Proc. Indian Acad. Sci. (Math. Sci.) (2018) 128:38
https://doi.org/10.1007/s12044-018-0404-5

Property of reflexivity for multiplication operators on Banach function spaces

BAHMANN YOUSEFI and FATEMEH ZANGENEH*

Department of Mathematics, Payame Noor University, P.O. Box 19395-3697, Tehran, Iran
*Corresponding author.
E-mail: b_yousefi@pnu.ac.ir; f.zangeneh88@yahoo.com

MS received 7 January 2016; revised 17 January 2017; accepted 10 August 2017; published online 15 June 2018

Abstract. In this paper, we give conditions under which the powers of the multiplication operator M_z are reflexive on a Banach space of functions analytic on a plane domain.

Keywords. Banach spaces of analytic functions; multiplication operators; reflexive operator; multipliers; Caratheodory hull; bounded point evaluation; spectral set.

Mathematics Subject Classification. 47B37, 47A25.

1. Introduction

For any set E and any function $f : E \to \mathbb{C}$, define $\|f\|_E$ by

$$\|f\|_E = \sup\{|f(z)| : z \in E\}.$$

If B is a bounded domain in the plane, then the Caratheodory hull (or $\mathbb{C}$-hull) of B is the complement of the closure of the unbounded component of the complement of the closure of B. The $\mathbb{C}$-hull of B is denoted by B^*. Intuitively, B^* can be described as the interior of the outer boundary of B, and in analytic terms it can be defined as the interior of the set of all points z_0 in the plane such that $|p(z_0)| \leq \sup\{|p(z)| : z \in B\}$ for all polynomials p. The components of B^* are simply connected; in fact, one can easily see that each of these components has a connected complement. The component of B^* that contains B is denoted by B_1. Note that for all polynomials p, $\|p\|_B = \|p\|_{B_1}$. Since B_1 is a Caratheodory domain, so by the Farrel–Rubel–Shields theorem [2, Theorem 5.1, p. 151], each bounded analytic function on B_1 can be approximated by a sequence of polynomials pointwise boundedly.

For the algebra $\mathcal{B}(\mathcal{X})$ of all bounded linear operators on a Banach space $\mathcal{X}$, the weak operator topology (WOT) is the one in which a net A_α converges to A if $A_\alpha x \to Ax$ weakly, $x \in \mathcal{X}$. Also, the strong operator topology (SOT) is the one in which a net A_α converges to A if $A_\alpha x \to Ax$, $x \in \mathcal{X}$.

Recall that if $A \in \mathcal{B}(\mathcal{X})$, then $\mathrm{Lat}(A)$ is by definition the lattice of all invariant subspaces of A, and $\mathrm{AlgLat}(A)$ is the algebra of all operators B in $\mathcal{B}(\mathcal{X})$ such that $\mathrm{Lat}(A) \subset \mathrm{Lat}(B)$. An operator A in $\mathcal{B}(\mathcal{X})$ is said to be *reflexive* if $\mathrm{AlgLat}(A) = W(A)$, where $W(A)$ is the smallest subalgebra of $\mathcal{B}(\mathcal{X})$ that contains A and the identity I is closed in the weak operator topology.

In [8], it is shown that any powers of the operator M_z is reflexive on Banach spaces of formal Laurent series. Also, reflexivity of the operator M_z on Hilbert function spaces has been investigated in [3,6] and for the case of Banach function spaces, see [7]. Here we give some sufficient conditions so that the powers of the operator M_z, acting on Banach function spaces becomes reflexive. As usual, for a good basic source of reflexivity we refer to [1, Chapter 8].

Consider a Banach space $\mathcal{X}$ of functions analytic on a plane domain G, such that for each $\lambda \in G$, the linear functional e_λ of evaluation at λ (defined by $e_\lambda(f) = f(\lambda)$) is bounded on $\mathcal{X}$. A complex-valued function φ on G for which $\varphi f \in \mathcal{X}$ for every $f \in \mathcal{X}$ is called a *multiplier* of $\mathcal{X}$ and the collection of all these multipliers is denoted by $\mathcal{M}(\mathcal{X})$. Each multiplier φ of $\mathcal{X}$ determines a multiplication operator M_φ on $\mathcal{X}$ by $M_\varphi f = \varphi f$, $f \in \mathcal{X}$. It is well-known that each multiplier is a bounded analytic function on G, in fact $\|\varphi\|_G \le \|M_\varphi\|$. The notation $\|\varphi\|_\infty = \|M_\varphi\|$ is usually used for the norm of the operator M_φ.

By $H(G)$ and $H^\infty(G)$ we will mean respectively the set of analytic functions on a plane domain G and the set of bounded analytic functions on G. Also, by $\mathcal{P}(G)$ we mean the uniform closure in $C(G, \mathbb{C})$ (the space of continuous functions from G into $\mathbb{C}$) of the polynomials. Note that $f \in \mathcal{P}(G)$ if and only if there exists a sequence of polynomials $\{p_n\}_n$ that converges uniformly to f on every compact subset of G.

2. Main results

We investigate the reflexivity of the powers of the multiplication operator M_z acting on a Banach function space.

Recall that a sequence $\{x_n\}_n$ in a Banach space $\mathcal{X}$ is called a Schauder basis of $\mathcal{X}$ if for every $x \in \mathcal{X}$ there is a unique sequence of scalars $\{a_n\}_n$ so that $x = \sum_n a_n x_n$. In this case, the closed linear span of $\{x_n\}_n$ is all of $\mathcal{X}$. Also, for every integer n, the linear functional x_n^* on $\mathcal{X}$ defined by $x_n^*(\sum_i a_i x_i) = a_i$ is a bounded linear functional. These functionals $\{x_n^*\}_n$, which are characterized by the relation $x_n^*(x_m) = \delta_m(n)$, are called the biorthogonal functionals associated to the basis $\{x_n\}$. In the weak* topology, $x^* = \sum_n x^*(x_n)x_n^*$ for every $x^* \in \mathcal{X}^*$, and we have convergence in norm for every $x^* \in \mathcal{X}^*$ if and only if the sequence $\{x_n^*\}_n$ is a Schauder basis of $\mathcal{X}^*$. For this to happen, $\mathcal{X}^*$ must, in particular, be separable. On the other hand, this is always the case if $\mathcal{X}$ is reflexive.

From now on, let Ω be a domain in the complex plane such that Ω_1 is equal to the open unit disc $\mathbb{D}$. Also, suppose that the Banach space $\mathcal{X}$ under consideration satisfy the following axioms:

Axiom 1. $\mathcal{X}$ is a subspace of the space of all analytic functions on Ω that are continuous on $\bar{\Omega}$.

Axiom 2. For each $\lambda \in \bar{\Omega}$, the linear functional of evaluation at λ, e_λ, is bounded on $\mathcal{X}$.

Axiom 3. The sequences $\{f_k\}_k$ and $\{f_k^*\}_k$ are Schauder basis for $\mathcal{X}$ and $\mathcal{X}^*$ respectively, where $f_k(z) = z^k$ for all integers k and $\{f_k^*\}_k$ is also the biorthogonal functionals associated to $\{f_k\}_k$.

For $h = \sum_n \hat{h}(n)z^n \in H(\mathbb{D}) \cap \mathcal{M}(\mathcal{X})$ and $w \in \partial\mathbb{D}$, define h_w by $h_w(z) = h(wz)$. Then $h_w = \sum_n \hat{h}_w(n)z^n$ where $\hat{h}_w(n) = w^n\hat{h}(n)$ for all n. Note that $H(\mathbb{D}) \cap \mathcal{M}(\mathcal{X})$ is nonempty since $1, z \in H(\mathbb{D}) \cap \mathcal{M}(\mathcal{X})$.

DEFINITION 2.1

We say that $H(\mathbb{D}) \cap \mathcal{M}(\mathcal{X})$ is bi-isometrically rotation invariant whenever $\varphi \in H(\mathbb{D}) \cap \mathcal{M}(\mathcal{X})$, then $\varphi_{e^{-i\theta}} \in H(\mathbb{D}) \cap \mathcal{M}(\mathcal{X})$, $\|\varphi\|_\infty = \|\varphi_{e^{-i\theta}}\|_\infty$ and $\|\varphi\| = \|\varphi_{e^{-i\theta}}\|$ for all $\theta \in \mathbb{R}$.

Furthermore, we assume that $\mathcal{X}$ holds in the following axiom:

Axiom 4. $z \in \mathcal{M}(\mathcal{X})$ and $H(\mathbb{D}) \cap \mathcal{M}(\mathcal{X})$ is bi-isometrically rotation invariant.

In the following, we give an example of a Banach space which holds in the Axioms 1 through 4.

Example 2.2. Let $\{\beta(n)\}_{n=-\infty}^{\infty}$ be a sequence of positive numbers satisfying $\beta(0) = 1$ and $1 < p < \infty$. The space $L^p(\beta)$ consists of all formal Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$$

such that the norm

$$\|f\| = \|f\|_\beta = \left(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p\right)^{\frac{1}{p}}$$

is finite. Note that when n ranges on $\mathbb{N} \cup \{0\}$, they are called formal power series and are denoted by $H^p(\beta)$. It is well-known that these are Banach spaces with the norm $\|\cdot\|_\beta$. Let $\hat{f}_k(n) = \delta_k(n)$, so $f_k(z) = z^k$ and $\|f_k\| = \beta(k)$. We denote the set of multipliers

$$\left\{\varphi \in L^p(\beta) : \ \varphi L^p(\beta) \subseteq L^p(\beta)\right\}$$

by $L^p_\infty(\beta)$ and the linear operator of multiplication by φ on $L^p(\beta)$ by M_φ.

We say that a complex number λ is a *bounded point evaluation* on $L^p(\beta)$ if the functional $e_\lambda : L^p(\beta) \longrightarrow \mathbb{C}$ defined by $e_\lambda(f) = f(\lambda)$ is bounded. We note that

$$||e_\lambda||^q = \sum_{n=-\infty}^{\infty} \frac{|\lambda|^{nq}}{\beta(n)^q}.$$

Let $\frac{1}{p} + \frac{1}{q} = 1$. Then it is well-known that $L^p(\beta)^* = L^q(\beta^{\frac{p}{q}})$ and if $f(z) = \sum_n \hat{f}(n)z^n \in L^p(\beta)$, $g(z) = \sum_n \hat{g}(n)z^n \in L^q(\beta^{\frac{p}{q}})$, then

$$g(f) = \sum_n \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^p$$

[14, 16, 17]. Let M_z be bounded on $L^p(\beta)$ and consider the following notations:

$$r_0 = \overline{\lim}\beta(-n)^{\frac{-1}{n}};\ \Omega_0 = \{z \in \mathbb{C} : |z| > r_0\}$$
$$r_1 = \underline{\lim}\beta(n)^{\frac{1}{n}};\ \Omega_1 = \{z \in \mathbb{C} : |z| < r_1\}$$
$$\Omega = \Omega_0 \cap \Omega_1 = \{z \in \mathbb{C} : r_0 < |z| < r_1\}.$$

Assume that $r_0 < r_1 = 1$. Then, Ω is an annulus with $\Omega_1 = \mathbb{D}$ and $L^p(\beta) \subset H(\Omega)$. If

$$\sum_{n=-\infty}^{-1} r_0^{nq}/\beta(n)^q < \infty;\quad \sum_{n=0}^{\infty} 1/\beta(n)^q < \infty,$$

then each $\lambda \in \bar{\Omega}$ is a bounded point evaluation and we have $L^p(\beta) \subset H(\Omega) \cap C(\bar{\Omega})$ [9]. So Axioms 1 and 2 are consistent. Clearly, $\{f_k\}_{k\in\mathbb{Z}}$ is a basis for both $L^p(\beta)$ and it's dual $L^q(\beta^{p/q})$. Put $f_j^* = f_j$ for all $j \in \mathbb{Z}$. Then $f_j^*(f_k) = \delta_j(k)$, where $f_j^* \in L^p(\beta)^* = L^q(\beta^{p/q})$ and $f_k \in L^p(\beta)$), so Axiom 3 holds. By a similar method used in the proof of Proposition 28 in [5, page 88], we can see that $H(\mathbb{D}) \cap \mathcal{M}(\mathcal{X})$ is bi-isometrically rotation invariant. Thus Axiom 4 also holds.

The following Lemma extends a result obtained by Allen Shields [5, page 88] that have been proved only for the special case where $\mathcal{X}$ is $H^2(\beta)$, the Hilbert space of formal power series.

Lemma 2.3. *Let $\varphi \in H(\mathbb{D}) \cap \mathcal{M}(\mathcal{X})$. Then for the sequence $\{r_n = \sum_j \hat{r_n}(j)z^j\}$ such that $\hat{r_n}(j) = (1 - \frac{j}{n+1})\hat{\varphi}(j)$ whenever $j = 0, \ldots, n$ and is 0 otherwise, we have $M_{r_n} \to M_\varphi$ in the weak operator topology.*

Proof. First note that by Axiom 4, $H(\mathbb{D}) \cap \mathcal{M}(\mathcal{X})$ is bi-isometrically rotation invariant. Let $\varphi \in H(\mathbb{D}) \cap \mathcal{M}(\mathcal{X})$. Since $\Omega_1 = \mathbb{D}$, we can represent φ by a power series $\sum_{k=0}^{\infty} \hat{\varphi}(k)z^k$. Put

$$P_n(\varphi) = \sum_{k=0}^{n}\left(1 - \frac{k}{n+1}\right)\hat{\varphi}(k)z^k, \qquad n \geq 0$$

and

$$K_n(w) = \sum_{|k|\leq n}\left(1 - \frac{|k|}{n+1}\right)w^k, \qquad w \in \partial U,\ \ n \geq 0.$$

Then

$$\int_{\partial\mathbb{D}} \varphi_w K_n(\bar{w})\mathrm{d}\lambda = M_{P_n(\varphi)}, \qquad n \geq 0.$$

Note that $K_n \geq 0$ and

$$\int_{\partial\mathbb{D}} K_n \mathrm{d}\lambda = 1.$$

Clearly, for all $n \geq 0$, $P_n(\varphi) \in \mathcal{M}(\mathcal{X})$. Since $H(\mathbb{D}) \cap \mathcal{M}(\mathcal{X})$ is bi-isometrically rotation invariant, by the same method used by Shields in [5, page 88], we can see that

$$||M_{P_n(\varphi)}|| \leq ||M_\varphi|| \int_{\partial\mathbb{D}} K_n \mathrm{d}\lambda = ||M_\varphi||.$$

Put $r_n = P_n(\varphi)$. Note that M_{r_n} is represented by the matrix whose $(i, \; j)$-th entry with respect to the base elements $\{f_k\}_k$ and $\{f_k^*\}_k$ (respectively in $\mathcal{X}$ and $\mathcal{X}^*$) is

$$\begin{aligned} f_i^*(M_{r_n} f_j) &= f_i^* \left(\sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) \hat{\varphi}(k) f_{k+j}(z) \right) \\ &= \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) \hat{\varphi}(k) f_i^*(f_{k+j}) \\ &= \left(1 - \frac{i-j}{n}\right) \hat{\varphi}(i-j). \end{aligned}$$

Hence

$$\lim_n f_i^*(M_{r_n} f_j) = f_i^*(M_\varphi f_j)$$

for all base elements f_j and f_i^*, respectively in $\mathcal{X}$ and $\mathcal{X}^*$. By the boundedness of the sequence $\{M_{r_n}\}$, we have $M_{r_n} \to M_\varphi$ in the weak operator topology. This completes the proof. □

Theorem 2.4. *If $\mathcal{P}(\Omega) \subset \mathcal{M}(\mathcal{X})$, then M_{z^k} is reflexive on $\mathcal{X}$ for all $k \geq 1$.*

Proof. The boundedness of point evaluations and the closed graph theorem ensure that multiplication by z, M_z is a bounded operator on $\mathcal{X}$. Let $k \in \mathbb{N}$ and note that $W(M_{z^k}) \subset \mathrm{AlgLat}(M_{z^k})$. On the other hand, let $A \in \mathrm{AlgLat}(M_{z^k})$. Since $\mathrm{Lat}(M_z) \subset \mathrm{Lat}(M_{z^k})$, thus we have $\mathrm{Lat}(M_z) \subset \mathrm{Lat}(A)$. This implies that $A \in \mathrm{AlgLat}(M_z)$. Now by a similar method used in the proof of Theorem 3 in [4], we can see that $A = M_\varphi$ where $\varphi \in \mathcal{M}(\mathcal{X})$, hence $\varphi \in H^\infty(\Omega)$.

Now put $\mathcal{N} = H^\infty(\Omega_1) \cap \mathcal{X}$. Then $\mathcal{N} \neq \emptyset$, because $1 \in \mathcal{N}$. Note that since by the assumption $\mathcal{P}(\Omega) \subset \mathcal{M}(\mathcal{X})$, we can see that $\mathcal{N} \subset \mathcal{M}(\mathcal{X})$. To see this, let $f \in H^\infty(\Omega_1)$. Since Ω_1 is a Caratheodory domain, by the Farrel–Rubel–Shields theorem, there is a sequence $\{p_n\}$ of polynomials converging to f such that for all n, $\|p_n\|_\Omega = \|p_n\|_{\Omega_1} \leq c$ for some $c > 0$. So $\{p_n\}_n$ is a normal family in $H^\infty(\Omega)$ and by passing to a subsequence if necessary, we may suppose that for some function g, $p_n \longrightarrow g$ uniformly on compact subsets of Ω. This implies that indeed $g = f$. Hence by our assumption, $f \in \mathcal{M}(\mathcal{X})$ and so $\mathcal{N} \subset \mathcal{M}(\mathcal{X})$. Also, it is a closed subspace of $\mathcal{X}$, since if $\{h_n\}_n \subset \mathcal{N}$ and $h_n \longrightarrow f$ in $\mathcal{X}$, then for all n, $\|h_n\|_\mathcal{X} \leq c_1$ for some $c_1 > 0$. By Axiom 2, the functionals of point evaluations, at each point of $\bar{\Omega}$ are bounded. So for all λ in Ω, we have

$$h_n(\lambda) = e_\lambda(h_n) \longrightarrow e_\lambda(f) = f(\lambda).$$

Also, we note that for all λ in Ω,

$$|h_n(\lambda)| = |e_\lambda(h_n)| \leq \|h_n\|_\mathcal{X} \|e_\lambda\|.$$

Note that if $f \in \mathcal{X}$, then

$$\sup\{|e_\lambda(f)| : \lambda \in \Omega\} = \sup\{|f(\lambda)| : \lambda \in \Omega\} = ||f||_\Omega = ||f||_{\bar{\Omega}} < \infty.$$

So by the principle of boundedness theorem, $\{e_\lambda : \lambda \in \Omega\}$ is bounded. Put $c_2 = \sup\{\|e_\lambda\| : \lambda \in \Omega\}$, thus we get $|h_n(\lambda)| \leq c_2\|h_n\|_{\mathcal{X}}$ and so

$$\|h_n\|_\Omega \leq c_2\|h_n\|_{\mathcal{X}} \leq c_1c_2$$

for all n. Since $h_n \in H^\infty(\Omega_1)$, $\|h_n\|_{\Omega_1} = \|h_n\|_\Omega$ and so $\|h_n\|_{\Omega_1} \leq c_1c_2$ for all n. This implies that $\{h_n\}_n$ is a normal family in $H^\infty(\Omega_1)$ and we may assume that for some function g, $h_n \longrightarrow g$ uniformly on compact subsets of Ω_1. Thus $g \in H^\infty(\Omega_1)$. But by pointwise convergence $f = g$ on Ω and so f can be extended to a bounded analytic function on Ω_1, i.e., $f \in H^\infty(\Omega_1)$ and so $\mathcal{N}$ is indeed a closed subspace of $\mathcal{X}$. Now, clearly $\mathcal{N} \in \mathrm{Lat}(M_z)$, thus $A\mathcal{N} \subset \mathcal{N}$. Since $1 \in \mathcal{N}$, we get

$$A1 = \varphi \in \mathcal{N} \subset H^\infty(\Omega_1).$$

But $\Omega_1 = \mathbb{D}$, thus $\varphi \in H(\mathbb{D}) \cap \mathcal{M}(\mathcal{X})$ and so by the same method used in the proof of Lemma 2.1 in [8], we can see that there exists a sequence of polynomials $\{r_n\}$ (indeed $r_n = P_n(\varphi)$) such that $M_{r_n} \to M_\varphi$ in the weak operator topology. Now let $\mathcal{M}_k$ be the closed linear span of the set $\{f_{nk} : n \geq 0\}$ (recall that by Axiom 3, $f_i(z) = z^i$ for all i). We have

$$M_{z^k} f_{nk} = f_{(n+1)k} \in \mathcal{M}_k$$

for all $n \geq 0$. Thus $\mathcal{M}_k \in \mathrm{Lat}(M_{z^k})$ and so $\mathcal{M}_k \in \mathrm{Lat}(M_\varphi)$. Let

$$\varphi(z) = \sum_{n=0}^{\infty} \hat{\varphi}(n) z^n.$$

Since $1 \in \mathcal{M}_k$, $M_\varphi 1 = \varphi \in \mathcal{M}_k$. Hence $\hat{\varphi}(i) = 0$ for all $i \neq nk$, $n \geq 0$. Now, by a consequence of the particular construction of r_n used in Lemma 2.3, each r_n should be a polynomial in z^k, i.e., $r_n(z) = q_n(z^k)$ for some polynomial q_n. Thus

$$M_{r_n} = r_n(M_z) = q_n(M_{z^k}) \to A$$

in the weak operator topology. Hence $A \in W(M_{z^k})$. Thus M_{z^k} is reflexive and this completes the proof. □

In the proof of Theorem 2.4, we used the assumption $\mathcal{P}(\Omega) \subset \mathcal{M}(\mathcal{X})$ to show that $H^\infty(\Omega_1) \cap \mathcal{X} \subset \mathcal{M}(\mathcal{X})$. So the following corollary is an immediate consequence of the proof of Theorem 2.4.

COROLLARY 2.5

If $z \in \mathcal{M}(\mathcal{X})$ and $H^\infty(\Omega_1) \cap \mathcal{X} \subset \mathcal{M}(\mathcal{X})$, then M_{z^k} is reflexive on $\mathcal{X}$ for all $k \geq 1$.

Recall that M_z is called polynomially bounded on $\mathcal{X} \subset H(\Omega)$ if there exists $c > 0$ such that $||p(M_z)|| \leq c||p||_\Omega$ for all polynomials p.

Theorem 2.6. *If M_z is polynomially bounded on $\mathcal{X}$, then M_{z^k} is reflexive on $\mathcal{X}$ for all $k \geq 1$.*

Proof. Since M_z is polynomially bounded, there exists $c > 0$ such that $||p(M_z)|| \leq c||p||_\Omega$ for all polynomials p. By Corollary 2.5, it is sufficient to show that $H^\infty(\Omega_1) \cap \mathcal{X} \subset \mathcal{M}(\mathcal{X})$. For this, let $f \in H^\infty(\Omega_1) \cap \mathcal{X}$. By the Farrel–Rubel–Shields theorem, there exists a sequence $\{p_n\}$ of polynomials converging to f such that for all n, $\|p_n\|_\Omega = \|p_n\|_{\Omega_1} \leq d$ for some $d > 0$. So we obtain

$$\|M_{p_n}\| \leq c\|p_n\|_{\Omega_1} \leq cd$$

for all n. Since $\mathcal{X}$ is reflexive, the unit ball of $\mathcal{X}$ is weakly compact. Therefore ball $B(\mathcal{X})$ is compact in the weak operator topology and so by passing to a subsequence, if necessary, we may assume that for some $A \in B(\mathcal{X})$, $M_{p_n} \longrightarrow A$ in the weak operator topology. Using the fact that $M^*_{p_n} \longrightarrow A^*$ in the weak operator topology and acting these operators on e_λ we obtain

$$p_n(\lambda)e_\lambda = M^*_{p_n} e_\lambda \longrightarrow A^* e_\lambda$$

weakly. Since $p_n(\lambda) \longrightarrow f(\lambda)$, we see that

$$A^* e_\lambda = f(\lambda)e_\lambda.$$

Because the closed linear span of $\{e_\lambda : \lambda \in \Omega\}$ is weak star dense in $\mathcal{X}^*$, we conclude that $A = M_f$ and so $f \in \mathcal{M}(\mathcal{X})$. Thus indeed $H^\infty(\Omega_1) \cap \mathcal{X} \subset \mathcal{M}(\mathcal{X})$. □

Remark 2.7. We used Axioms 1, 2 in the proof of Theorem 2.4 to show that $\{e_\lambda : \lambda \in \Omega\}$ is bounded. So Theorems 2.4, 2.6 and Corollary 2.5 remain true if we substitute Axioms 1, 2 by the following axioms:

Axiom (*i*). $\mathcal{X}$ is a subspace of the space of all analytic functions on Ω.

Axiom (*ii*). The set $\{e_\lambda : \lambda \in \Omega\}$ is norm bounded.

In the following, we give examples satisfying Axioms (i) and (ii).

Example 2.8. Let $\{\beta(n)\}_{n=-\infty}^{\infty}$ be a sequence of positive numbers satisfying $\beta(0) = 1$ and $1 < p < \infty$. Consider the space $L^p(\beta)$ and let the notations $r_0,\ r_1$ and Ω are defined as in Example 2.2 and $r_0 < r_1 = 1$. Then clearly $L^p(\beta) \subset H(\Omega)$. Also, note that if $\lambda \in \Omega$, then

$$||e_\lambda||^q = \sum_{n=-\infty}^{\infty} \frac{|\lambda|^{nq}}{\beta(n)^q}.$$

So, if

$$\sum_{n=-\infty}^{-1} r_0^{nq}/\beta(n)^q < \infty; \quad \sum_{n=0}^{\infty} 1/\beta(n)^q < \infty,$$

then each $\lambda \in \bar{\Omega}$ is a bounded point evaluation and clearly $\{e_\lambda : \lambda \in \Omega\}$ is norm bounded.

Example 2.9. Let $\{\beta(n)\}_{n=-\infty}^{\infty}$ be a sequence of positive numbers satisfying $\beta(0) = 1$ and $1 < p < \infty$. Consider the space $L^p(\beta)$ and let the notations r_0, r_1 and Ω be defined as in Example 2.2 and $r_0 < r_1 = 1$. Then clearly $L^p(\beta) \subset H(\Omega)$. If $L^p(\beta) = L^p_\infty(\beta)$, then $L^p(\beta)$ is a Banach algebra and since for each $\lambda \in \Omega$, e_λ is a homomorphism on $L^p(\beta)$. Thus it should be $||e_\lambda|| = 1$ for all $\lambda \in \Omega$. Hence Axiom (ii) holds.

COROLLARY 2.10

If $L^p(\beta) = L^p_\infty(\beta)$, then M_{z^k} is reflexive on $L^p(\beta)$ for all positive integers k.

Proof. It is sufficient only to show that M_z is polynomially bounded. Put $\mathcal{M} = H^\infty(\Omega_1) \cap L^p_\infty(\beta)$. Then $\mathcal{M} \neq \emptyset$, since $1 \in \mathcal{M}$. By the same method used earlier in Theorem 2.4, we can see that $\mathcal{M}$ is closed. Now we show that

$$L : (\mathcal{M}, \|.\|_{\Omega_1}) \longrightarrow B(L^p(\beta))$$

be given by $L(\varphi) = M_\varphi$ is continuous. Suppose that the sequence $\{\varphi_n\}_n$ converges to φ in $\mathcal{M}$ and $L(\varphi_n) = M_{\varphi_n}$ converges to A in $B(L^p(\beta))$. Then for each f in $L^p(\beta)$,

$$Af = \lim_n M_{\varphi_n} f = \lim_n \varphi_n f$$

and so $\{\varphi_n f\}_n$ is convergent in $L^p(\beta)$. Note that by the continuity of point evaluations $\varphi_n f$ converges pointwise to φf. Thus Af is analytic and agree with φf on Ω. Hence $A = M_\varphi$ and so L is continuous. This implies that there is a constant $d > 0$ such that $\|M_\varphi\| \leq d\|\varphi\|_\Omega$ for all φ in $\mathcal{M}$. In the special case, we get $||M_p|| \leq d||p||_\Omega$ for all polynomials p and so M_z is polynomially bounded. Thus indeed M_{z^k} is reflexive on $L^p(\beta)$ for all positive integers k.

COROLLARY 2.11

If M_z is invertible and M_{z^k} is reflexive on $L^p(\beta)$ for all $k \in \mathbb{N}$, then M_{z^k} is reflexive for all $k \in \mathbb{Z}$.

Proof. Let M_z be invertible on $L^p(\beta)$ and note that since $M_z f_m = f_{m+1}$, we have $M_z^{-1} f_m = f_{m-1}$ for all m. Let $f'_m = f_{-m}$. Then $M_z^{-1} f'_m = f'_{m+1}$ for all m. So $\{f'_m\}$ is shifted (forward) by M_z^{-1}. Hence M_z^{-k} is reflexive for all $k \geq 1$. But the identity operator is also reflexive, thus indeed M_{z^k} is reflexive for all integers k. □

Remark 2.12. We have proved the main theorem under some restrictions on Ω. So it is interesting to ask about conditions under which any power of the multiplication operator M_z is reflexive as an operator acting on a Banach space $\mathcal{X} \subset H(\Omega)$, where Ω is an arbitrary complex plane domain.

Remark 2.13. A norm-closed subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{X})$ is called hereditary reflexive if, for every weakly closed subspace $\mathcal{L} \subset \mathcal{A}$, every $x \in \mathcal{X}$ and $T \in \mathcal{B}(\mathcal{X})$, we have $T \in \mathcal{L}$ whenever Tx belongs to the norm closure of the set $\{Lx : L \in \mathcal{L}\}$. Now since in the main theorem, reflexivity of M_z^k is considered for any positive integer k and since an operator M_z^m belongs to the algebra generated by M_z^k if k divides m, it is natural and interesting to investigate about conditions for hereditary reflexivity.

Acknowledgements

The authors are grateful to referees for helpful suggestions. This paper is a part of the second author's Doctoral thesis, written at Payame Noor University under the direction of the first author.

References

[1] Conway J B, A Course in Operator Theory, Amer. Math. Soc. (2000)
[2] Gamelin T, Uniform Algebra (1984) (NY: Chelsea)
[3] Jahedi K, On the multiplication operators on analytic function spaces, *Iranian J Sci. Technol.*, **37(4)** (2013) 449–452
[4] Seddighi K and Yousefi B, On the reflexivity of operators on function spaces, *Proc. Amer. Math. Soc.*, **116** (1992) 45–52
[5] Shields A L, Weighted shift operators and analytic functions theory, Math. Surveys, A.M.S. Providence 13 (1974) 49–128
[6] Yousefi B, Multiplication operators on Hilbert spaces of analytic functions, *Archiv der Mathematik*, **83(6)** (2004) 536–539
[7] Yousefi B and Khaksari A, Multiplication operators on analytic functional spaces, *Taiwanese J. Math.*, **13(4)** (2009) 1159–1165
[8] Yousefi B and Khoshdel Sh, Reflexivity of powers of the multiplication operator on special function spaces, *Acta Mathematica Scientia*, **32B(6)** (2012) 2279–2284
[9] Zorboska N, Cyclic composition operators on smooth weighted Hardy spaces, *Rocky Mountain J. Math.*, **29(2)** (1999) 725–740

COMMUNICATING EDITOR: Gadadhar Misra