Proc. Indian Acad. Sci. (Math. Sci.) (2018) 128:39
https://doi.org/10.1007/s12044-018-0410-7

Stabilization of the higher order nonlinear Schrödinger equation with constant coefficients

MO CHEN

School of Mathematics and Statistics, Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University, Changchun 130024,
People's Republic of China
E-mail: chenmochenmo.good@163.com

MS received 7 April 2016; revised 18 July 2017; accepted 31 October 2017; published online 15 June 2018

Abstract. We study the internal stabilization of the higher order nonlinear Schrödinger equation with constant coefficients. Combining multiplier techniques, a fixed point argument and nonlinear interpolation theory, we can obtain the well-posedness. Then, applying compactness arguments and a unique continuation property, we prove that the solution of the higher-order nonlinear Schrödinger equation with a damping term decays exponentially.

Keywords. Stabilization; higher order nonlinear Schrödinger equation; Korteweg–de Vries equation.

2010 Mathematics Subject Classification. 93D15, 35Q55, 35Q53.

1. Introduction

In this paper, we consider the higher-order nonlinear Schrödinger (HNLS) equation on a bounded interval $I = (0, L)$ $(L > 0)$ in the presence of a localized damping

$$\begin{cases} iu_t + i\beta u_{xxx} + \alpha u_{xx} + i\delta u_x + |u|^2 u + iau = 0 & x \in I, t > 0, \\ u(0,t) = u(L,t) = u_x(L,t) = 0 & t > 0, \\ u(x,0) = u_0(x) & x \in I. \end{cases} \tag{1.1}$$

Here α, β and δ are real parameters, $u = u(x,t)$ is a complex-valued function and $a = a(x)$ is a nonnegative real-valued function belonging to $H^1(I)$.

The original HNLS equation was first derived [11] as follows:

$$iu_t + i\beta u_{xxx} + \alpha u_{xx} + \gamma |u|^2 u + i\mu(|u|^2 u)_x + i\epsilon u(u^2)_x = 0. \tag{1.2}$$

This equation describes the propagation of a signal in an optic fiber [13]. In the above equation, u is the slowly varying envelope of the pulse, the subscripts t and x denote the temporal and spatial partial derivatives, and α, β, γ, μ and ϵ mean the real parameters related to group velocity dispersion, third-order dispersion, self-phase modulation, selfsteepening and delayed nonlinear response effect arising from stimulated Raman scattering, respectively.

Equation (1.2) also arose in the optical communications (see [1,12,22]). They can be applied to long-distance communications and ultrafast signal routing systems. The HNLS models can describe the propagation of ultrashort light pulses which are shorter than $\sim 10^{-13}$ s.

In practical applications, the stabilization of the HNLS equation can be viewed as a good mathematical representation sometimes when we want to prevent the propagation of some signal in optical fibre.

To state the main result in this paper, we recall some results for the Korteweg–de Vries (KdV) equation

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0 & x \in I, t \in (0,T), \\ u(0,t) = u(L,t) = u_x(L,t) = 0 & t \in (0,T), \\ u(x,0) = u_0(x) & x \in I. \end{cases} \tag{1.3}$$

In [18], Rosier obtained the existence of a critical set

$$\bar{\mathcal{N}} = \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}} : k, l \in \mathbb{N}^* \right\}.$$

He proved that the solution of the linear system associated with (1.3) does not decay when $L \in \bar{\mathcal{N}}$. Therefore, it is not easy to obtain the decay of the solution of (1.3) without the damping term. To stabilize the solution, Perla Menzala *et al.* [17] introduced a damping term $a(x)u$ which is active simultaneously in a neighborhood of both extremes of the interval $(0, L)$. Subsequently, Pazoto [16] extended this result to the case in which the damping term $a(x)u$ is active in any open subset of $(0, L)$. Recently, Chu *et al.* [8] proved that without the damping term, the origin is (locally)asymptotically stable for (1.3) by using the dynamics on the center manifold.

For the controllability of the control system,

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0 & x \in I, t \in (0,T), \\ u(0,t) = u(L,t) = 0, \quad u_x(L,t) = h(t) & t \in (0,T), \\ u(x,0) = u_0(x) & x \in I, \end{cases} \tag{1.4}$$

Rosier [18] obtained the local exact controllability of (1.4) under the assumption that $L \notin \bar{\mathcal{N}}$. Then, Coron and Crépeau in [9] obtained the local exact controllability of (1.4) for the critical lengths $L = 2k\pi$ with $k \in \mathbb{N}^*$ satisfying

$$\nexists (m,n) \in \mathbb{N}^* \times \mathbb{N}^* \text{ with } m^2 + mn + n^2 = 3k^2 \text{ and } m \neq n.$$

The method is a power series expansion of the solution along this direction. Later on, in [6,7], this method is applied to address the case of any critical length.

Inspired by the results of the KdV equation, in [5], Ceballos *et al.* studied the exact boundary controllability of

$$\begin{cases} iu_t + i\beta u_{xxx} + \alpha u_{xx} + i\delta u_x + u|u|^2 = 0 & x \in I, t \in (0,T), \\ u(0,t) = u(L,t) = 0, \ \ u_x(L,t) = h(t) & t \in (0,T), \\ u(x,0) = u_0(x) & x \in I, \end{cases}$$

where $T > 0$ and h is the control. They proved that the above system is exact controllable if

$$L \notin \mathcal{N} = \left\{ 2\pi\beta \frac{k^2 + kl + l^2}{3\beta\delta + \alpha^2} \colon k, l \in \mathbb{N}^* \right\}.$$

This also means that if L belongs to $\mathcal{N}$, one can find initial data $u_0 \in L^2(I)$ such that the solution u of the linear system associated with (1.1) satisfies $u_x(0,\cdot) \equiv 0$, which combines with the energy estimate and shows that

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_I |u(x,t)|^2 \mathrm{d}x = 0.$$

This implies that the solution u of the linear system associated with (1.1) does not decay.

In order to handle the case of $L \in \mathcal{N}$ and to have the solution of (1.1) with large amplitude stabilized, we add the extra damping term $a(x)u$ as in [16] to the system. We assume that $a(x) \geq a_0 > 0$ a.e. in an open, nonempty subset ω of I. Therefore, the damping term acts effectively in ω. System (1.1) was also investigated in [2], where we obtain the stabilization of this system with a different method.

The main result in this paper is the following theorem.

Theorem 1.1. *Let $\beta > 0$. For any $r > 0$, there exist $C = C(r) > 0$ and $\mu = \mu(r) > 0$ such that*

$$\|u(t)\|_{L^2(I)} \leq C\|u_0\|_{L^2(I)} \mathrm{e}^{-\mu t}$$

for all $t \geq 0$ and any solution of (1.1) *with $u_0 \in L^2(I)$ such that $\|u_0\|_{L^2(I)} \leq r$.*

The stabilization of the nonlinear Schrödinger equation and the Korteweg–de Vries equation have been intensively studied (see, for instance, [4,10,14,15,17,19]). However as far as we know, there are a few results concerning the stabilization of the Korteweg–de Vries and Schrödinger type equations.

The rest of the paper is organized as follows: In section 2, we consider the well-posedness of (1.1). Section 3 is devoted to the proof of Theorem 1.1. First, we need to prove a unique continuation property by a Carleman estimate, then combining this unique continuation property and compactness arguments, we can prove Theorem 1.1.

2. Well-posedness

In this section, we pay attention to the well-posedness of (1.1). First, we investigate the linear system, then by using a fixed point argument and nonlinear interpolation theory, we obtain the well-posedness of the nonlinear system.

2.1 *Linear system*

Now we investigate the following linear system:

$$\begin{cases} iu_t + i\beta u_{xxx} + \alpha u_{xx} + i\delta u_x = f & (x,t) \in I \times (0,+\infty), \\ u(0,t) = u(L,t) = u_x(L,t) = 0 & t \in (0,+\infty), \\ u(x,0) = u_0(x) & x \in I. \end{cases} \tag{2.1}$$

PROPOSITION 2.1

Let $\beta > 0$, $u_0 \in L^2(I)$ and $f \in L^1(0,+\infty; L^2(I))$. For any $T > 0$, (2.1) admits a unique mild solution $u \in C([0,T]; L^2(I)) \cap L^2(0,T; H^1(I))$ satisfying

$$\|u\|_{L^\infty(0,T;L^2(I))} + \|u\|_{L^2(0,T;H^1(I))} \le C(T^{1/2}+1)(\|u_0\|_{L^2(I)} + \|f\|_{L^1(0,T;L^2(I))}). \tag{2.2}$$

Moreover, $u_x(0,\cdot)$ makes sense in $L^2(0,T)$.

Remark 2.1. Here and throughout this paper, unless otherwise specified, C denotes a generic positive constant whose value can change from line to line. If it is essential, the dependence of a constant C on some parameters, say '$\cdot$', will be written by $C(\cdot)$.

Proof. Let A be the linear operator defined by

$$Aw = -\beta w''' + i\alpha w'' - \delta w'$$

with the domain

$$\mathcal{D}(A) = \{w \in H^3(0,L) \colon w(0) = w(L) = w'(L) = 0\}.$$

As it is proved in [5], operator A generates a strongly continuous semigroup of contractions on $L^2(I)$. We denote by $\{S(t)\}_{t\ge 0}$ the semigroup of contractions with A. It follows from semigroup theory that (2.1) admits a unique solution $u \in C([0,T]; L^2(I))$ for any $T > 0$.

We first assume that $u_0 \in \mathcal{D}(A)$ and $f \in L^1(0,T;\mathcal{D}(A))$. According to semigroup theory, $u \in C([0,T];\mathcal{D}(A)) \cap C^1([0,T]; L^2(0,L))$. Multiplying the first equation in (2.1) by $\bar{u}$ and conjugating, we obtain

$$\begin{aligned} iu_t\bar{u} + i\beta u_{xxx}\bar{u} + \alpha u_{xx}\bar{u} + i\delta u_x\bar{u} &= f\bar{u}, \\ -i\bar{u}_t u - i\beta \bar{u}_{xxx}u + \alpha\bar{u}_{xx}u - i\delta\bar{u}_x u &= \bar{f}u. \end{aligned}$$

Subtracting and integrating over $I \times (0, t)$, we have

$$\int_0^t \int_I [i(|u|^2)_t + i\beta(u_{xxx}\bar{u} + \bar{u}_{xxx}u) + \alpha(u_{xx}\bar{u} - \bar{u}_{xx}u) + i\delta(u_x\bar{u} + \bar{u}_x u)]\mathrm{d}x\mathrm{d}s = \int_0^t \int_I (f\bar{u} - \bar{f}u)\mathrm{d}x\mathrm{d}s.$$

Direct calculation yields

$$\begin{aligned}
&\int_I |u(x,t)|^2\mathrm{d}x + \beta\int_0^t |u_x(0,s)|^2\mathrm{d}s \\
&= \int_I |u_0(x)|^2\mathrm{d}x + \int_0^t \int_I (f\bar{u} - \bar{f}u)\mathrm{d}x\mathrm{d}s \\
&\le \int_I |u_0(x)|^2\mathrm{d}x + C\int_0^T \int_I |f||u|\mathrm{d}x\mathrm{d}s \\
&\le \|u_0\|^2_{L^2(I)} + C\int_0^T \|f\|_{L^2(I)}\|u\|_{L^2(I)}\mathrm{d}s \\
&\le \|u_0\|^2_{L^2(I)} + \varepsilon \sup_{0\le s\le T}\|u(\cdot,s)\|^2_{L^2(I)} + C(\varepsilon)\|f\|^2_{L^1(0,T;L^2(I))}
\end{aligned}$$

for any $\varepsilon > 0$. This implies that

$$\|u\|_{L^\infty(0,T;L^2(I))} + \|u_x(0,\cdot)\|_{L^2(0,T)} \le C(\|u_0\|_{L^2(I)} + \|f\|_{L^1(0,T;L^2(I))}). \tag{2.3}$$

Then, multiplying the first equation in (2.1) by $x\bar{u}$ and conjugating, we have

$$\begin{aligned}
ixu_t\bar{u} + i\beta xu_{xxx}\bar{u} + \alpha xu_{xx}\bar{u} + i\delta xu_x\bar{u} &= xf\bar{u}, \\
-ix\bar{u}_t u - i\beta x\bar{u}_{xxx}u + \alpha x\bar{u}_{xx}u - i\delta x\bar{u}_x u &= x\bar{f}u.
\end{aligned}$$

Subtracting and integrating over $I \times (0, T)$, it is easy to obtain that

$$\int_0^T \int_I [ix(|u|^2)_t + i\beta x(u_{xxx}\bar{u} + \bar{u}_{xxx}u) + \alpha x(u_{xx}\bar{u} - \bar{u}_{xx}u) + i\delta x(u_x\bar{u} + \bar{u}_x u)]\mathrm{d}x\mathrm{d}t = \int_0^T \int_I x(f\bar{u} - \bar{f}u)\mathrm{d}x\mathrm{d}t.$$

After integrations by parts, for any $\varepsilon > 0$, we can deduce that

$$\begin{aligned}
&\int_I x|u(x,T)|^2 \mathrm{d}x + 3\beta \int_0^T \int_I |u_x|^2 \mathrm{d}x\mathrm{d}t \\
&\quad = \int_I x|u_0(x)|^2 \mathrm{d}x - 2\alpha \,\mathrm{Im} \int_0^T \int_I \bar{u}_x u \mathrm{d}x\mathrm{d}t + \delta \int_0^T \int_I |u|^2 \mathrm{d}x\mathrm{d}t \\
&\qquad + \int_0^T \int_I x(f\bar{u} - \bar{f}u)\mathrm{d}x\mathrm{d}t \\
&\quad \le \int_I x|u_0(x)|^2 \mathrm{d}x + \varepsilon \int_0^T \int_I |u_x|^2 \mathrm{d}x\mathrm{d}t + C(\varepsilon) \int_0^T \int_I |u|^2 \mathrm{d}x\mathrm{d}t \\
&\qquad + C \int_0^T \int_I |f||u| \mathrm{d}x\mathrm{d}t \le C\|u_0\|^2_{L^2(I)} + \varepsilon \int_0^T \int_I |u_x|^2 \mathrm{d}x\mathrm{d}t \\
&\qquad + C(\varepsilon)(T+1)\|u\|^2_{L^\infty(0,T;L^2(I))} + C\|f\|^2_{L^1(0,T;L^2(I))}.
\end{aligned}$$

Taking $\varepsilon > 0$ small enough, it follows from (2.3) that

$$\|u\|_{L^2(0,T;H^1(I))} \le C(T^{1/2}+1)(\|u_0\|_{L^2(I)} + \|f\|_{L^1(0,T;L^2(I))}). \tag{2.4}$$

Combining (2.3), (2.4) and the density of $\mathcal{D}(A)$ in $L^2(0, L)$, the proof is complete. □

2.2 *Nonlinear system*

Here, we investigate the well-posedness of (1.1). We first introduce some spaces which will be used later.

For $0 \le s \le 3$, let X_s be the collection of all functions h in the space $H^s(I)$ satisfying the compatibility conditions

$$\begin{cases} h(0) = h(L) = 0 & \text{when } \ 1/2 < s \le 3/2, \\ h(0) = h(L) = h'(L) = 0 & \text{when } \ 3/2 < s \le 3 \end{cases}$$

equipped with its natural norm. For any $T > 0$, define

$$Y_{s,T} = C([0,T]; X_s) \cap L^2(0,T; H^{s+1}(I))$$

equipped with its natural norm.

The main result in this section is the following theorem.

Theorem 2.1. *Let $\beta > 0$, $T > 0$ and $0 \le s \le 3$. Assume that $a \in H^1(I)$. Then, for any $u_0 \in X_s$, system (1.1) admits a unique solution $u \in Y_{s,T}$. Moreover, there exists a nondecreasing continuous function $\gamma_s \colon \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|u\|_{Y_{s,T}} \le \gamma_s(\|u_0\|_{X_0})\|u_0\|_{X_s}.$$

Proof. The proof is similar to that of Theorem 2.9 in [19], so we just sketch it.

Step 1. $s = 0$. Let us define the closed ball of $Y_{0,\theta}$:

$$B_R^\theta = \{v \in Y_{0,\theta} \mid \|v\|_{Y_{0,\theta}} \leq R\},$$

where θ and R will be specified later. Then we introduce the map Γ on B_R^θ defined by

$$\Gamma(v)(t) = S(t)u_0 + i\int_0^t S(t-s)(|v|^2 v)(s)\mathrm{d}s - \int_0^t S(t-s)av(s)\mathrm{d}s.$$

It is not difficult to obtain

$$\begin{aligned}\||v|^2 v\|_{L^1(0,\theta;L^2(I))} &= \int_0^\theta \||v|^2 v\|_{L^2(I)}\mathrm{d}t \leq \int_0^\theta \|v\|_{L^\infty(I)}^2 \|v\|_{L^2(I)}\mathrm{d}t \\ &\leq C\int_0^\theta \|v\|_{L^2(I)}^2 \|v_x\|_{L^2(I)}\mathrm{d}t \\ &\leq C\Big(\int_0^\theta \|v\|_{L^2(I)}^4 \mathrm{d}t\Big)^{1/2}\Big(\int_0^\theta \|v_x\|_{L^2(I)}^2 \mathrm{d}t\Big)^{1/2} \\ &\leq C\theta^{1/2}\|v\|_{Y_{0,\theta}}^3.\end{aligned}$$

In the computation above, we have used the inequality

$$\|v\|_{L^\infty(I)} \leq C\|v\|_{L^2(I)}^{1/2}\|v_x\|_{L^2(I)}^{1/2}.$$

It is not difficult to obtain

$$\|av\|_{L^1(0,\theta;L^2(I))} \leq C\theta^{1/2}\|a\|_{L^2(I)}\|v\|_{Y_{0,\theta}}.$$

Therefore, applying (2.2), we can show that

$$\begin{aligned}\|\Gamma(v)\|_{Y_{0,\theta}} &\leq C(\theta^{1/2}+1)(\|u_0\|_{X_0} + \||v|^2 v\|_{L^1(0,\theta;L^2(I))} + \|av\|_{L^1(0,\theta;L^2(I))}) \\ &\leq C_1(\theta^{1/2}+1)\|u_0\|_{X_0} + C_2(\theta+\theta^{1/2})\|v\|_{Y_{0,\theta}}^3 \\ &\quad + C_3(\theta+\theta^{1/2})\|v\|_{Y_{0,\theta}}.\end{aligned}$$

Similarly, for $v_1, v_2 \in B_R^\theta$, we have

$$\begin{aligned}\|\Gamma(v_1)-\Gamma(v_2)\|_{Y_{0,\theta}} &\leq C_2(\theta+\theta^{1/2})(\|v_1\|_{Y_{0,\theta}}^2 + \|v_2\|_{Y_{0,\theta}}^2)\|v_1-v_2\|_{Y_{0,\theta}} \\ &\quad + C_3(\theta+\theta^{1/2})\|v_1-v_2\|_{Y_{0,\theta}}.\end{aligned}$$

Let

$$R = 2C_1(\theta^{1/2}+1)\|u_0\|_{X_0}$$

and take θ small enough such that

$$2C_2(\theta+\theta^{1/2})R^2 + C_3(\theta+\theta^{1/2}) \leq \frac{1}{2}.$$

It follows immediately that

$$\|\Gamma(v)\|_{Y_{0,T}} \leq R,$$
$$\|\Gamma(v_1) - \Gamma(v_2)\|_{Y_{0,T}} \leq \frac{1}{2}\|v_1 - v_2\|_{Y_{0,T}}$$

for all $v, v_1, v_2 \in B_R^\theta$. Thus, with such a choice of R and θ, Γ is a contraction mapping of B_R^θ. Its fixed point $u = \Gamma(u)$ is the unique solution of (1.1) in B_R^θ. Note that θ depends only on $\|u_0\|_{X_0}$.

Multiplying the first equation in (1.1) by $\bar{u}$ and proceeding as in the proof of Proposition 2.1, we obtain

$$\sup_{0\leq t\leq\theta} \|u(\cdot, t)\|_{L^2(I)} \leq \|u_0\|_{X_0}.$$

By the standard extension argument, one may extend θ to T.

Consequently, we can find a nondecreasing continuous function γ_0: $\mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|u\|_{Y_{0,T}} \leq \gamma_0(\|u_0\|_{X_0})\|u_0\|_{X_0},$$
$$\|u - v\|_{Y_{0,T}} \leq \gamma_0(\|u_0\|_{X_0}^2 + \|v_0\|_{X_0}^2)\|u_0 - v_0\|_{X_0},$$

where v is the solution of (1.1) when $u_0 = v_0$.

Step 2. $s = 3$. Let $w = u_t$, then w satisfies

$$\begin{cases} iw_t + i\beta w_{xxx} + \alpha w_{xx} + i\delta w_x + 2|u|^2 w + u^2\bar{w} + iaw = 0 & (x,t) \in I \times (0,T), \\ w(0,t) = w(L,t) = w_x(L,t) = 0 & t \in (0,T), \\ w(x,0) = w_0(x) & x \in I, \end{cases}$$

where

$$w_0(x) = -\beta u_0'''(x) + i\alpha u_0''(x) - \delta u_0'(x) + i|u_0(x)|^2 u_0(x) - a(x)u_0(x).$$

It is not difficult to see that

$$\|w_0\|_{X_0} \leq C(\|u_0\|_{X_0})\|u_0\|_{X_3}.$$

Following the methods used in [19] and using the result in Step 1, we obtain $w \in Y_{0,T}$. This implies that $u \in H^1(0, T; H^1(I)) \subset C([0, T]; C(\overline{I}))$. It then follows from

$$u_{xxx} = -\frac{1}{\beta}u_t + i\frac{\alpha}{\beta}u_{xx} - \frac{\delta}{\beta}u_x + i\frac{1}{\beta}|u|^2 u - \frac{1}{\beta}au \tag{2.5}$$

that $u \in L^2(0, T; H^2(I))$. Applying (2.5) again, we can see that $u \in L^2(0, T; H^3(I)) \cap C([0, T]; H^2(I))$. Since $a \in H^1(I)$, it is easy to show that each term in (2.5) belongs to $C([0, T]; L^2(I)) \cap L^2(0, T; H^1(I))$. We then have $u \in Y_{3,T}$ and u satisfies

$$\|u\|_{Y_{3,T}} \leq \gamma_3(\|u_0\|_{X_0})\|u_0\|_{X_3},$$

where $\gamma_3\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function.

Step 3. $0 \leq s \leq 3$. According to the results in Step 1 and Step 2, the well-posedness of (1.1) in X_s $(0 \leq s \leq 3)$ can be obtained from nonlinear interpolation theory [21]. □

Applying Theorem 2.1, we can obtain a strong smoothing property of system (1.1) by a similar method as in [19, Corollary 2.10].

COROLLARY 2.1

Assume that $a \in H^1(I)$. For any $T > 0$ and $u_0 \in L^2(I)$, the corresponding solution u of (1.1) *satisfies*

$$u \in C(\varepsilon, T; H^3(I)) \cap L^2(\varepsilon, T; H^4(I))$$

for any $0 < \varepsilon < T$.

3. Exponential stability

3.1 *Unique continuation property*

The goal is to prove a unique continuation property for a modified system

$$\begin{cases} iv_t + i\beta v_{xxx} + \alpha v_{xx} + i\delta v_x + \lambda^2|v|^2 v + iav = 0 & (x,t) \in I \times (0,T), \\ v(0,t) = v(L,t) = v_x(L,t) = 0 & t \in (0,T), \\ v(x,0) = v_0(x) & x \in I, \end{cases} \tag{3.1}$$

where $T > 0$ and λ are real constants. The well-posedness result in section 2 also adapts to (3.1).

First, let us recall a Carleman estimate for the following system:

$$\begin{cases} q_t + \beta q_{xxx} = g & (x,t) \in I \times (0,T), \\ q(0,t) = q(L,t) = q_x(L,t) = 0 & t \in (0,T), \\ q(x,0) = q_0(x) & x \in I, \end{cases} \tag{3.2}$$

where q, g and q_0 are all real functions.

Choose $(l_1, l_2) \subset \omega$ with $0 < l_1 < l_2 < L$. Pick any function $\psi \in C^3(\bar{I})$ such that

$$\begin{cases} \psi > 0 \ \text{ in } \ \bar{I}; \\ |\psi'| > 0, \ \psi'' < 0 \ \text{ and } \ \psi'\psi''' < 0 \ \text{ in } \ \bar{I}\backslash(l_1, l_2); \\ \psi'(0) < 0 \ \text{ and } \ \psi'(L) > 0; \\ \min\limits_{x\in[l_1,l_2]} \psi(x) = \psi(l_3) < \max\limits_{x\in[l_1,l_2]} \psi(x) = \psi(l_1) = \psi(l_2), \\ \max\limits_{x\in\bar{I}} \psi(x) = \psi(0) = \psi(L) \ \text{ and } \ \psi(0) < \dfrac{4}{3}\psi(l_3) \ \text{ for some } l_3 \in (l_1, l_2). \end{cases}$$

The existence of ψ can be found in [3]. Set

$$\varphi_\varepsilon(x,t) = \frac{\psi(x)}{(t-\varepsilon)(T-t)}.$$

Following the methods developed in [3] with minor changes, we have as follows.

PROPOSITION 3.1

Pick any $T > 0$ and $\beta > 0$. There exist two constants $C > 0$ and $s_0 > 0$ such that for any $g \in L^2(0,T; L^2(I))$, any $q_0 \in L^2(I)$ and any $s \geq s_0$, the solution q of (3.2) fulfills

$$\begin{aligned}\int_\varepsilon^T \int_I (s\varphi_\varepsilon |q_{xx}|^2 + s^3\varphi_\varepsilon^3 |q_x|^2 + s^5\varphi_\varepsilon^5 |q|^2)\mathrm{e}^{-2s\varphi_\varepsilon}\mathrm{d}x\mathrm{d}t \\ \leq C\Big(\int_\varepsilon^T \int_I |g|^2 \mathrm{e}^{-2s\varphi_\varepsilon}\mathrm{d}x\mathrm{d}t + \int_\varepsilon^T \int_\omega (s\varphi_\varepsilon |q_{xx}|^2 + s^3\varphi_\varepsilon^3 |q_x|^2 \\ + s^5\varphi_\varepsilon^5 |q|^2)\mathrm{e}^{-2s\varphi_\varepsilon}\mathrm{d}x\mathrm{d}t\Big),\end{aligned}$$

where $0 < \varepsilon < T$.

Now, we can prove the unique continuation property for (3.1).

PROPOSITION 3.2

Let $T > 0$, $\beta > 0$, $\lambda \geq 0$ and $a \in H^1(I)$. If the solution v of (3.1) satisfies

$$v \equiv 0 \text{ in } \omega \times (0,T),$$

then $v \equiv 0$ in $I \times (0,T)$.

Proof. It is not difficult to prove that Proposition 3.1 holds for the following complex equation:

$$\begin{cases} v_t + \beta v_{xxx} = f & (x,t) \in I \times (0,T), \\ v(0,t) = v(L,t) = v_x(L,t) = 0 & t \in (0,T), \\ v(x,0) = v_0(x) & x \in I. \end{cases}$$

Namely, we have

$$\begin{aligned}\int_\varepsilon^T \int_I (s\varphi_\varepsilon |v_{xx}|^2 + s^3\varphi_\varepsilon^3 |v_x|^2 + s^5\varphi_\varepsilon^5 |v|^2)\mathrm{e}^{-2s\varphi_\varepsilon}\mathrm{d}x\mathrm{d}t \\ \leq C\Big(\int_\varepsilon^T \int_I |f|^2 \mathrm{e}^{-2s\varphi_\varepsilon}\mathrm{d}x\mathrm{d}t \\ + \int_\varepsilon^T \int_\omega (s\varphi_\varepsilon |v_{xx}|^2 + s^3\varphi_\varepsilon^3 |v_x|^2 + s^5\varphi_\varepsilon^5 |v|^2)\mathrm{e}^{-2s\varphi_\varepsilon}\mathrm{d}x\mathrm{d}t\Big). \qquad (3.3)\end{aligned}$$

Note that

$$f = i\alpha v_{xx} - \delta v_x + i\lambda^2 |v|^2 v - av.$$

It follows from Corollary 2.1 that

$$\begin{aligned}
&\int_\varepsilon^T \int_I |f|^2 \mathrm{e}^{-2s\varphi_\varepsilon} \mathrm{d}x\mathrm{d}t \\
&\quad \le C \int_\varepsilon^T \int_I (|v_{xx}|^2 + |v_x|^2 + |v|^6 + |v|^2) \mathrm{e}^{-2s\varphi_\varepsilon} \mathrm{d}x\mathrm{d}t \\
&\quad \le C(1 + \|v\|^4_{L^\infty(I\times(\varepsilon,T))}) \int_\varepsilon^T \int_I (|v_{xx}|^2 + |v_x|^2 + |v|^2) \mathrm{e}^{-2s\varphi_\varepsilon} \mathrm{d}x\mathrm{d}t \\
&\quad \le C(\|v_0\|_{L^2(I)}) \int_\varepsilon^T \int_I (|v_{xx}|^2 + |v_x|^2 + |v|^2) \mathrm{e}^{-2s\varphi_\varepsilon} \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Taking s_0 large enough, $C(\|v_0\|_{L^2(I)}) \int_\varepsilon^T \int_I (|v_{xx}|^2 + |v_x|^2 + |v|^2) \mathrm{e}^{-2s\varphi_\varepsilon} \mathrm{d}x\mathrm{d}t$ can be absorbed by the left-hand side of (3.3). Therefore, we have

$$\begin{aligned}
&\int_\varepsilon^T \int_I (s\varphi_\varepsilon |v_{xx}|^2 + s^3\varphi_\varepsilon^3 |v_x|^2 + s^5\varphi_\varepsilon^5 |v|^2) \mathrm{e}^{-2s\varphi_\varepsilon} \mathrm{d}x\mathrm{d}t \\
&\quad \le C \int_\varepsilon^T \int_\omega (s\varphi_\varepsilon |v_{xx}|^2 + s^3\varphi_\varepsilon^3 |v_x|^2 + s^5\varphi_\varepsilon^5 |v|^2) \mathrm{e}^{-2s\varphi_\varepsilon} \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Note that $v \equiv 0$ in $\omega \times (0, T)$. This implies that $v \equiv 0$ in $I \times (\varepsilon, T)$. Since $\varepsilon > 0$ is arbitrary, we have $v \equiv 0$ in $I \times (0, T)$. □

3.2 *Proof of Theorem* 1.1

Now we can prove Theorem 1.1. The proof of Theorem 1.1 is motivated by [17,19].

Proof. Multiplying the first equation in (1.1) by $\bar{u}$ and proceeding as in the proof of Proposition 2.1, the following equality holds:

$$\int_I |u(x,T)|^2 \mathrm{d}x + \beta \int_0^T |u_x(0,t)|^2 \mathrm{d}t + 2\int_0^T \int_I a|u|^2 \mathrm{d}x\mathrm{d}t = \int_I |u_0(x)|^2 \mathrm{d}x. \tag{3.4}$$

Then multiplying the first equation in (1.1) by $(T-t)\bar{u}$ and conjugating, we have

$$\begin{aligned}
&i(T-t)u_t\bar{u} + i\beta(T-t)u_{xxx}\bar{u} + \alpha(T-t)u_{xx}\bar{u} \\
&\qquad + i\delta(T-t)u_x\bar{u} + (T-t)|u|^4 + i(T-t)a|u|^2 = 0, \\
&-i(T-t)\bar{u}_t u - i\beta(T-t)\bar{u}_{xxx}u + \alpha(T-t)\bar{u}_{xx}u \\
&\qquad - i\delta(T-t)\bar{u}_x u + (T-t)|u|^4 - i(T-t)a|u|^2 = 0.
\end{aligned}$$

Subtracting and integrating over $(0, T) \times I$, it is easy to obtain that

$$\int_I |u_0(x)|^2 \mathrm{d}x \leq \frac{1}{T}\int_0^T \int_I |u|^2 \mathrm{d}x\mathrm{d}t + \beta \int_0^T |u_x(0,t)|^2 \mathrm{d}t + 2\int_0^T \int_I a|u|^2 \mathrm{d}x\mathrm{d}t. \tag{3.5}$$

Now it remains only to prove that for any $u_0 \in L^2(I)$ with $\|u_0\|_{L^2(I)} \leq r$, there exists a constant $C = C(r)$ such that

$$\int_0^T \int_I |u|^2 \mathrm{d}x\mathrm{d}t \leq C\Big(\beta \int_0^T |u_x(0,t)|^2 \mathrm{d}t + 2\int_0^T \int_I a|u|^2 \mathrm{d}x\mathrm{d}t\Big). \tag{3.6}$$

In fact, if (3.6) holds, it follows from (3.5) that

$$\int_I |u_0(x)|^2 \mathrm{d}x \leq C\Big(\beta \int_0^T |u_x(0,t)|^2 \mathrm{d}t + 2\int_0^T \int_I a|u|^2 \mathrm{d}x\mathrm{d}t\Big).$$

Taking (3.4) into consideration, we get

$$\int_I |u(x,T)|^2 \mathrm{d}x \leq \gamma \int_I |u_0(x)|^2 \mathrm{d}x \quad \text{with } 0 < \gamma = \frac{C}{C+1} < 1.$$

Combining the semigroup property, we can prove Theorem 1.1.

To prove (3.6), we argue by contradiction the so-called 'compactness-uniqueness' argument. Suppose that (3.6) is not valid. Then there exist a sequence of functions $u_n \in C([0,T]; L^2(I)) \cap L^2(0,T; H^1(I))$ that solve (1.1), satisfying $\|u_n(\cdot,0)\|_{L^2(I)} \leq r$ such that

$$\lim_{n\to+\infty} \frac{\|u_n\|^2_{L^2(0,T;L^2(I))}}{\beta \int_0^T |(u_n)_x(0,t)|^2 \mathrm{d}t + 2\int_0^T \int_I a|u_n|^2 \mathrm{d}x\mathrm{d}t} = +\infty. \tag{3.7}$$

Let $\lambda_n = \|u_n\|_{L^2(0,T;L^2(I))}$ and define $v_n(x,t) = u_n(x,t)/\lambda_n$. Then v_n satisfies

$$\|v_n\|_{L^2(0,T;L^2(I))} = 1$$

and

$$\lim_{n\to+\infty} \left(\beta \int_0^T |(v_n)_x(0,t)|^2 \mathrm{d}t + 2\int_0^T \int_I a|v_n|^2 \mathrm{d}x\mathrm{d}t\right) = 0.$$

For each $n \in \mathbb{N}^*$, v_n solves

$$\begin{cases} i(v_n)_t + i\beta(v_n)_{xxx} + \alpha(v_n)_{xx} + i\delta(v_n)_x + \lambda_n^2|v_n|^2 v_n + iav_n = 0 & (x,t) \in I \times (0,T), \\ v_n(0,t) = v_n(L,t) = (v_n)_x(L,t) = 0 & t \in (0,T). \end{cases}$$

According to (3.5) and (3.7), we obtain

$$\|v_n(\cdot,0)\|_{L^2(I)} = \|u_n(\cdot,0)\|_{L^2(I)}/\lambda_n \leq C.$$

Since $\|u_0\|_{L^2(I)} \leq r$, it is easy to see that $\{\lambda_n\}$ is a bounded sequence. Arguing as in the proof of Theorem 2.1, we have

$$\|v_n\|_{Y_{0,T}} \leq C. \tag{3.8}$$

This implies that

$$\||v_n|^2 v_n\|_{L^2(0,T;L^2(I))} \leq C.$$

Thus $\{(v_n)_t\}$ is bounded in $L^2(0,T;H^{-2}(I))$. Combining (3.8) and the classical compactness results (see [20]), we can extract a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) such that

$$v_n \to v \ \text{ strongly in } L^2(0,T;L^2(I)).$$

Moreover v satisfies

$$\|v\|_{L^2(0,T;L^2(I))} = 1 \tag{3.9}$$

and

$$\begin{aligned}&\beta\int_0^T |v_x(0,t)|^2\mathrm{d}t + 2\int_0^T\int_I a|v|^2\mathrm{d}x\mathrm{d}t\\ &\quad\leq \liminf_{n\to+\infty}\beta\int_0^T |(v_n)_x(0,t)|^2\mathrm{d}t + 2\int_0^T\int_I a|v_n|^2\mathrm{d}x\mathrm{d}t = 0.\end{aligned}$$

This gives $v \equiv 0$ in $\omega\times(0,T)$.

Extracting a subsequence of $\{\lambda_n\}$ (still denoted by $\{\lambda_n\}$) such that

$$\lambda_n \to \lambda,$$

then v solves

$$\begin{cases} iv_t + i\beta v_{xxx} + \alpha v_{xx} + i\delta v_x + \lambda_n^2|v|^2 v + iav = 0 & (x,t)\in I\times(0,T),\\ v(0,t) = v(L,t) = v_x(L,t) = 0 & t\in(0,T).\end{cases}$$

It is not difficult to obtain that $v(\cdot,0)\in L^2(I)$. Therefore, applying Proposition 3.2, we have

$$v \equiv 0 \ \text{ in } \ I\times(0,T),$$

which contradicts (3.9). □

4. Open problems

In this section, we state some open problems concerning stabilization for the HNLS equation.

- We show an exponential decay of the solution for the HNLS equation with a localized damping term. It is an interesting problem that whether the solution decays without damping term for any $L \in \mathcal{N}$. As we mentioned in the Introduction, the solution of the linearized system does not decay for $L \in \mathcal{N}$. We wonder if the nonlinearity gives us stability in the critical cases. A similar result is obtained for the KdV equation in [8]. However, applying the methods in [8] to the HNLS equation, we can only obtain the existence of the center manifold. Maybe we need some new methods to deal with this problem.
- In Theorem 1.1, we only know that the decay rate $\mu(r)$ depends on the radius r for the initial data. A challenging problem is how $\mu(r)$ depends on r? Furthermore, can we obtain some estimate of $\mu(r)$? To estimate $\mu(r)$, the key point is to estimate the constant $C = C(r)$ in (3.6). The main difficulty is proving (3.6) by direct estimates instead of by contradiction.
- For any $\mu > 0$, does there exist constants $r = r(\mu)$, $C = C(\mu)$ and a localized feedback control $F(u)$ (instead of damping term au) such that for

  $$\|u_0\|_{L^2(I)} \leq r,$$

  the solution u will satisfy

  $$\|u(\cdot, t)\|_{L^2(I)} \leq C\mathrm{e}^{-\mu t}, \quad t > 0?$$

  In order to answer this question, a really nonlinear method is needed because with a first-order approximation, one obtains the linear system which has some solution conserving its L^2-norm. So power series expansion method used by Coron [4] may help to conclude some results in this direction.

Acknowledgements

The author would sincerely like to thank the referees for their interesting comments. He also thanks Prof. Yong Li for many useful suggestions and help. This work was supported by NSFC Grant (11701078), China Post-doctoral Science Foundation (2017M611292), the Fundamental Research Funds for the Central Universities (2412017QD002), NSFC Grants (11626043 and 11601073).

References

[1] Agrawal G P, Nonlinear fiber optics, 3rd ed. (2001) (San Diego: Academic Press)

[2] Bisognin E, Bisognin V and Vera Villagrán O, Stabilization of solutions to higher-order nonlinear Schrödinger equation with localized damping, *Electron. J. Differ. Equ.*, **6** (2007) 18 pp

[3] Capistrano-Filho R, Pazoto A and Rosier L, Internal controllability of the Korteweg–de Vries equation on a bounded domain, *ESAIM Control Optim. Calc. Var.*, **21** (2015) 1076–1107

[4] Cavalcanti M M, Domingos Cavalcanti V N, Soriano J and Natali F, Qualitative aspects for the cubic nonlinear Schrödinger equations with localized damping: exponential and polynomial stabilization, *J. Diff. Equat.*, **248** (2010) 2955–2971

[5] Ceballos V, Pavez F R, Villagrán V and Paulo O, Exact boundary controllability for higher order nonlinear Schrödinger equations with constant coefficients, *Electron. J. Differ. Equ.*, **122** (2005) 1–31

[6] Cerpa E, Exact controllability of a nonlinear Korteweg-de Vries equation on a critical spatial domain, *SIAM J. Control Optim.*, **46** (2007) 877–899

[7] Cerpa E and Crépeau E, Boundary controllability for the nonlinear Korteweg–de Vries equation on any critical domain, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **26** (2009) 457–475

[8] Chu J, Coron J M and Shang P, Asymptotic stability of a nonlinear Korteweg–de Vries equation with critical lengths, *J. Differ. Equ.*, **259** (2015) 4045–4085

[9] Coron J M and Crépeau E, Exact boundary controllability of a nonlinear KdV equation with critical lengths, *J. Eur. Math. Soc.*, **6** (2004) 367–398

[10] Fabio N, Exponential stabilization for the nonlinear Schrödinger equation with localized damping, *J Dyn. Control Syst.*, **21** (2015) 461–474

[11] Hasegawa A and Kodama Y, Higher order pulse propagation in a monomode dielectric guide, *IEEE J. Quant. Elect.*, **23** (1987) 510–524

[12] Kivshar Y S and Agrawal G P, optical solitons: from fibers to photonic crystals (2003) (San Diego: Academic Press)

[13] Kodama Y, Optical solitons in a monomode fiber, *J. Phys. Stat.*, **39** (1985) 596–614

[14] Laurent C, Global controllability and stabilization for the nonlinear Schrödinger equation on an interval, *ESAIM Control Optim Calc Var.*, **16** (2010) 356–379

[15] Laurent C, Rosier L and Zhang B-Y, Control and stabilization of the Korteweg–de Vries equation on a periodic domain, *Comm. Partial Differ. Equ.* **35** (2010) 707–744

[16] Pazoto A F, Unique continuation and decay for the Korteweg–de Vries equation with localized damping, *ESAIM Control Optim. Calc. Var.*, **11** (2005) 473–486

[17] Perla Menzala G, Vasconcellos C F and Zuazua E, Stabilization of the Korteweg–de Vries equation with localized damping, *Quart. Appl. Math.*, **60** (2002) 111–129

[18] Rosier L, Exact boundary controllability for the Korteweg-de Vries equation on a bonded domain, *ESAIM Control Optim. Calc. Var.*, **2** (1997) 33–55

[19] Rosier L and Zhang B-Y, Global stabilization of the generalized Korteweg–de Vries equation posed on a finite domain, *SIAM J. Control Optim.*, **45** (2006) 927–956

[20] Simon J, Compact sets in the space $L^p(0, T; B)$, *Ann. Mat. Pura Appl.*, **4** (1987) 65–96

[21] Tartar L, Interpolation non linéaire et régularité, *J. Funct. Anal.*, **9** (1972) 469–489

[22] Tian B and Gao Y T, Variable-coefficient higher-order nonlinear Schrödinger model in optical fibers: New transformation with burstons, brightons and symbolic computation, *Phys. Lett. A*, **359** (2006) 241–248

Communicating Editor: S Kesavan