\frac{n-2-3i}{2}\right].$$

Now, that number should be added to the number of partitions ending with 2, and as earlier, there are

$$p_{\ldots+2}\,(n, 4) = p\,(n-5, 3) = \sum\nolimits_{i \ge 0} \left[\frac{n-6-3i}{2}\right].$$

Similarly, we conclude that the number of partitions of the fourth class ending with j, for the number n is

$$p_{\dots+j}(n,4) = p(n-1-4j,3) = \sum_{i\geq 0}\left[\frac{n-2-4j-3i}{2}\right].$$

By combining the previous conclusions, the total number of partitions of the fourth class of number n is equal to

$$\begin{aligned} p(n,4) &= p(n-1,3) + p(n-5,3) + p(n-9,3) \\ &\quad + \cdots = \sum_{i\geq 0}\left[\frac{n-2-3i}{2}\right] + \sum_{i\geq 0}\left[\frac{n-6-3i}{2}\right] \\ &\quad + \sum_{i\geq 0}\left[\frac{n-10-3i}{2}\right] + \cdots . \end{aligned}$$

Relying on (2) the last equality can be rewritten as

$$p(n,4) = \sum_{j\geq 0} p(n-1-4j,3) = \sum_{j\geq 0}\sum_{i\geq 0}\left[\frac{n-2-4j-3i}{2}\right].$$

Now we are able to generalize the results. To determine $p(n,m)$, it is sufficient to determine all $p(n-1,m-1), p(n-1-m,m-1), p(n-1-2m,m-1), \dots$, while $n-k\cdot m-1\geq 0$, $k\in\mathbb{N}$. Summing all these values we obtain the required value. This property enables calculating the number of partitions of the k-th class of the number n if the number of partitions of the $(k-1)$-th class of the number$n$ is given by the recurrent formula. Thus we introduce the following theorem about a vertical k-collecting.

Theorem 1 **(A vertical *k*-collecting).** *Let $n, m\in\mathbb{N}$. For $n\geq m\geq 1$, it is true that*

$$\begin{aligned} p(n,m) &= p(n-1,m-1) + p(n-1-m,m-1) + \cdots \\ &= \sum_{j\geq 0} p(n-1-j\cdot m, m-1). \end{aligned} \tag{3}$$

Proof. Let n and m be given such that $n\geq m\geq 1$. There is a natural number k and non-negative integer number r, $0\leq r<m$ such that $n=k\cdot m+r$. Suppose that all the partitions of the number n of the m-th class are placed in the matrix of size $p(n,m)\times m$. Then sort the rows of the matrix so that the first rows end with elements 1, then with 2,..., and last with m. The first row is consisted of the elements of the partition:

$$(n-m+1) + \underbrace{1+1+\cdots+1}_{m-1},$$

while the last row consists of the elements of the partition

$$\underbrace{\underbrace{(k+1)+\cdots+(k+1)}_{r} + k + k + \cdots + k}_{m}.$$

Now, we recount the number of rows that have the same last part, to determine the exact size of the $p(n,m)$.

In the rows ending with 1 (delete all of 'end part'), obtained rows actually become all partitions of the $(m-1)$-th class of the number $(n-1)$ and their total number is $p\,(n-1, m-1)$.

If $k=1$, Theorem 1 is proved. If $k>1$, then there are rows ending with the number 2. In the rows ending with 2 decrease all addends by 1 in each of the rows, and then apply the previous procedure. We remove the last part, which is also the smallest part. Note that all rows produced in this way are derived from the partitions of the $(m-1)$-th class of the number $(n-1-m)$, so their total number equals $p\,(n-1-m, m-1)$.

If $k=2$, the proof is completed. If $k>2$, we repeat the previous procedure in the following manner. There are rows that have the last part equal to 3. Let us decrease each element in every such row by 2 and thus, we deduce that their number is $p\,(n-1-2m, m-1)$.

The previous procedure can be used exactly as $\left[\frac{n}{m}\right]=k$. If we encompass all the conclusions obtained, we get the statement in (3). □

The following theorem is already known [6], and we state it because it is simply proved by Theorem 1 and will be used in future work.

Theorem 2 (A horizontal total collecting). *For the classes of partition* $p(n,m)$, $n,m\in\mathbb{N}$, *the following is valid*:

$$p\,(n+m, m) = p\,(n,m) + p\,(n, m-1) + \cdots + p\,(n,2) + p\,(n,1)\,. \tag{4}$$

Proof. Let n be an arbitrary natural number greater than 1. We will prove Theorem 2, using mathematical induction, that for every $m\geq 1$, assertion (4) is valid.

For $m=1$, Theorem 1 is obviously true because $p\,(n+1,1)=p\,(n,1)=1$, for each $n\in\mathbb{N}$. Suppose the assertion (4) is true for $m=k$, i.e. the following is valid:

$$p\,(n+k, k) = p\,(n,k) + p\,(n, k-1) + \cdots + p\,(n,2) + p\,(n,1)\,.$$

Let us prove that the assertion (4) for $m=k+1$, i.e.,

$$p\,(n+k+1, k+1) = p\,(n,k+1) + p\,(n, k) + \cdots + p\,(n,2) + p\,(n,1)\,.$$

Applying Theorem 1 on the left-hand side of the previous equality, we have

$$\begin{aligned} &p(n+k+1, k+1)\\ &\quad = p(n+k,k) + \underbrace{p\,(n-1,k) + p(n-k-2,k) + p(n-2k-3,k) + \cdots}_{p(n,k+1).}. \end{aligned} \tag{5}$$

Applying Theorem 1 to the selected term on the right-hand side in (5), we conclude as follows:

$$p\,(n+k+1, k+1) = p\,(n+k,k) + p\,(n,k+1)\,.$$

Based on (5) and regrouping in the last equality, we obtain

$$p\,(n+k+1, k+1) = p\,(n,k+1) + p\,(n, k) + \cdots + p\,(n,2) + p\,(n,1)\,,$$

that has to be proved. □

The following consequences are already known [6] and we are referring to them because they are very important for this paper.

Consequence 1. The total number of partitions of number n is equal to the number of partitions of the same class twice that number, i.e., $p\,(n) = p\,(2n, n)$.

The proof of this fact is a direct consequence of Theorem 2. It is enough to observe that

$$p\,(2n, n) = p\,(n + n, n) = p\,(n, n) + p\,(n, n-1) + \cdots + p\,(n, 1) = p\,(n)\,.$$

Consequence 2. For each $m \in \mathbb{N}$ such that $m < n \leq 2m$, it is true that $p(n-m) = p(n, m)$. To prove, apply Theorem 2 on $p(n, m) = p((n-m) + m,\ n)$,

$$\begin{aligned} p(n-m+m,\ m) &= p(n-m,\ m) + p(n-m,\ m-1) + \cdots + p(n-m,\ 1) \\ &= 0 + \cdots + p(n-m,\ n-m) + \cdots +\ p(n-m,\ 1) \\ &= p(n-m). \end{aligned}$$

From Consequence 1 it follows that if we find the general form of the function that calculates the values of the k-th class, it would be possible to determine the general form of the $p\,(n)$. We are not interested in the general form of all $p\,(n, k)$, but only those for which $k \leq \left[\frac{n}{2}\right]$. The classes for which $k > \left[\frac{n}{2}\right]$ in Consequence 2 have the general form of the function $p\,(n-k)$ which requires a step more, since it would be additionally necessary to introduce a substitution of $n-k$ with m afterwards in the acquired form. Therefore, in future, it will always be assumed that $k \leq \left[\frac{n}{2}\right]$.

2.1 *The general case of $p(n, k)$*

Here we derive the conclusion that $p(n, m)$ can be recursively expressed by $p(n, 2)$. In this regard, the following theorem is true.

Theorem 3. *The number of partitions of the number n of the m-th class $(n \geq m \geq 2)$ is given by*

$$\begin{aligned} p\,(n, m) = {} & \sum\nolimits_{j_1 \geq 0} \sum\nolimits_{j_2 \geq 0} \cdots \sum\nolimits_{j_{m-3} \geq 0} \sum\nolimits_{j_{m-2} \geq 0} \\ & \times \left[\frac{n-(m-2)-m \cdot j_1 - (m-1)\,j_2 - \cdots - 4\,j_{m-3} - 3\,j_{m-2}}{2}\right]. \end{aligned} \tag{6}$$

The addition in (6) *is performed only for those indices $j_1, j_2, \ldots, j_{m-2}$ for which the value of the integer part is positive. Only the first class, whose value is always* 1, *is not included with* (6).

Proof. Using the property of 'a vertical k-collecting' of all classes of partitions (proved by the Theorem 1), it follows that

$$p\,(n, m) = \sum\nolimits_{j_1 \geq 0} p\,(n - 1 - m \cdot j_1, m - 1)\,. \tag{7}$$

The resulting relation shows that the total number of partitions of the m-th class of the number n can be calculated from the number of partitions of some elements in the preceding class. Therefore, to calculate the order of any class of the number of partitions of the number n, it is necessary to continue this procedure, stepping back to the previous class, until we reach the second class.

The result that we have obtained in Theorem 1 can be transformed, calculating the values of the $(m-1)$-th class from the previous class values, as follows:

$$\begin{aligned} p\,(n,m) &= \sum\nolimits_{j_1 \geq 0} p\,(n-1-m\cdot j_1, m-1) \\ &= \sum\nolimits_{j_1 \geq 0} \sum\nolimits_{j_2 \geq 0} p\,(n-2-m\cdot j_1-(m-1)\,j_2, m-2)\,. \end{aligned}$$

We extend this to the recurrent relation, until we reach $m = 2$ or calculation of $p(n, 2)$ where we deduce (7).

In the previous formula, we have descended to the value of the partitions of the second class. All that remains is to combine the results. The formulas, when combined together, make the total number of partitions of the number n $(n \geq 1)$ only with values of the second class. This is to be written as

$$\begin{aligned} p\,(n) &= \sum\nolimits_{k=1}^{n} p\,(n,k) = 1 + \sum\nolimits_{k=2}^{n} \left(\sum\nolimits_{j_{k-2} \geq 0} \sum\nolimits_{j_{k-3} \geq 0} \cdots \sum\nolimits_{j_1 \geq 0} \right. \\ &\quad \left. \times \left[\frac{n-(k-2)-k\cdot j_1-(k-1)\,j_2-\cdots-3\,j_{k-2}}{2} \right] \right). \end{aligned} \tag{8}$$

Some of the sum-operators in (8) do not have upper limits, only because positive values should be added, and upper boundaries are not important. □

Let us use Consequence 1 to obtain a somewhat simplified formula for calculating the number of the partitions of every natural number. Having this as an objective, let us discuss values of the classes having the greatest values in (7). Substituting $m = n, m = n-1, \ldots,$ where $m \in \mathbb{N}$, $m \geq \left[\frac{n}{2}\right]$, which holds for $p\,(n,m) = p\,(n-m)$, we find that

$$\begin{aligned} p\,(n,n) = p\,(0) &= \sum\nolimits_{j_1 \geq 0} \sum\nolimits_{j_2 \geq 0} \cdots \sum\nolimits_{j_{n-3} \geq 0} \sum\nolimits_{j_{n-2} \geq 0} \\ &\quad \times \left[\frac{2-n\cdot j_1-(n-1)\,j_2-\cdots-4\,j_{n-3}-3\,j_{n-2}}{2} \right] = 1, \\ p\,(n,n-1) = p\,(1) &= \sum\nolimits_{j_1 \geq 0} \sum\nolimits_{j_2 \geq 0} \cdots \sum\nolimits_{j_{n-3} \geq 0} \sum\nolimits_{j_{n-2} \geq 0} \\ &\quad \times \left[\frac{3-n\cdot j_1-(n-1)\,j_2-\cdots-4\,j_{n-3}-3\,j_{n-2}}{2} \right] = 1. \end{aligned}$$

If we use the result of Consequence 1, we obtain the general formula

$$\begin{aligned} p\,(2n,n) = p\,(n) &= \sum\nolimits_{j_1 \geq 0} \sum\nolimits_{j_2 \geq 0} \cdots \sum\nolimits_{j_{n-3} \geq 0} \sum\nolimits_{j_{n-2} \geq 0} \\ &\quad \times \left[\frac{n+2-n\cdot j_1-(n-1)\,j_2-\cdots-4\,j_{n-3}-3\,j_{n-2}}{2} \right]. \end{aligned} \tag{9}$$

Remark 1. The formula (9) could be transformed to the values of partitions of the first class. Then (9) takes the form

$$p(n) = \sum_{k=1}^{n} \sum_{j_{k-1}\geq 0} \sum_{j_{k-2}\geq 0} \cdots \sum_{j_2\geq 0} \sum_{j_1\geq 0} sgn\,(n+1-k\cdot j_1-(k-1)\,j_2 - \cdots - 3\,j_{k-2} - 2\,j_{k-1}). \tag{10}$$

The proof is in the fact that the theorem of k-collecting applies on all the classes. The value of $p(n)$ is calculated according to the following properties: The value of $p(n)$ is a sum $\sum_{k=1}^{n} p(n,k)$. Calculating the value of $p(n,k)$ has two essential steps. (1) The value in each class $p(n,k)$ are expressed using the value of the first class (it provides an expression: $n+1-k\cdots j_1-(k-1)\,j_2-\cdots-3\,j_{k-2}-2\,j_{k-1}$) and (2) determine these values (in the first class all values are one, it provides the function sgn(x)) and all of the positive values add up.

The obtained formulas (8), (9) and (10) are explicit universal formulas that calculate the number of all partitions of the arbitrary natural number n and the values of all classes relevant to it. Although (8), (9) and (10) appear complicated, it is possible to find the general form of the results.

3. Fractal form of partitions function of classes

Many attempts were made to determine the functions by which the number of partitions of the number n was calculated up to k parts as well as to exactly k parts. For example, Ekhad [2] calculated all the polynomials required to calculate the number of the partition for a number n at most k parts, until $k = 60$. The results given in [2] are not unified in a unique form and it is not clear whether this is possible. More importantly, it is unclear if there is a general form and how to get there. We show that values in each class are calculated with only one fractal polynomial. Then, the process by which we calculate fractal polynomials for the first few classes allow inferences about the general form of the fractal polynomial arbitrary class and enables finding the general form of its coefficients, which was not possible even in [2] nor in [3] or [5].

We will prove that for class k, it takes at most the LCM $(2, 3, \ldots, k)$ different polynomials for calculating its values. The degree of each polynomial is $k-1$. The first $k-\left[\frac{k}{2}\right]-1$ coefficients (with the highest degrees) are in common. The coefficients in the remaining degrees will vary. The coefficient with the higher degree of the variable pary has two values, the next six, and so on, until the free member has a variability up to LCM $(2, 3, \ldots, k)$.

3.1 *The first few partitions function of classes*

The number of partitions of all classes and the total number of partitions of all natural numbers can be explicitly expressed in a polynomial form. One of the methods for obtaining the general form for partitions function of classes is to know Theorem 3, and also the generally known Lemma A.

Lemma A. Let $S_n^j = 1^j + 2^j + \cdots + n^j$ denote the sum of the j-th degrees of all the natural numbers from 1 to n, where $j \in \mathbb{N}$. Then the sums $\{S_n^j\}$, $1 \leq j \leq m, m \in N$ satisfy the recurrence formula

$$S_n^m = \frac{(n+1)^{m+1} - (n+1)}{m+1} - \frac{1}{2}\binom{m}{1} S_n^{m-1} - \frac{1}{3}\binom{m}{2} S_n^{m-2} - \frac{1}{m}\binom{m}{m} S_n^1. \tag{13}$$

Proof. Let us start with the following equalities:

$$\begin{aligned}
(1+1)^{m+1} - 1 &= \binom{m+1}{1} 1^m + \binom{m+1}{2} 1^{m-1} + \cdots \\
&\quad + \binom{m+1}{m} 1 + 1,\ (2+1)^{m+1} - 2^{m+1} = \binom{m+1}{1} 2^m \\
&\quad + \binom{m+1}{2} 2^{m-1}, + \cdots + \binom{m+1}{m} 2 + 1, \\
&\quad \cdots \\
&\quad (n+1)^{m+1} - n^{m+1} = \binom{m+1}{1} n^m + \binom{m+1}{2} n^{m-1} \\
&\qquad + \cdots + \binom{m+1}{m} n + 1.
\end{aligned}$$

If we add up all of the previous equations and introduce a notation for the sum of all integers of the same degree, as in Lemma A, we obtain the specified recurrent relation (13).

It is now possible to prove the accuracy of the following lemmas. Only the first lemma is commonly known in the given form. Others, even when given, are not given in the form as in this paper. □

Lemma 1. *The number of partitions of the third class of the number* $n \in \mathbb{N}$*, is calculated by the fractal polynomial*

$$p(n,3) = \frac{n^2 + w_j}{12}, \quad w_j \in \{0, -1, -4, 3, -4, -1\}, \quad n \equiv j \mod 6. \tag{11}$$

The values for w_j are listed in order from $j = 0$ to $j = 5$, as is the case in all of the following examples.

Proof. To determine the number of partitions for the third class, four different polynomials of the second degree must be used. For the explicit computation of the formula (7), the following cases should be differentiated: $n = 6m$, $n = 6m + 1$, $n = 6m + 2$, $n = 6m + 3$, $n = 6m + 4$, $n = 6m + 5$.

Let $n = 6m$. In this case (7) can be written as

$$\begin{aligned}
&\left[\frac{n-1}{2}\right] + \left[\frac{n-4}{2}\right] + \left[\frac{n-7}{2}\right] + \cdots = (3m-1) + (3m-2) \\
&+ (3m-4) + \cdots = (1 + 2 + \cdots + (3m-1)) \\
&- (3 + 6 + \cdots + (3m-3)) = 3m^2 = \frac{n^2}{12}.
\end{aligned}$$

In other cases, we would proceed in a similar manner. If we combine the results, we get (11), which has to be proved. □

Let us make the fractal polynomial for $p\,(n, 4)$. In this case, to calculate the number of partitions of the fourth class, it is necessary to use nine different polynomials of degree three.

Lemma 2. *The number of partitions of the fourth class of the number $n \in \mathbb{N}$, is calculated using the fractal polynomial*

$$p\,(n, 4) = \frac{1}{144}n^3 + \frac{1}{48}n^2 + \begin{cases} \frac{w_j}{144}, n\ even, \\ -\frac{1}{16}n + \frac{w_j}{144}, n\ odd, \end{cases} \quad n \equiv j \bmod 12, \quad (12)$$

$w_j \in \{0, 5, -20, -27, 32, -11, -36, 5, 16, -27, -4, -11\}$.

Proof. Let us start with the sum (7), which can be rewritten in the following form:

$$p\,(n, 4) = p\,(n-1, 3) + p\,(n-5, 3) + p\,(n-9, 3) + \cdots .$$

In order to calculate the sum on the right-hand side, it is necessary to distinguish remainders of the division of number n by 12. Because of the length of the recorded evidence, we shall consider only one branch.

Suppose that $n = 12m + 1$ (other eleven cases are analogous). In this case, according to Lemma 1, the following is true:

$$\begin{aligned} &p\,(12m+1, 4) \\ &\quad = \frac{(12m)^2 - 4 + (12m-4)^2 - 4 + (12m-8)^2 - 4 + \cdots + 8^2 - 4 + 4^2 - 4}{12} \\ &\quad = 12m^3 + 6m^2. \end{aligned}$$

If it is written by the number n,

$$p\,(n, 4) = \frac{n^3 + 3n^2 - 9n + 5}{144}, \quad n = 12m + 1.$$

In a similar way, to distinguish between the twelve cases of n being divided by 12, the formula $p(n, 4)$ can be used to express all other cases. It is possible to combine all the obtained cases in one formula. This proves (12). □

All of this can be expressed for the partition of the fifth class. By analysis as in the previous case, starting from $p\,(n, 5) = \sum_{i\geq 0} p\,(n - 1 - 5i, 4)$, the sum of the values of all non-negative expressions in brackets can be obtained. Through the previous procedure, the sum of the cubes of every fifth integer should be found. In addition, one must distinguish 60 cases, due to the divisibility of the number n by 60. It can also be shown that all cases can be combined with the following assertion.

Lemma 3. *The number of partitions of the fifth class is calculated using the fractal polynomial*

$$p(n,5) = \frac{1}{2880}n^4 + \frac{1}{288}n^3 + \frac{1}{288}n^2 + \begin{cases} -\frac{1}{96}n + \frac{w_j}{2880}, n\ odd \\ -\frac{1}{24}n + \frac{w_j}{2880}, n\ even \end{cases} \quad n \equiv j \bmod 60.$$

Here w_j are the numeric constants that can be precisely determined. In this case, these constants are the following numbers respectively: 0, 9, 104, −351, −576, 905, −216, −351, −256, 9, 360, −31, −576, 9, 104, 225, −576, 329, −216, −351, 320, 9, −216, −31, −576, 585, 104, −351, −576, 329, 360, −351, −256, 9, −216, 545, −576, 9, 104, −351, 0, 329, −216, −351, −256, 585, −216, −31, −576, 9, 680, −351, −576, 329, −216, 225, −256, 9, −216, −31.

They correspond to the remainders from 0 to 59 produced when dividing the number n by the number 60.

Proof. Given these cases there are exactly 18 mutually different polynomials of the fourth degree which determine the values of $p(n,5)$ (fifth class of partitions) for all natural numbers.

The proof is completely analogous to the previous two. In this case, it is necessary to consider 60 different cases. In the process of summing, the following sums appear: the fourth, third and second degrees as well as the sum of odd integers. In all these cases, one should find the sum of each: fifth, fourth and third degree, and squares from 1 to $n-1$. That way, the sum is considered separately for each case of n given its divisibility by 60. For example, for $n = 60m + 1$, we have

$$p(n,5) = p(60m,4) + p(55m,4) + \cdots + p(0,4).$$

The resulting sum can be calculated without distinguishing even and odd numbers n. Since only odd members have a linear sum, the total should be reduced only for the sum of odd members from 5 to $60m$ multiplied by -9. Thus it is found that

$$5^3 + 10^3 + \cdots + (60m)^3 = \frac{n^4 + 6n^3 + n^2 - 24n + 16}{20};$$

$$3\left(5^2 + 10^2 + \cdots + (60m)^2\right) = \frac{2n^3 + 9n^2 + n - 12}{10};$$

$$-9(5 + 15 + \cdots + (60m - 5)) = \frac{-9n^2 + 18n - 9}{20}.$$

The sum obtained depends on the number of remainders while dividing n by 60 and is of particular importance. Of particular importance is the sum which is obtained depending on the number of remainders in the division of n with 60. In this case, the sum can be found when all of the following free members from Lemma 2 are lined up 5 times together one after another and form a series of length 60. The total sum of free members depending on forms of the number n can be started from different places in that series adding to the sum every fifth element. In this case, we find that the sum is equal to the sum of all free members given in Lemma 2 and it is equal to -78. Therefore, the sum of ω_j in this case is equal to $-78\frac{n-1}{60} = \frac{-13n+13}{10}$. In other cases, the value is always the sum of all but one member. When all is united, we get

$$p\,(n,5)=\frac{n^4+10n^3+10n^2-30n+9}{2880},\quad n=60m+1.$$

The same procedure is to be followed in other cases too, which proves the validity of Lemma 3. □

Lemma 4. *The number of partitions of the sixth class is calculated using the fractal polynomial*

$$p\,(n,6)=\frac{1}{86400}n^5+\frac{1}{3840}n^4+\frac{19}{12960}n^3+\begin{cases}0+w_{6,1}\,(n)\,,n\ even\\ -\frac{1}{384}n^2+w_{6,1}\,(n)\,,n\ odd\end{cases}$$

where

$$w_{6,1}=\begin{cases}\frac{1}{180}n+w_j\,,n=6m,\\ -\frac{629}{17280}n+w_j\,,n=6m\pm 1,\\ -\frac{7}{540}n+w_j\,,n=6m\pm 2,\\ -\frac{103}{5760}n+w_j\,,n=6m+3.\end{cases}\qquad n\equiv j \bmod 60.$$

The free members of these polynomials w_j show the highest variability. It can take one of sixty different values, depending on the index j. All values are given in the following:

$$w_j\in\left\{\begin{array}{l}0,\frac{19319}{518400},\frac{313}{32400},\frac{87}{6400},\frac{-244}{2025},\frac{-2801}{20736},\frac{89}{400},\frac{30983}{518400},\frac{-188}{2025},\\ \quad\frac{-569}{6400},\frac{-127}{1296},\frac{45319}{518400},\frac{3}{25},\frac{-22153}{518400},\frac{-2279}{32400},\frac{-17}{256},\\ \frac{-1}{2025},\frac{-7817}{518400},\frac{57}{400},\frac{-10489}{518400},\frac{-14}{81},\frac{199}{6400},\frac{713}{32400},\frac{3847}{518400},\frac{1}{25},\frac{-2545}{20736},\\ \quad\frac{1609}{32400},\frac{343}{6400},\frac{-163}{2025},\frac{-49289}{518400},\frac{1}{16},\\ \frac{51719}{518400},\frac{-107}{2025},\frac{-313}{6400},\frac{-1879}{32400},\frac{-1505}{20736},\frac{4}{25},\frac{-1417}{518400},\frac{-983}{32400},\frac{-169}{6400},\\ \quad\frac{-13}{81},\frac{12919}{518400},\frac{73}{400},\frac{10247}{518400},\frac{-269}{2025},\frac{-33}{256},\\ \frac{2009}{32400},\frac{24583}{518400},\frac{2}{25},\frac{-42899}{518400},\frac{-143}{1296},\frac{599}{6400},\frac{-82}{2025},\frac{-28553}{518400},\frac{41}{400},\\ \quad\frac{-1249}{20736},\frac{-26}{2025},\frac{-57}{6400},\frac{-583}{32400},\frac{-16889}{518400}.\end{array}\right\}.$$

Lemma 5. *The variable part for the fractal polynomial of the seventh class is a second degree polynomial*

$$p\,(n,7)=\frac{1}{362880}n^6+\frac{1}{86400}n^5+\frac{1}{6480}n^4+\frac{7}{12960}n^3+\begin{cases}-\frac{101}{57600}n^2+w_{7,1}\,(n)\,,n\,even,\\ -\frac{11}{3600}n^2+w_{7,1}\,(n)\,,n\,odd.\end{cases}$$

The linear members show greater variability with

$$w_{7,1}(n) = \begin{cases} \frac{-1}{36}n + w_j, n = 6d \\ \frac{29}{10368}n + w_j, n = 6d+1 \\ \frac{-7}{324}n + w_j, n = 6d+2 \\ \frac{-11}{1152}n + w_j, n = 6d+3 \\ \frac{-5}{234}n + w_j, n = 6d+4 \\ \frac{-35}{10368}n + w_j, n = 6d+5 \end{cases} \quad n \equiv j \bmod 420.$$

The free members have the highest variability expressed by the variability of the length of 420 *members. The first members are*

$$w_j \in \left\{0, -\frac{127}{72576}, \frac{76}{1575}, \frac{321}{22400}, \frac{334}{14175}, \frac{1159}{8064}, -\frac{1}{7}, -\frac{11}{175}, \frac{73511}{259200}, \frac{13}{1575}, -\frac{183}{27400}, -\frac{32}{567}, -\frac{655}{8064}, -\frac{11}{75}, \frac{69401}{1814400}, \frac{43}{225}, \ldots\right\}.$$

□

3.2 *General global properties of partitions function of classes*

It is now possible to formulate and prove the following theorem.

Theorem 4. *The function $p(n,k)$ that calculates the number of partitions of each class k, is a fractal polynomial with the following general properties*: (a) *The entire fractal polynomial $p(n,k)$ is made up of at most* LCM $(2,3,\ldots,k)$ *different polynomials*; (b) *The degree of each polynomial $p(n,k)$, $\left(1 \le k \le \left[\frac{n}{2}\right]\right)$ is $k-1$*; (c) *Coefficients of all the highest degrees conclusively with the coefficient to the degree $k-\left[\frac{k}{2}\right]-1$ in common*; (d) *The variability of the coefficient increases as the degree decreases. The degree $k-\left[\frac{k}{2}\right]-2$ has the first variable coefficient and in all polynomials it can have exactly two values that depend on the parity of n. The next coefficient can have up to six different values, and so on, up to a free member whose variability is up to* LCM $(2,3,\ldots,k)$.

Proof.

(a) We prove the assertion by the method of mathematical induction by k. For $k=1$, $p(n,1)=1$ and for $k=2$, $p(n,2)=\left[\frac{n}{2}\right]$. The assertion is true, even though all statements of the theorem are yet to be manifested. The statements are true for $k=3$, $k=4,\ldots,k=7$ shown by Lemmas 1–5. Thus, we can assume that the assertion is true for some class $k=k_0$. Let us prove that it is true for the (k_0+1)-th class.

The theorem of a vertical k-collecting states that within $(k+1)$-th class all its values are the sums of each $(k+1)$-th value of the class k, given with the formula (7). Let us denote by $d=\mathrm{LCM}(2,3,\ldots,k_0)$, the lowest common multiple of $2,3,\ldots,k_0$. Within the k_0-th class, by inductive assumption, each d-th value is obtained by calculating from the same polynomial. For each value in the class k_0+1, we make the sum with step

$(k_0 + 1)$ in the previous class. The question is: how many different sums within the class $k_0 + 1$ can be obtained using d different polynomials for the class k_0 which can be obtained with vertical k-collecting? The answer is $d_1 = \text{LCM}\,(d, k_0 + 1)$. Therefore, most of the addition of different combinations of values within a class k_0 to its different polynomials may be $\text{LCM}\,(2, 3, \ldots, k_0 + 1)$, which was to be proven.

Statement (b) is a direct consequence of Lemma A. We prove the assertion by the method of mathematical induction by k. For $k = 1$, $p\,(n, 1) = 1$ and for $k = 2$, $p\,(n, 2) = \left[\frac{n}{2}\right]$ assertion is true. This assertion is true for $k = 3, \ldots, 7$ shown by Lemmas 1–5. So, we can assume that the assertion is true for some class k_0, $k_0 \in \mathbb{N}$.

Let us suppose that for the k_0-th class the fractal polynomial $p\,(n, k_0) = a_1 n^{k_0 - 1} + a_2 n^{k_0 - 2} + \cdots$ whose degree is $k_0 - 1$. Let us prove that assertion is true for the class $k_0 + 1$.

The idea of the proof is to prove that each degree of $p\,(n, k)$ increases exactly for one as well as for all classes of k, $1 \le k \le \left[\frac{n}{2}\right]$ when we increase the class. By induction, it is trivially proved that the sum of all j-th degrees from 1 to n is calculated using the polynomial of degree $j + 1$. The proof is directly followed by (13). This also applies to 'diluted sums' of the same type. It has been shown (Theorem 1) that the values of a class are calculated by the value of the class preceding it. Based on the theorem of a vertical k-collecting, it follows that

$$p\,(n + 1, k_0 + 1) = \sum\nolimits_{m \ge 0} p\,(n - m\,(k_0 + 1)\,, k_0)\,. \tag{14}$$

Using (14) to calculate the values of the $k_0 + 1$ class, sums are obtained at each step m, $1 \le m \le k_0 + 1$. Let us note such a sum with the highest degree in (14). The shape of this sum depends on the remainder when dividing the number n from $k_0 + 1$. For $n = t \cdot (k_0 + 1) + r$, $t \in \mathbb{N}$, $0 \le r < k_0 + 1$, all of these sums are of the form

$$((k_0 + 1) - 1)^{k_0 - 1} + (2\,(k_0 + 1) - 1)^{k_0 - 1} + \cdots + (t\,(k_0 + 1) - 1)^{k_0 - 1} = K_{n = t(k_0 + 1)}$$
$$(k_0 + 1)^{k_0 - 1} + (2\,(k_0 + 1))^{k_0 - 1} + \cdots + (t\,(k_0 + 1))^{k_0 - 1} = K_{n = t(k_0 + 1) + 1}$$
$$\ldots$$
$$1 + (2k_0)^{k_0 - 1} + (3\,(k_0 + 1) + k_0 - 1)^{k_0 - 1} + \cdots + (t\,(k_0 + 1) + k_0 - 1)^{k_0 - 1}$$
$$= K_{n = t(k_0 + 1) + k_0}.$$

Each of the previous sums can be represented by a polynomial of degree k_0, e.g.,

$$K_{n = r(k_0 + 1) + 1} = (k_0 + 1)^{k_0 - 1}\,(1 + 2^{k_0 - 1} + \cdots + t^{k_0 - 1}),$$

where the sum in the second bracket is a polynomial of degree k_0, because $t = \frac{n}{k_0 + 1}$. Hence, the degree of polynomial on the left-hand side (14) is one larger than each individual on the right.

We prove the assertion (c) by the method of mathematical induction by k. For $k = 3, \ldots, 7$ the statement is true as shown in Lemmas 1–5. The evidence is based on the fact that the first member variable part of partition functions of classes depends on the parity of n. Suppose that the statement is true for all fractal polynomials up to the class $2k - 1$. According to the inductive hypothesis the degree of the last fixed member of the fractal polynomial for the class $2k - 1$ is $2k - 1 - \left[\frac{2k - 1}{2}\right] - 1 = k - 1$. The first variable member has degree $k - 2$ and there are only two values that depend on the parity of n. When we calculate the values of a fractal polynomial for the number with the same parity, the number of fixed coefficients increases by one.

The fractal polynomial of class $2k$ is determined by Theorem 1 using the recurrent formula which can also be written for odd numbers in a shorter form [8] with $p\,(2n+1, 2k) = p\,(2n, 2k-1) + p\,(2n-2k, 2k-1)$. From this formula, it can be seen that for odd numbers in the $(2k)$-th class, the number of partitions are obtained exclusively by adding the values of the even numbers of the previous class. By an inductive assumption, for all the same parity numbers in a fractal polynomial, the number of fixed coefficients increases by one. The values for even numbers of the $(2k)$-th class is $p\,(2n, 2k) = p\,(2n-1, 2k-1) + p\,(2n-2k, 2k-1)$. For the first term in the above, the formula still applies $p\,(2n-1, 2k-1) = p\,(2n-2, 2k-2) + p\,(2n-2k-2, 2k-2)$. The values for even numbers of the $2k$-th class are obtained only from the values for even numbers of the $(2k-1)$-th class and the $(2k-2)$-th class. Therefore, the number of fixed parts of a fractal polynomial increases by one and the number of variable parts does not increase. This is also indicated by the formula for calculating the degree of the last fixed member in a fractal polynomial for the $(2k)$-th class $2k - \left[\frac{2k}{2}\right] - 1 = k - 1$. The first variable member should be of degree $k-2$, which proves the assertion.

Let us now consider the crossing from an even class to an odd class, from class $2k$ to class $2k+1$. From the formula $p\,(2n, 2k+1) = p\,(2n-1, 2k) + p\,(2n-2k-1, 2k)$, we conclude that values of even numbers are formed only from values of odd numbers of the previous class. The values for odd numbers in the $(2n+1)$-th class are obtained by $p\,(2n+1, 2k+1) = p\,(2n, 2k) + p\,(2n-2k, 2k)$, only from values of even numbers of the previous class. In this case, the number of variable coefficients increases by one. Thus, the degree of variable parts increases only when passing through two consecutive classes. This is confirmed by the formula $2k + 1 - \left[\frac{2k+1}{2}\right] - 1 = k$.

In this way, the degree of the fixed part of a fractal polynomial increases by one when crossing to the next class, while the degree of the variabile part increases by one when crossing two consecutive classes.

Statement (d) is derived directly from the previous statements. □

Remark 2. It is interesting to note how quickly the sum of (8) returns to the value of number of partitions of an integer n. The speed can be very slow when the sum is reduced to the value of the first class. The values of the second class are of order $O(n)$. The values of the third class are obtained by summing every third element from the second class so they are of order $O(n^2)$. The values of the fourth class are obtained from the sum of every fourth element in the third class so they are of order $O(n^3)$. This result cannot be generalized in the same way because of Consequence 2 (for $2m < n$).

In formula (8), by increasing the class we reduce computation, thus increasing the speed. When calculating the classes of order larger than m-th reducing its value on the $(m-1)$-th order class, the speed of convergence of the formula (7) is $O(n^{m-1})$. For example, if we count the number of partitions reducing it to the fourth class, it will be as follows:

$$p(n) = \sum_{k=5}^{n} \sum_{j_1 \geq 0} \sum_{j_2 \geq 0} \cdots \sum_{j_{k-5} \geq 0} \sum_{j_{k-4} \geq 0} p\,(n - (k-4) - kj_1 - (k-1)\,j_2 - \cdots - 6\,j_{k-5} - 5\,j_{k-4}, 4) + \ p\,(n, 3) + \left[\frac{n}{2}\right] + 1.$$

Then the convergence velocity is $O(n^3)$. Of course, the assumption is that the polynomials that calculate all values of the partitions of that class are known.

Remark 3. Bringman and Ono in [7] have stated as follows: "I can take any number, plug it into P, and instantly calculate the partitions of that number. P does not return gruesome numbers with infinitely many decimal places. It is the finite, algebraic formula that we have all been looking for." The same can be stated for the formulas given here and in an elementary form.

Consequences 1 and 2 allow as follows:

$$p\,(3) = p\,(6,3) = p\,(7,4) = p\,(8,5) \\ = p\,(9,6) = \cdots = p\,(n, n-3) = \cdots = 3.$$

Based on Lemmas 1–5, we have

$$p\,(6,3) = \left(\frac{n^2}{12}\right) \underbrace{=}_{n=6k} \frac{6^2}{12} = 3,$$

$$p\,(7,4) = \left(\frac{n^3+3n^2-9n+5}{144}\right)\Bigg|_{n=12k+7} = \frac{7^3+3\cdot 7^2-9\cdot 7+5}{144} = 3,$$

$$p\,(8,5) = \left(\frac{n^4+10n^3+10n^2-120n-256}{2880}\right)\Bigg|_{n=60k+8}$$
$$= \frac{8^4+10\cdot 8^3+10\cdot 8^2-120\cdot 8-256}{2880} = 3,$$

$$p\,(9,6) = \left(\frac{1}{86400}n^5 + \frac{1}{3840}n^4 + \frac{19}{12960}n^3 - \frac{1}{384}n^2 - \frac{103}{5760}n - \frac{569}{6400}\right)\Bigg|_{n=60k+9}$$
$$= \frac{9^5}{86400} + \frac{9^4}{3840} + \frac{19\cdot 9^3}{12960} - \frac{9^2}{384} - \frac{103\cdot 9}{5760} - \frac{569}{6400} = 3, \ldots.$$

4. Conclusion

Determining the general properties of the partition functions of classes, it is possible to determine the general form of its first, second, third, … coefficients. This will demonstrate their vertical connection. It is also possible to determine the general form of the coefficients of the same fractal polynomial, which will represent their horizontal connectivity. That is precisely the task of the next work.

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