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Some new estimates for the Helgason Fourier transform on rank 1 symmetric spaces

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Abstract. New estimates are proved for the Helgason Fourier transform in the space $L^2(X)$ on certain classes of functions characterized by the spherical modulus of continuity.

Keywords. Symmetric space; Helgason Fourier transform; spherical modulus of continuity; generalized translation operator.

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1. Introduction

In [6], Abilov *et al.* proved two useful estimates for the Fourier transform in the space of square integrable functions on certain classes of functions characterized by the generalized continuity modulus, using a translation operator.

In this paper, we prove new estimates in certain classes of functions characterized by a spherical modulus of continuity and connected with the Helgason Fourier transform associated to Δ in $L^2(X)$ analogs of the statements proved in [6]. For this purpose, we use a generalized translation operator.

In section 2, we give some definitions and preliminaries concerning the Helgason Fourier transform. The new estimates are proved in section 3.

2. Helgason Fourier transformation on symmetric spaces

Here we collect the necessary facts about the Fourier transformation on symmetric spaces and the spherical Fourier transformation (see [1,2]). For the required properties of semisimple Lie groups and symmetric spaces, we refer the reader to [3,4]. An arbitrary Riemannian symmetric space X of noncompact type can be represented as the factor space G/K, where G is a connected noncompact semisimple Lie group with finite center, and K is a maximal compact subgroup of G. On $X = G/K$, the group G acts transitively by left shifts, and K coincides with the stabilizer of the point $o = eK$ (e is the unity of G). Let $G = NAK$ be an Iwasawa decomposition for G, and let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$ be the Lie algebras of the groups

G, K, A, N, respectively. We denote by M the centralizer of the subgroup A in K and put $B = K/M$. Let dx be *a* G-invariant measure on X; the symbols db and dk denote the normalized K-invariant measures on B and K, respectively.

We denote by $\mathfrak{a}^*$ the real dual to $\mathfrak{a}$, and by W the finite Weyl group acting on $\mathfrak{a}^*$. Let $\sum$ be the set of restricted roots ($\sum \subset \mathfrak{a}^*$), Let $\sum^+$ be the set of restricted positive roots, and let

$$\mathfrak{a}^+ = \left\{ h \in \mathfrak{a} : \alpha(h) > 0, \alpha \in \sum^+ \right\},$$

be the positive Weyl chamber. If ρ is the half-sum of the positive roots (with multiplicity), then $\rho \in \mathfrak{a}^*$. Let $\langle , \rangle$ be the Killing form on the Lie algebra $\mathfrak{g}$. This form is positive definite on $\mathfrak{a}$. For $\lambda \in \mathfrak{a}^*$, let H_λ denote a vector in $\mathfrak{a}$ such that $\lambda(H) = \langle H_\lambda, H \rangle$ for all $H \in \mathfrak{a}$. For $\lambda, \mu \in \mathfrak{a}^*$, we put $\langle \lambda, \mu \rangle := \langle H_\lambda, H_\mu \rangle$. The correspondence $\lambda \mapsto H_\lambda$ enables us to identify $\mathfrak{a}^*$ and $\mathfrak{a}$. By this identification, the action of the Weyl group W can be transferred to $\mathfrak{a}$. Let

$$\mathfrak{a}^*_+ = \{\lambda \in \mathfrak{a}^* : H_\lambda \in \mathfrak{a}^+\}.$$

If X is a symmetric space of rank 1, then $\dim \mathfrak{a}^* = 1$, and the set $\sum^+$ consists of the roots α and 2α with some multiplicities a and b depending on X (see [1]). In this case, we identify the set $\mathfrak{a}^*$ with $\mathbb{R}$ via the correspondence $\lambda \leftrightarrow \lambda\alpha$, $\lambda \in \mathbb{R}$. Upon this identification, positive numbers correspond to the set $\mathfrak{a}^*_+$. The numbers m_α, $m_{2\alpha}$ occur frequently in various formulas for rank 1 symmetric spaces. For example, the area of a sphere of radius t on X is equal to

$$S(t) = c(\sinh t)^{m_\alpha}(\sinh 2t)^{m_{2\alpha}},$$

where c is some constant; the dimension of X is equal to

$$\dim X = m_\alpha + m_{2\alpha} + 1.$$

We return to the case in which $X = G/K$ is an arbitrary symmetric space. Given $g \in G$, denote by $A(g) \in \mathfrak{a}$ the unique element satisfying

$$g = n \cdot \exp A(g) \cdot u,$$

where $u \in K$ and $n \in N$. For $x = gK \in X$ and $b = kM \in B = K/M$, we put

$$A(x, b) := A(k^{-1}g).$$

We denote by $\mathcal{D}(X)$ and $\mathcal{D}(G)$ the set of infinitely differentiable compactly-supported functions on X and G. Let dg be the element of the Haar measure on G. We assume that the Haar measure on G is normalized so that

$$\int_X f(x)\mathrm{d}x = \int_G f(g \cdot o)\mathrm{d}g, \quad f \in \mathcal{D}(X).$$

For a function $f(x) \in \mathcal{D}(X)$, the Helgason Fourier transform on X was introduced by Helgason (see [2] or [5]) and is defined by the formula

$$\hat{f}(\lambda, b) := \int_X f(x)\mathrm{e}^{(i\lambda+\rho)(A(x,b))}\mathrm{d}x, \quad \lambda \in \mathfrak{a}^*, \quad b \in B = K/M.$$

We can norm the measure on X so that the inversion formula for the Helgason Fourier transform on X would have the form

$$f(x) = \frac{1}{|W|} \int_{\mathfrak{a}^* \times B} \hat{f}(\lambda, b)\mathrm{e}^{(i\lambda+\rho)(A(x,b))}|c(\lambda)|^{-2}\mathrm{d}\lambda\mathrm{d}b,$$

where $|W|$ is the order of the Weyl group, $d\lambda$ is the element of the Euclidean measure on $\mathfrak{a}^*$, and $c(\lambda)$ is the Harish Chandra c-function. Henceforth, for brevity, we use the notation

$$\mathrm{d}\mu(\lambda) := |c(\lambda)|^{-2}\mathrm{d}\lambda.$$

Also, the Plancherel formula is valid:

$$\|f\|_2^2 := \int_X |f(x)|^2\mathrm{d}x = \frac{1}{|W|}\int_{\mathfrak{a}^*\times B} |\hat{f}(\lambda, b)|^2\mathrm{d}\mu(\lambda)db$$
$$= \int_{\mathfrak{a}_+^*\times B} |\hat{f}(\lambda, b)|^2\mathrm{d}\mu(\lambda)\mathrm{d}b.$$

By continuity, the mapping $f(x) \mapsto \hat{f}(\lambda, b)$ extends from $\mathcal{D}(X)$ to an isomorphism of the Hilbert space $L^2(X) = L^2(X, \mathrm{d}x)$ onto the Hilbert space $L^2(\mathfrak{a}_+^* \times B, \mathrm{d}\mu(\lambda)\mathrm{d}b)$.

Let $n = \dim X$. Denote by $d(x, y)$ the distance between points $x, y \in X$ and let

$$\sigma(x; t) = \{y \in X : d(x, y) = t\}$$

be the sphere of radius $t > 0$ on X centered at x. Let $d\sigma_x(y)$ be the $(n-1)$-dimensional area element of the sphere $\sigma(x; t)$ and let $|\sigma(t)|$ be the area of the whole sphere $\sigma(x; t)$ (it is independent of the point x). We denote by $C_0(X)$ the set of all continuous compactly-supported functions on X. Given $f(x) \in C_0(X)$, define the generalized translation operator S^t by the formula

$$(S^t f)(x) = \frac{1}{|\sigma(t)|}\int_{\sigma(x;t)} f(y)\mathrm{d}\sigma_x(y), \quad t > 0;$$

i.e., $(S^t f)(x)$ is the average of f over $\sigma(x; t)$.

Lemma 2.1 [8]. *The following inequality is valid for every function $f \in L^2(X)$ and every $t \in \mathbb{R}_+ = [0; +\infty)$:*

$$\|S^t f\|_2 \le \|f\|_2.$$

An important role in harmonic analysis on symmetric spaces is played by spherical functions (see [1]*). For $\lambda \in \mathfrak{a}^*$, let $\varphi_\lambda(t)$ denote the zonal spherical function on G defined by the Harish-Chandra formula*

$$\varphi_\lambda(g) = \int_K \mathrm{e}^{(i\lambda+\rho)(A(kg))}\mathrm{d}k, \quad g \in G.$$

We list some properties of the spherical functions to be used later on:

$$\varphi_\lambda(u_1 g u_2) = \varphi_\lambda(g), \quad u_1, u_2 \in K,$$
$$\varphi_\lambda(e) = 1,$$
$$\Delta\varphi_\lambda = -(\lambda^2 + \rho^2)\varphi_\lambda,$$

where Δ is the Laplace–Beltrami operator on X, and

$$\int_K \varphi_\lambda(gkh)\mathrm{d}k = \varphi_\lambda(g)\varphi_\lambda(h), \quad g, h \in G.$$

Lemma 2.2 [8]. *If* $f \in L^2(X)$, *then*

$$\widehat{S^t f}(\lambda, b) = \varphi_\lambda(t)\hat{f}(\lambda, b), \quad \lambda, t \in \mathbb{R}_+ = [0; +\infty).$$

Lemma 2.3 [7]. *The following inequalities are valid for a spherical function* $\varphi_\lambda(t)$ $(\lambda, t \in \mathbb{R}_+)$:

(i) $|\varphi_\lambda(t)| \leq 1$,
(ii) $1 - \varphi_\lambda(t) \leq t^2(\lambda^2 + \rho^2)$,
(iii) *there is a constant* $c > 0$ *such that*

$$1 - \varphi_\lambda(t) \geq c,$$

for $\lambda t \geq 1$.

Lemma 2.4 [8]. *Suppose that* $f(x)$ *and* $\Delta f(x)$ *belong to* $L^2(X)$ (*the action of the Laplace–Beltrami operator* Δ *on* f *is understood in the sense of distributions*). *Then*

$$\widehat{\Delta f}(\lambda,b) = -(\lambda^2 + \rho^2)\hat{f}(\lambda,b).$$

For $\alpha > \frac{-1}{2}$, *we introduce the Bessel normalized function of the first kind* j_α *defined by*

$$j_\alpha(x) = \Gamma(\alpha + 1)\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{2n}}{n!\Gamma(n + \alpha + 1)}, \quad x \in \mathbb{R}.$$

In the terms of $j_\alpha(x)$, *we have* (*see* [10])

$$\sqrt{tx}J_\alpha(tx) = O(1), \quad tx \geq 0, \tag{1}$$

where $J_\alpha(x)$ *is Bessel function of the first kind, which is related to* $j_\alpha(x)$ *by the formula*

$$j_\alpha(x) = \frac{2^\alpha\Gamma(\alpha + 1)}{x^\alpha}J_\alpha(x). \tag{2}$$

Lemma 2.5 [9]. *Let* $\alpha > \frac{-1}{2}$. *Then for* $|\nu| \leq \rho$, *there exists a positive constant* c_0 *such that*

$$|1 - \varphi_{\lambda+i\nu}(t)| \geq c_0|1 - j_\alpha(\lambda t)|.$$

For $f \in L^2(X)$, *we define the finite differences of first and higher order as follows*:

$$\Delta_t^1 f = \Delta_t f = (I - S^t)f,$$
$$\Delta_t^k f = \Delta_t(\Delta_t^{k-1} f) = (I - S^t)^k f, \quad k = 2,3,\ldots,$$

where I *is the identity operator in the space* $L^2(X)$.

The spherical modulus of continuity of a function $f \in L^2(X)$ is defined by

$$\omega(f, \delta)_{2,k} = \sup_{0<t\leq\delta}\|\Delta_t^k f\|_2, \quad \delta > 0.$$

Denote by $\mathcal{W}_2^r(\Delta)$, $r = 0, 1, 2, \ldots$, the class of functions $f \in L^2(X)$ that have generalized derivatives in the sense of distributions satisfying $\Delta^r f \in L^2(X)$, i.e.,

$$\mathcal{W}_2^r(\Delta) = \{f \in L^2(X) : \Delta^r f \in L^2(X)\},$$

where $\Delta^0 f = f$, $\Delta^r f = \Delta(\Delta^{r-1} f)$; $r = 1, 2, \ldots$.

3. Main result

The goal of this work is to prove several new estimates for the integral

$$J_N^2(f) = \int_N^{+\infty} \int_B |\hat{f}(\lambda, b)|^2 \mathrm{d}\mu(\lambda)\mathrm{d}b,$$

in certain classes of functions in $L^2(X)$.

Lemma 3.1. *If* $f \in \mathcal{W}_2^r(\Delta)$, *then*

$$\|\Delta_t^k \Delta^r f\|_2^2 = \int_0^{+\infty} \int_B (\lambda^2 + \rho^2)^{2r} |1 - \varphi_\lambda(t)|^{2k} |\hat{f}(\lambda, b)|^2 \mathrm{d}\mu(\lambda)\mathrm{d}b.$$

Proof. From Lemma 2.2, we have

$$\widehat{\Delta_t^k f(\lambda, b))} = (1 - \varphi_\lambda(t))^k \hat{f}(\lambda, b), \quad \lambda, t \in \mathbb{R}_+ = [0; +\infty).$$

Furthermore, we obtain by Lemma 2.4,

$$\widehat{\Delta^r f}(\lambda, b) = (-1)^r (\lambda^2 + \rho^2)^r \hat{f}(\lambda, b).$$

Now by Plancherel formula, we have the result. □

Theorem 3.2. *Given* k, r *and* $f \in \mathcal{W}_2^r(\Delta)$, *there exists a constant* $c > 0$ *such that, for all* $N > 0$,

$$J_N^2(f) = O\big(N^{-4r} \omega(\Delta^r f, cN^{-1})_{2,k}^2\big).$$

Proof. Firstly, we have

$$J_N^2(f) \le \int_N^{+\infty} |j_\alpha(\lambda t)| \mathrm{d}\eta(\lambda) + \int_N^{+\infty} |1 - j_\alpha(\lambda t)| \mathrm{d}\eta(\lambda), \tag{3}$$

with $\mathrm{d}\eta(\lambda) = \int_B |\hat{f}(\lambda, b)|^2 \mathrm{d}b\mathrm{d}\mu(\lambda)$. The parameter $t > 0$ will be chosen later.

In view of formulas (1) and (2), there exists a constant $c_1 > 0$ such that

$$|j_\alpha(\lambda t)| \le c_1 (\lambda t)^{-\alpha - \frac{1}{2}}.$$

Then

$$\int_N^{+\infty} |j_\alpha(\lambda t)| \mathrm{d}\eta(\lambda) \le c_1 (tN)^{-\alpha - \frac{1}{2}} J_N^2(f).$$

Choose a constant c_2 such that the number $c_3 = 1 - c_1 c_2^{-\alpha-\frac{1}{2}}$ is positive. Setting $t = c_2/N$ in the inequality (3), we have

$$c_3 J_N^2(f) \leq \int_N^{+\infty} |1 - j_\alpha(\lambda t)| \mathrm{d}\eta(\lambda). \tag{4}$$

By Hölder inequality and Lemma 2.5, the second term in (4) satisfies

$$\begin{aligned}
&\int_N^{+\infty} |1 - j_\alpha(\lambda t)| \mathrm{d}\eta(\lambda) = \int_N^{+\infty} |1 - j_\alpha(\lambda t)| \cdot 1 \cdot \mathrm{d}\eta(\lambda) \\
&\leq \left(\int_N^{+\infty} |1 - j_\alpha(\lambda t)|^{2k} \mathrm{d}\eta(\lambda) \right)^{1/2k} \left(\int_N^{+\infty} \mathrm{d}\eta(\lambda) \right)^{1-1/2k} \\
&\leq \left(\int_N^{+\infty} (\lambda^2 + \rho^2)^{-2r} |1 - j_\alpha(\lambda t)|^{2k} (\lambda^2 + \rho^2)^{2r} \mathrm{d}\eta(\lambda) \right)^{1/2k} (J_N(f))^{2-1/k} \\
&\leq (N^2 + \rho^2)^{-r/k} \left(\int_N^{+\infty} |1 - j_\alpha(\lambda t)|^{2k} (\lambda^2 + \rho^2)^{2r} \mathrm{d}\eta(\lambda) \right)^{1/2k} (J_N(f))^{2-1/k} \\
&\leq \frac{N^{-2r/k}}{c_0} \left(\int_N^{+\infty} (\lambda^2 + \rho^2)^{2r} |1 - \varphi_\lambda(t)|^{2k} \mathrm{d}\eta(\lambda) \right)^{1/2k} (J_N(f))^{2-1/k}.
\end{aligned}$$

From Lemma 3.1, we conclude that

$$\int_N^{+\infty} (\lambda^2 + \rho^2)^{2r} |1 - \varphi_\lambda(t)|^{2k} \mathrm{d}\eta(\lambda) \leq \|\Delta_t^k \Delta^r f\|_2^2.$$

Therefore

$$\int_N^{+\infty} |1 - j_\alpha(\lambda t)| \mathrm{d}\eta(\lambda) \leq \frac{N^{-2r/k}}{c_0} \|\Delta_t^k \Delta^r f\|_2^{1/k} (J_N(f))^{2-1/k}.$$

For $t = c_2/N$, we obtain

$$c_3 J_N^2(f) \leq \frac{N^{-2r/k}}{c_0} \omega(\Delta^r f, c_2/N)_{2,k}^{1/k} (J_N(f))^{2-1/k}.$$

Consequently by raising both sides to the power k and simplifying by $(J_N(f))^{2k}$, we finally obtain

$$c_0^k c_3^k J_N(f) \leq N^{-2r} \omega(\Delta^r f, c_2/N)_{2,k}.$$

for all $N > 0$. The theorem is proved with $c = c_2$. □

Theorem 3.3. *Let $f \in L^2(X)$. Then, for all $N > 0$,*

$$\omega(f, N^{-1})_{2,k} = O\left(N^{-2k} \left(\sum_{l=0}^{N-1} (l+1)^{4k-1} J_l^2(f) \right)^{\frac{1}{2}} \right).$$

Proof. From Lemma 3.1, we have

$$\|\Delta_t^k f\|_2^2 = \int_0^{+\infty}\int_B |1-\varphi_\lambda(t)|^{2k}|\hat{f}(\lambda,b)|^2 \mathrm{d}\mu(\lambda)\mathrm{d}b.$$

This integral is divided into two:

$$\int_0^{+\infty}\int_B = \int_0^N\int_B + \int_N^{+\infty}\int_B = I_1 + I_2,$$

where $N = [t^{-1}]$. We estimate them separately. From (i) of Lemma 2.3, we have the estimate

$$I_2 \le c_4 \int_N^{+\infty}\int_B |\hat{f}(\lambda,b)|^2 \mathrm{d}\mu(\lambda)\mathrm{d}b = c_4 J_N^2(f).$$

Now, we estimate I_1. From formula (ii) of Lemma 2.3, we have

$$\begin{aligned} I_1 &\le t^{4k}\int_0^N\int_B (\lambda+\rho)^{4k}|\hat{f}(\lambda,b)|^2 \mathrm{d}\mu(\lambda)\mathrm{d}b \\ &= t^{4k}\sum_{l=0}^{N-1}\int_l^{l+1}\int_B (\lambda+\rho)^{4k}|\hat{f}(\lambda,b)|^2 \mathrm{d}\mu(\lambda)\mathrm{d}b \\ &\le t^{4k}\sum_{l=0}^{N-1}(l+\rho+1)^{4k}\left(J_l^2(f) - J_{l+1}^2(f)\right), \end{aligned}$$

From the inequality $l+\rho+1 \le (\rho+1)(l+1)$, we conclude

$$I_1 \le (\rho+1)^{4k} t^{4k}\sum_{l=0}^{N-1} a_l\big(J_l^2(f) - J_{l+1}^2(f)\big),$$

with $a_l = (l+1)^{4k}$.

For all integers $m \ge 1$, by rearranging terms analogous to summation by parts, we obtain

$$\begin{aligned} \sum_{l=0}^{m} a_l\left(J_l^2(f) - J_{l+1}^2(f)\right) &= a_0 J_0^2(f) + \sum_{l=1}^{m}(a_l - a_{l-1})\,J_l^2(f) - a_m J_{m+1}^2(f) \\ &\le a_0 J_0^2(f) + \sum_{l=1}^{m}(a_l - a_{l-1})\,J_l^2(f), \end{aligned}$$

because $a_m J_{m+1}^2(f) \ge 0$. Hence

$$I_1 \le (\rho+1)^{4k} N^{-4k}\left(J_0^2(f) + \sum_{l=1}^{N-1}\left((l+1)^{4k} - l^{4k}\right)J_l^2(f) - N^{4k}J_N^2(f)\right).$$

Moreover by the Lagrange's mean value theorem, we have

$$(l+1)^{4k} - l^{4k} \leq 4k(l+1)^{4k-1}.$$

Then

$$I_1 \leq (\rho+1)^{4k} N^{-4k} \left(J_0^2(f) + 4k \sum_{l=1}^{N-1} (l+1)^{4k-1} J_l^2(f) - N^{4k} J_N^2(f) \right).$$

Combining the estimates for I_1 and I_2 gives

$$\|\Delta_t^k f\|_2^2 = O\left(N^{-4k} \sum_{l=0}^{N-1} (l+1)^{4k-1} J_l^2(f) \right),$$

which implies that

$$\omega(f, N^{-1})_{2,k} = O\left(N^{-2k} \left(\sum_{l=0}^{N-1} (l+1)^{4k-1} J_l^2(f) \right)^{\frac{1}{2}} \right),$$

and this completes the proof. □

Theorem 3.4. *Let $f \in L^2(X)$. If the series*

$$\sum_{l=1}^{\infty} l^{2r-1} J_l(f), \quad r = 1, 2, \ldots$$

converges, then $f \in \mathcal{W}_2^r(\Delta)$ and, for all $N > 0$,

$$\omega(\Delta^r f, N^{-1})_{2,k} = O\left(N^{-4k} \sum_{l=0}^{N-1} (l+1)^{4r+4k-1} J_l^2(f) \right)^{\frac{1}{2}} + O\left(\sum_{l=[\frac{N}{2}]}^{\infty} l^{2r-1} J_l(f) \right).$$

Proof. Let $f \in L^2(X)$. By Lemma 2.4 and Plancherel theorem, we have

$$\begin{aligned}
\|\Delta^r f\|_2^2 &= \int_0^{+\infty} \int_B (\lambda^2+\rho^2)^{2r} |\hat{f}(\lambda, b)|^2 \mathrm{d}\mu(\lambda)\mathrm{d}b \\
&= \sum_{l=0}^{\infty} \int_l^{l+1} \int_B (\lambda^2+\rho^2)^{2r} |\hat{f}(\lambda, b)|^2 \mathrm{d}\mu(\lambda)\mathrm{d}b \\
&\leq \sum_{l=0}^{\infty} \int_l^{l+1} \int_B (\lambda+\rho)^{4r} |\hat{f}(\lambda, b)|^2 \mathrm{d}\mu(\lambda)\mathrm{d}b.
\end{aligned}$$

By rearranging terms analogous to summation by parts, we obtain

$$\|\Delta^r f\|_2^2 \leq J_0^2(f) + 4r\sum_{l=1}^{\infty}(l+\rho+1)^{4r-1}J_l^2(f).$$

From the inequality $l+\rho+1 \leq (l+1)(\rho+1) \leq 2l(\rho+1)$, we conclude

$$\|\Delta^r f\|_2^2 \leq c_5\left(J_0^2(f) + \sum_{l=1}^{\infty}(l+\rho)^{4r-1}J_l^2(f)\right).$$

Hence

$$\|\Delta^r f\|_2 = O\left(\sum_{l=1}^{\infty} l^{2r-1}J_l(f)\right).$$

Since the series

$$\sum_{l=1}^{\infty} l^{2r-1}J_l(f), \quad r = 1, 2, \ldots$$

converges, we see that $f \in \mathcal{W}_2^r(\Delta)$.

From Lemma 3.1, we have

$$\|\Delta_t^k \Delta^r f\|_2^2 = \int_0^{+\infty}\int_B (\lambda^2+\rho^2)^{2r}|1-\varphi_\lambda(t)|^{2k}|\hat{f}(\lambda,b)|^2 \mathrm{d}\mu(\lambda)\mathrm{d}b.$$

This integral is divided into two:

$$\int_0^{+\infty}\int_B = \int_0^N\int_B + \int_N^{+\infty}\int_B = K_1 + K_2,$$

where $N = [t^{-1}]$. We estimate them separately.

By rearranging terms analogous to summation by parts and proceeding as with I_1, we obtain

$$\begin{aligned}
K_1 &= \int_0^N\int_B (\lambda^2+\rho^2)^{2r}|1-\varphi_\lambda(t)|^{2k}|\hat{f}(\lambda,b)|^2 \mathrm{d}\mu(\lambda)\mathrm{d}b \\
&\leq h^{4k}\int_0^N\int_B (\lambda+\rho)^{4r+4k}|\hat{f}(\lambda,b)|^2 \mathrm{d}\mu(\lambda)\mathrm{d}b \\
&\leq h^{4k}\sum_{l=0}^{N-1}\int_l^{l+1}\int_B (\lambda+\rho)^{4r+4k}|\hat{f}(\lambda,b)|^2 \mathrm{d}\mu(\lambda)\mathrm{d}b \\
&\leq c_6 N^{-4k}\sum_{l=0}^{N-1}(l+1)^{4r+4k-1}J_l^2(f).
\end{aligned}$$

Now we estimate K_2 by formula (i) of Lemma 2.3. We obtain

$$\begin{aligned}K_2 &= \int_N^{+\infty}\int_B (\lambda^2+\rho^2)^{2r}|1-\varphi_\lambda(t)|^{2k}|\hat{f}(\lambda,b)|^2\mathrm{d}\mu(\lambda)\mathrm{d}b\\ &= O\left(\int_N^{+\infty}\int_B (\lambda+\rho)^{4r}|\hat{f}(\lambda,b)|^2\mathrm{d}\mu(\lambda)\mathrm{d}b\right)\\ &= O\left(\sum_{m=1}^{\infty}\int_{2^{m-1}N}^{2^mN}\int_B \lambda^{4r}|\hat{f}(\lambda,b)|^2\mathrm{d}\mu(\lambda)\mathrm{d}b\right)\\ &= O\left(\sum_{m=1}^{\infty}(2^mN)^{4r}\int_{2^{m-1}N}^{2^mN}\int_B |\hat{f}(\lambda,b)|^2\mathrm{d}\mu(\lambda)\mathrm{d}b\right)\\ &= O\left(\sum_{m=1}^{\infty}(2^mN)^{4r}J^2_{2^{m-1}N}(f)\right),\end{aligned}$$

i.e.,

$$(K_2)^{\frac{1}{2}} = O\left(\sum_{m=1}^{\infty}(2^mN)^{2r}J_{2^{m-1}N}(f)\right).$$

Taking account of the fact that

$$\begin{aligned}&2^{4r}\sum_{l=2^{m-2}N+1}^{2^{m-1}N} l^{2r-1}J_l(f) \geq 2^{4r}(2^{m-2}N)^{(2r-1)}J_{2^{m-1}N}(f)2^{m-2}N\\ &\quad = (2^mN)^{2r}J_{2^{m-1}N}(f),\end{aligned}$$

we obtain the estimate

$$\begin{aligned}(K_2)^{\frac{1}{2}} &= O\left(\sum_{m=1}^{\infty}\sum_{l=2^{m-2}N+1}^{2^{m-1}N} l^{2r-1}J_l(f)\right)\\ &= O\left(\sum_{l=[\frac{N}{2}]}^{\infty} l^{2r-1}J_l(f)\right).\end{aligned}$$

Combining the estimates for K_1 and K_2 gives

$$\begin{aligned}\|\Delta_t^k\Delta^r f\|_2 &= O\left(N^{-4k}\sum_{l=0}^{N-1}(l+1)^{4r+4k-1}J_l^2(f)\right)^{\frac{1}{2}}\\ &\quad + O\left(\sum_{l=[\frac{N}{2}]}^{\infty} l^{2r-1}J_l(f)\right),\end{aligned}$$

which implies that

$$\omega(\Delta^r f, N^{-1})_{2,k} = O\left(N^{-4k} \sum_{l=0}^{N-1} (l+1)^{4r+4k-1} J_l^2(f)\right)^{\frac{1}{2}} + O\left(\sum_{l=[\frac{N}{2}]}^{\infty} l^{2r-1} J_l(f)\right)$$

and this completes the proof. □

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