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Approximate controllability of a non-autonomous differential equation

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Abstract. In this paper, we establish the approximate controllability results for a non-autonomous functional differential equation using the theory of linear evolution system, Schauder fixed point theorem, and by making use of resolvent operators. The results obtained in this paper, improve the existing ones in this direction, to a considerable extent. An example is also given to illustrate the abstract results.

Keywords. Approximate controllability; Schauder fixed point theorem; resolvent operators.

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1. Introduction

The approximate controllability of the nonlinear systems were studied by many mathematicians, we refer to the survey paper [1] and references therein. Carmichael and Quinn [3] studied the abstract nonlinear systems and used the Banach fixed point theorem to obtain the local exact controllability in the case of nonlinearities with small Lipschitz constant. Joshi *et al.* [14] and George [12] obtained the controllability results of a semilinear control system by assuming monotonicity conditions on the nonlinear part. Dauer and Mahmudov [7] studied the approximate controllability of functional differential equation by using the Schauder fixed point theorem when the semigroup is compact, and the Banach fixed theorem when the semigroup is not compact. Later, Sakthivel *et al.* [11] studied the controllability of a semilinear integrodifferential equation in Banach space.

Kumar and Sukavanam [15] studied the approximate controllability of fractional order semilinear systems with bounded delay. Later Shukla *et al.* [18] studied similar results for the fractional differential equation of order $\alpha \in (1, 2]$, in Hilbert spaces. These days many mathematicians have increasing interest in studying the approximate controllability of various functional differential equations in abstract spaces, see for e.g. [2,4,5,7–9,13, 15,18,19] and references therein.

In [8], Fan characterizes the properties of resolvent operators and also provides some applications of it. Recently, Fan [9] studied the approximate controllability results, using the resolvent operators for fractional differential equations in Hilbert space. Motivated by them, we study the approximate controllability of a non-autonomous evolution equation using the method of resolvent operators.

Let X and $\mathcal{U}$ be two Hilbert spaces. We study the approximate controllability of the following non-autonomous functional differential system:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t, x(t)) + Bu(t),\ t \in J = [0, b], \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $A(t)$ generates an evolution family on a Hilbert space X, the function f is defined from $f : J \times X \to X$. Let the state $x(\cdot)$ take values in X, control function $u(\cdot)$ is given in space $L^2([0, b]; \mathcal{U})$: a Hilbert space of admissible control functions. The operator $B : \mathcal{U} \to X$ is a bounded linear operator and $x_0 \in X$.

2. Preliminaries

In this section, we introduce some notations and basic definitions, which will be used in a sequel throughout the paper.

Notations

- X and $\mathcal{U}$ are Hilbert spaces.
- $\mathbb{R}$ denotes the set of real numbers.
- $J = [0, b], b > 0$.
- $\mathcal{L}(X)$ denotes the set of all bounded linear operators on X.

Let $\{A(t) : t \in [0, b]\}$ be a family of operators satisfying the following properties:

(A1) The domain $D(A(t)) = D$ of $A(t)$ is dense in X and independent of t, $A(t)$ is a closed linear operator.

(A2) For each $t \in J$, the resolvent $R(\lambda, A(t))$ exists for all λ with $\operatorname{Re}\lambda \geq 0$ and there exists $K > 0$ such that $\|R(\lambda, A(t))\| \leq \dfrac{K}{|\lambda| + 1}$.

(A3) There exists constants $H > 0$ and $0 < \alpha \leq 1$ such that for $t, s, \tau \in [0, b]$,

$$\|(A(t) - A(s))A^{-1}(\tau)\| \leq H|t - s|^{\alpha}. \tag{2.1}$$

(A4) For each $t \in [0, b]$ and some $\lambda \in \rho(A(t))$, the resolvent $R(\lambda, A(t))$ is a compact operator.

DEFINITION 2.1

A two-parameter family of bounded linear operators $\{U(t, s) :\ 0 \leq s \leq t \leq b\}$ on X is called an evolution system if the following two conditions are satisfied:

(i) $U(s, s) = I,\ U(t, r)U(r, s) = U(t, s)$ for $0 \leq s \leq r \leq t \leq b$.

(ii) $(t, s) \mapsto U(t, s)$ is strongly continuous for $0 \leq s \leq t \leq b$.

DEFINITION 2.2

A function $x \in C([0, b], X)$ is called a mild solution of (1.1), for each $0 \leq t < b$ and $s \in [0, t)$, if it satisfies the following equation:

$$x(t) = U(t, 0)x_0 + \int_0^t U(t, s)f(s, x(s))\mathrm{d}s + \int_0^t U(t, s)Bu(s)\mathrm{d}s.$$

Now we mention below some lemmas, which will be very useful to prove our main results.

Lemma 2.3 (*see* [17, *Theorem* 6.1]). *Under the assumptions*, (*A*1)–(*A*3), *there is a unique evolution system* $U(t,s)$ *on* $0 \leq s \leq t \leq b$, *satisfying the following conditions:*

(*i*) *For* $0 \leq s \leq t \leq b$, *we have*

$$\|U(t,s)\| \leq C.$$

(*ii*) *For* $0 \leq s \leq t \leq b$, $U(t,s) : X \to D$ *and* $t \to U(t,s)$ *is strongly differentiable in* X. *The derivative* $\frac{\partial}{\partial t}U(t,s) \in \mathcal{L}(X)$ *and is strongly continuous on* $0 \leq s \leq t \leq b$. *Moreover, we also have*

$$\frac{\partial}{\partial t}U(t,s) + A(t)U(t,s) = 0, \quad \textit{for } 0 \leq s \leq t \leq b,$$
$$\left\| \frac{\partial}{\partial t}U(t,s) \right\| = \|A(t)U(t,s)\| \leq \frac{C}{t-s}$$

and

$$\|A(t)U(t,s)A(s)^{-1}\| \leq C, \quad \textit{for } 0 \leq s \leq t \leq b.$$

(*iii*) *For every* $v \in D$ *and* $t \in [0,b]$, $U(t,s)v$ *is differentiable with respect to* s *on* $0 \leq s \leq t \leq b$ *and*

$$\frac{\partial}{\partial s}U(t,s)v = U(t,s)A(s)v.$$

Lemma 2.4 [10]. *Let* $\{A(t) : t \in [0,b]\}$ *satisfy the conditions* (*A*1)–(*A*4). *If* $\{U(t,s) : 0 \leq s \leq t \leq b\}$ *is the linear evolution system generated by the family of operators* $\{A(t) : t \in [0,b]\}$, *then* $\{U(t,s) : 0 \leq s \leq t \leq b\}$ *is a compact operator whenever* $t - s > 0$.

DEFINITION 2.5 (Reachable set)

A set $K_b(f)$ is called 'reachable set' of the system (1.1), defined as follows:

$$K_b(f) := \begin{cases} x(b) \in X : \ u \in L^2([0,b], \mathcal{U}), \text{ and} \\ x \text{ is a mild solution of (1.1) with control } u. \end{cases}$$

Now we define two relevant operators on X, which will be used in the sequel.

$$\Lambda_b = \int_0^b U(b,s)BB^*U(b,s)^* \mathrm{d}s,$$
$$R(\lambda, \Lambda_b) = \left(\lambda I + \Lambda_b\right)^{-1}, \quad \lambda > 0$$

where B^*, $U(b,s)^*$ denote the adjoint operators of B and $U(b,s)$, respectively.

The optimal control problem. In order to find the optimal control u; which will be used in the approximate control system, we consider a linear regulator problem, consisting of minimizing the following cost functional:

$$J(u) = \|x(b) - x_b\|^2 + \lambda \int_0^b \|u(t)\|^2 \mathrm{d}t, \tag{2.2}$$

where x is the solution of (1.1) with control u, $x_b \in X$ and $\lambda > 0$.

It is well known that the optimal control u^*, associated with the approximate controllability of an integer-order differential equation, is the unique solution of the above discussed optimal control problem. Using this idea we obtain below, that the expression for optimal control u^* is given by the feedback law.

Lemma 2.6 *Suppose that u is the optimal control satisfying* (1.1) *and minimizing the cost functional* (2.2). *Then u^* is given by*

$$u(t) = B^*U(b,t)^*R(\lambda, \Lambda_b)p(x(\cdot)), \qquad t \in [0,b]$$

with

$$p(x(\cdot)) = x_b - U(b,0)x_0 - \int_0^b U(b,s)f(s,x(s))\mathrm{d}s.$$

Proof. Let u be the optimal control of (2.2). Then $\varepsilon = 0$ is the critical point of

$$I(\varepsilon) = J(u + \varepsilon w)$$

with $w \in L^2([0,b], \mathcal{U})$.

Now computing variation of the functional J (defined in (2.2)), we obtain

$$\left[2\left\langle x(b) - x_b, \frac{\mathrm{d}}{\mathrm{d}\varepsilon}(x(b) - x_b)\right\rangle + 2\lambda \int_0^b \langle u(t) + \varepsilon w(t), w(t)\rangle_{\mathcal{U}}\, \mathrm{d}t\right]\Bigg|_{\varepsilon=0} = 0,$$

where $\langle\cdot,\cdot\rangle$, $\langle\cdot,\cdot\rangle_{\mathcal{U}}$ denote the inner products in X and $\mathcal{U}$ respectively. Thus we have the following:

$$\begin{aligned}
&\left\langle x(b) - x_b, \int_0^b U(b,t)Bw(t)\mathrm{d}t\right\rangle + \lambda \int_0^b \langle u(t), w(t)\rangle_{\mathcal{U}}\mathrm{d}t = 0\\
&\quad\Rightarrow \int_0^b \langle B^*U(b,t)^*[x(b) - x_b], w(t)\rangle_{\mathcal{U}}\mathrm{d}t + \lambda \int_0^b \langle u(t), w(t)\rangle_{\mathcal{U}} dt = 0\\
&\quad\Rightarrow \int_0^b \langle B^*U(b,t)^*[x(b) - x_b] + \lambda u(t),\ w(t)\rangle_{\mathcal{U}}\mathrm{d}t = 0.
\end{aligned} \tag{2.3}$$

Since $w \in L^2([0,b], \mathcal{U})$ is an arbitrary element, it follows that the optimal control is given by

$$u^*(t) = -\lambda^{-1}B^*U(b,t)^*[x(b) - x_b], \quad \text{for } t \in [0,b] \text{ a.e.}$$

in fact for all $t \in [0,b]$, follows from the continuity of u in $C([0,b], \mathcal{U})$. Therefore the state solution at b of the system (1.1), with the above control u^* is given by

$$\begin{aligned}
x(b) &= U(b,0)x_0 + \int_0^b U(b,s)f(s,x(s))\mathrm{d}s\\
&\qquad - \int_0^b \lambda^{-1}U(b,s)BB^*U(b,s)^*\big(x(b) - x_b\big)[x(b) - x_b]\mathrm{d}s\\
&= U(b,0)x_0 + \int_0^b U(b,s)f(s,x(s))\mathrm{d}s - \lambda^{-1}\Lambda_b[x(b) - x_b].
\end{aligned}$$

Let us define

$$p\big(x(\cdot)\big) := x_b - U(b,0)x_0 - \int_0^b U(b,s)f(s,x(s))\mathrm{d}s.$$

Then we have the following:

$$\begin{aligned} x(b) - x_b &= -p\big(x(\cdot)\big) - \lambda^{-1}\Lambda_b[x(b) - x_b] \\ &= -\lambda I(\lambda I + \Lambda_b)^{-1}p\big(x(\cdot)\big) = -\lambda R(\lambda, \Lambda_b)p(x(\cdot)). \end{aligned} \tag{2.4}$$

Consequently we have

$$u^*(t) = B^*U(b,t)^*R(\lambda, \Lambda_b)p\big(x(\cdot)\big), \qquad t \in [0,b].$$

□

As explained earlier, by adapting the idea of Lemma 2.6, for every $\lambda > 0$ and $x_b \in X$, we can construct the following integral system:

$$\begin{cases} x(t) = U(t,0)x_0 + \int_0^t U(t,s)[f(s,x(s)) + Bu(s)]\mathrm{d}s, \ 0 \le t \le b, \\ u^*(t) = B^*U(b,t)^*R(\lambda, \Lambda_b)p\big(x(\cdot)\big), \\ p(x(\cdot)) = x_b - U(b,0)x_0 - \int_0^b U(b,s)f(s,x(s))\mathrm{d}s. \end{cases} \tag{2.5}$$

Our aim in the section 3, is to prove the approximate controllability of the non-autonomous differential system (1.1) with the help of the above integral system (2.5). More precisely, we wish to approximate any fixed point $x_b \in X$ under appropriate conditions by using the final state of the solution x with the control u given in system (2.5).

3. Approximate controllability

In this section, we show the approximate controllability results for the system (1.1), where the control u appearing in (1.1) minimizes the cost functional (2.2). For that, we first show that for every $\lambda > 0$ and $x_b \in X$, integral system (2.5) has at least one mild solution (say x_λ). Then, given any $x_b \in X$ we can approximate it with these solutions $\{x_\lambda : \lambda > 0\}$. We need the following assumptions:

(H1) $f : [0,b] \times X \to X$ is continuous and there exists a positive constant K such that $\|f(t,x)\| \le K$ for all $(t,x) \in [0,b] \times X$.
(H2) $B : \mathcal{U} \to X$ is a bounded linear operator with $\|B\| = N$, $N > 0$.
(H3) $\lambda R(\lambda, \Lambda_b) \to 0$ as $\lambda \to 0$ in the strong operator topology.

Lemma 3.1 *Assume that* (A1)–(A4) *holds. Let* $G : C([0,b],X) \to C([0,b],X)$ *be the Cauchy operator defined by*

$$(Gh)(t) = \int_0^t U(t,s)h(s)\mathrm{d}s, \quad t \in [0,b]. \tag{3.1}$$

Then G is a compact operator.

Proof. We show G is a compact operator using Arzela–Ascoli theorem. Let the ball B_l defined as

$$B_l := \{h \in C([0, b], X) : \ \|h\| \le l\}, \tag{3.2}$$

be any bounded subset of $C([0, b], X)$. We first show that the set GB_l is a equicontinuous set on $C([0, b], X)$.

For $h \in B_l$ and $0 \le t_1 \le t_2 \le b$, we consider the following:

$$\begin{aligned} I &:= \|(Gf)(t_2) - (Gf)(t_1)\| \\ &= \left\| \int_0^{t_2} U(t_2, s)h(s)\mathrm{d}s - \int_0^{t_1} U(t_1, s)h(s)\mathrm{d}s \right\| \\ &\le \int_0^{t_1} \|U(t_2, s) - U(t_1, s)\| \|h(s)\|\mathrm{d}s + \int_{t_1}^{t_2} \|U(t_2, s)h(s)\|\mathrm{d}s \\ &\le l \int_0^{t_1} \|U(t_2, s) - U(t_1, s)\| \|h(s)\|\mathrm{d}s + Cl(t_2 - t_1). \end{aligned}$$

If $t_1 = 0$, it is easy to see that

$$\lim_{t_2 \to 0} I = 0, \quad \text{uniformly for } f \in B_l.$$

If $0 < t_1 < b$, for $0 < \delta < t_1$, we have

$$\begin{aligned} I \le l &\int_0^{t_1 - \delta} \|U(t_2, s) - U(t_1, s)\|\mathrm{d}s \\ &+ l \int_{t_1 - \delta}^{t_1} \|U(t_2, s) - U(t_1, s)\|\mathrm{d}s + Ml(t_2 - t_1). \end{aligned}$$

As $t_2 \to t_1$ and for δ sufficiently small, the right-hand side of the above inequality tends to zero, independent of $h \in B_l$, since $U(t, s)$ is a strongly continuous operator. Compactness of $U(t, s)$ for $t - s > 0$, implies the continuity in the uniform operator topology. From the arbitrariness of $\delta > 0$, we conclude that

$$\lim_{|t_1 - t_2| \to 0} I = 0, \text{ uniformly for } h \in B_l.$$

Thus GB_l is equicontinuous on $C([0, b], X)$.

Next, we show that $\{(Gh)(t) : \ h \in B_l\}$ is precompact in X for every $t \in [0, b]$. Let $B_l := \{h \in C([0, b], X) : \ \|h\| \le l\}$ be any bounded subset of $C([0, b], X)$.

For $0 < t \le b$ and $0 < \varepsilon < t$, we consider the following:

$$\begin{aligned} &\left\| U(t, t - \varepsilon) \int_0^{t - \varepsilon} U(t - \varepsilon, s)h(s)\mathrm{d}s - \int_0^t U(t, s)h(s)\mathrm{d}s \right\| \\ &\quad \le l \int_0^{t - \varepsilon} \|U(t, t - \varepsilon)U(t - \varepsilon, s) - U(t, s)\|\mathrm{d}s + l \int_{t - \varepsilon}^t \|U(t, s)\|\mathrm{d}s. \end{aligned}$$

Using the semigroup property of the evolution system $\{U(t, s)\}$ where $t - s > 0$, we conclude that

$$\left\| U(t, t-\varepsilon) \int_0^{t-\varepsilon} U(t-\varepsilon, s)h(s)\mathrm{d}s - \int_0^t U(t, s)h(s)\mathrm{d}s \right\| \leq Cl\varepsilon.$$

This shows that $\{(Gh)(t) : h \in B_l\}$ is precompact in X, by using the total boundedness and therefore G is compact in view of Arzela–Ascoli theorem. □

Theorem 3.2. *Assume that conditions* (*A*1)–(*A*4) *hold and* (*H*1)–(*H*2) *are satisfied. Then the integral system* (2.5) *has at least one mild solution on* $[0, b]$ *for every* $\lambda > 0$ *and for fixed* $x_b \in X$.

Proof. For fixed $\lambda > 0$ and given $x_b \in X$, let us define the solution operator $Q : C([0, b], X) \to C([0, b], X)$ as follows:

$$(Qx)(t) = U(t, 0)x_0 + \int_0^t U(t, s)[f(s, x(s)) + Bu(s)]\mathrm{d}s, \quad t \in [0, b], \tag{3.3}$$

with

$$u(t) = B^*U(b, t)^*R(\lambda, \Lambda_b)p\big(x(\cdot)\big) \quad \text{and}$$
$$p\big(x(\cdot)\big) = x_b - U(b, 0)x_0 - \int_0^b U(b, s)f(s, x(s))\mathrm{d}s.$$

It follows by definition of Q, that the fixed point of Q is a mild solution of the integral system (2.5). We get the fixed point of Q by using Schauder's fixed point theorem.

Firstly, we show that the mapping Q is continuous on $C([0, b], X)$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $C([0, b], X)$ with $\lim_{n\to\infty} x_n = x$ in $C([0, b], X)$. As f is continuous, $f(s, x_n(s))$ converges to $f(s, x(s))$ uniformly for $s \in [0, b]$ and we have

$$\|p\big(x_n(\cdot)\big) - p\big(x(\cdot)\big)\| \leq Mb \sup_{s\in[0,b]} \|f(s, x_n(s)) - f(s, x(s))\|.$$

Thus for $t \in [0, b]$, we have

$$\|(Qx_n)(t) - (Qx)(t)\|$$
$$\leq \left(Mb + \frac{1}{\lambda}C^3N^2b^2\right) \sup_{s\in[0,b]} \|f(s, x_n(s)) - f(s, x(s))\| \to 0,$$

as $n \to \infty$. This shows that Q is continuous on $C([0, b], X)$.

Following Fan [9], we wish to show that the operator $Q : C([0, b], X) \to C([0, b], X)$ defined by (3.3) is a compact operator. By Lemma 3.1, it suffices to show the compactness of $Q_1 : C([0, b], X) \to C([0, b], X)$, which is defined as

$$(Q_1x)(t) = \int_0^t U(t, s)Bu(s)\mathrm{d}s, \quad t \in [0, b],$$

with

$$u^*(t) = B^*U(b,t)^*R(\lambda, \Lambda_b)p\big(x(\cdot)\big),$$
$$p\big(x(\cdot)\big) = x_b - U(b,0)x_0 - \int_0^b U(b,s)f(s,x(s))\mathrm{d}s.$$

Next, we will show that Q_1 is compact using the Arzela–Ascoli theorem. Let B_l defined by (3.2) be a bounded subset of $C([0,b],X)$. For $0 \le t_1 \le t_2 \le b$ and $x \in B_l$, we consider the following:

$$\begin{aligned}
&\|(Q_1x)(t_2) - (Q_1x)(t_1)\| \\
&\quad \le \int_0^{t_1} \|U(t_2,s) - U(t_1,s)\| \|Bu(s)\|\mathrm{d}s + \int_{t_1}^{t_2} \|U(t_2,s)Bu(s)\| \\
&\quad \le \frac{1}{\lambda}N^2CL\int_0^{t_1} \|U(t_2,s) - U(t_1,s)\|\mathrm{d}s + \frac{1}{\lambda}N^2C^2L(t_2-t_1),
\end{aligned}$$

where $L = \|x_b\| + C\|x_0\| + CKb$ and the constant K comes from condition (H1). When $t_1 = 0$, it is easy to see that

$$\lim_{t_2\to 0} \|(Q_1x)(t_2) - (Q_1x)(t_1)\| = 0 \text{ uniformly for } x \in B_l.$$

For $0 < t_1 < b$ and for $0 < \delta < t_1$, we have

$$\begin{aligned}
&\|(Q_1x)(t_2) - (Q_1x)(t_1)\| \\
&\quad \le \frac{1}{\lambda}N^2CL\left[\int_0^{\delta} \|U(t_2,s) - U(t_1,s)\|\mathrm{d}s + \int_{\delta}^{t_1} \|U(t_2,s) - U(t_1,s)\|\mathrm{d}s\right] \\
&\qquad + \frac{1}{\lambda}N^2C^2L(t_2-t_1) \\
&\quad \le \frac{2\delta}{\lambda}N^2C^2L + \frac{1}{\lambda}N^2CL\int_{\delta}^{t_1} \|U(t_2,s) - U(t_1,s)\|\mathrm{d}s \\
&\qquad + \frac{1}{\lambda}N^2C^2L(t_2-t_1).
\end{aligned}$$

Since the evolution system $U(t,s)$ is compact for $t - s > 0$ (see Lemma 2.4), therefore, $U(t,s)$ is uniformly continuous in an operator norm for $\delta \le s < t \le b$. Hence, from the compactness of $U(t,s)$ and the arbitrariness of δ along with the above estimation, we have

$$\lim_{t_2\to t_1} \|(Q_1x)(t_2) - (Q_1x)(t_1)\| = 0, \quad \text{uniformly for } x \in B_l.$$

Thus, Q_1B_l is equicontinuous on $C([0,b],X)$.

For $t = 0$, one can easily observe that the set $\{(Q_1x)(0) : x \in B_l\}$ is precompact in X. Now, let $t \in [0,b]$ be given and $0 < \epsilon < t$. Since $U(t,t-\epsilon)$ is compact (see Lemma 2.4), therefore we have

$$\left\{U(t,t-\epsilon)\int_0^{t-\epsilon} U(t-\epsilon,s)Bu(s)\mathrm{d}s : x \in B_l\right\}$$

is precompact. Moreover, for $\epsilon < \delta < b$, we have

$$\begin{aligned}
&\left\| U(t, t-\epsilon)\int_0^{t-\epsilon} U(t-\epsilon, s)Bu(s)\mathrm{d}s - \int_0^{t-\epsilon} U(t, s)Bu(s)\mathrm{d}s \right\| \\
&\le \frac{1}{\lambda}N^2CL\int_0^{t-\delta} \|U(t, t-\epsilon)U(t-\epsilon, s) - U(t, s)\|\mathrm{d}s \\
&\quad + \frac{1}{\lambda}N^2CL\int_{t-\delta}^{t-\epsilon} \|U(t, t-\epsilon)U(t-\epsilon, s) - U(t, s)\|\mathrm{d}s \\
&\le \frac{\delta}{\lambda}N^2CL(C^2 + C) + \frac{1}{\lambda}N^2CL\int_0^{t-\delta} \|U(t, t-\epsilon)U(t-\epsilon, s) - U(t, s)\|\mathrm{d}s.
\end{aligned}$$

We note that $-\epsilon > -\delta$. Therefore $t-\epsilon > t-\delta$; which ensures the existence of the second term in the above inequality makes sense. Now, by semigroup property of the evolution operator $\{U(t, s) : t \ge s\}$, we have that the second term in the above integral goes to zero.

Therefore, the arbitrariness of δ and the Lebesgue dominated convergence theorem imply

$$\lim_{\epsilon\to 0}\left\| U(t, t-\epsilon)\int_0^{t-\epsilon} U(t-\epsilon, s)Bu(s)\mathrm{d}s - \int_0^{t-\epsilon} U(t, s)Bu(s)\mathrm{d}s \right\| = 0.$$

On the other hand, we have

$$\begin{aligned}
&\left\| U(t, t-\epsilon)\int_0^{t-\epsilon} U(t-\epsilon, s)Bu(s)\mathrm{d}s - \int_0^{t} U(t, s)Bu(s)\mathrm{d}s \right\| \\
&\le \left\| U(t, t-\epsilon)\int_0^{t-\epsilon} U(t-\epsilon, s)Bu(s)\mathrm{d}s - \int_0^{t-\epsilon} U(t, s)Bu(s)\mathrm{d}s \right\| \\
&\quad + \left\| \int_{t-\epsilon}^{t} U(t, s)Bu(s)\mathrm{d}s \right\| \\
&\le \frac{1}{\lambda}N^2CL\epsilon + \left\| U(t, t-\epsilon)\int_0^{t-\epsilon} U(t-\epsilon, s)Bu(s)\mathrm{d}s \right. \\
&\quad \left. - \int_0^{t-\epsilon} U(t, s)Bu(s)\mathrm{d}s \right\|.
\end{aligned}$$

Hence,

$$\lim_{\epsilon\to 0}\left\| U(t, t-\epsilon)\int_0^{t-\epsilon} U(t-\epsilon, s)Bu(s)\mathrm{d}s - \int_0^{t} U(t, s)Bu(s)\mathrm{d}s \right\| = 0.$$

It implies that $\{(Q_1x)(t) : x \in B_l\}$ is pre-compact in X by using the total boundedness. Thus, from Arzela–Ascoli theorem it follows that Q_1 is compact.

Finally we will show that there exists $l_0 > 0$ such that $QB_{l_0} \subseteq B_{l_0}$. Indeed, for all $x \in C([0, b], X)$, it follows from (3.3) that

$$\|Qx\| \le C\|x_0\| + CKb + \frac{1}{\lambda}N^2C^2Lb.$$

From the above inequality, we obtain that for large enough $l_0 > 0$, the inequality $\|Qx\| \le l_0$ holds for all $x \in C([0, b], X)$. Thus, we have $QB_{l_0} \subseteq B_{l_0}$.

Hence, by Schauder's fixed point theorem, the operator Q has a fixed point in B_{l_0}, which is just the mild solution of integral system (2.5). □

Theorem 3.3. *Assume that the conditions* (*H*1)–(*H*3) *hold true. Then the non-autonomous control system* (1.1) *is approximately controllable on* $[0, b]$.

Proof. It follows by Theorem 3.2, that for every $\lambda > 0$ and $x_b \in X$, there exists a mild solution $x_\lambda \in C([0, b], X)$ such that

$$x_\lambda(t) = U(t, 0)x_0 + \int_0^t U(t, s)\big[f(s, x(s)) + Bu(s)\big]\mathrm{d}s, \quad t \in [0, b],$$

with

$$\begin{aligned} u^*(t) &= B^* U(b, t)^* R(\lambda, \Lambda_b) p\big(x_\lambda(\cdot)\big), \\ p\big(x_\lambda(\cdot)\big) &= x_b - U(b, 0)x_0 - \int_0^b U(b, s) f(s, x(s))\mathrm{d}s. \end{aligned}$$

So we have

$$\begin{aligned} x_\lambda(b) &= U(b, 0)x_0 + \int_0^b U(b, s) f(s, x_\lambda(s))\mathrm{d}s + \Lambda_b R(\lambda, \Lambda_b) p\big(x_\lambda(\cdot)\big) \\ &\le x_b - p\big(x_\lambda(\cdot)\big) + \Lambda_b R(\lambda, \Lambda_b) p\big(x_\lambda(\cdot)\big) \\ &= x_b - (\lambda I + \Lambda_b) R(\lambda, \Lambda_b) p(x_\lambda(\cdot)) + \Lambda_b R(\lambda, \Lambda_b) p(x_\lambda(\cdot)) \\ &= x_b - \lambda R(\lambda, \Lambda_b) p(x_\lambda(\cdot)). \end{aligned} \tag{3.4}$$

It follows by (H1), that we have

$$\int_0^b \|f(s, x_\lambda(s))\|^2 \mathrm{d}s \le K^2 b,$$

which implies that the sequence $\{f(\cdot, x_\lambda(\cdot)) : \lambda > 0\}$ is a bounded sequence in the Hilbert space $L^2([0, b], X)$. Hence there exists a subsequence of $\{f(\cdot, x_\lambda(\cdot)) : \lambda > 0\}$, still denoted by itself, converging weakly to some point $g(\cdot) \in L^2([0, b], X)$. Let us define

$$\zeta = x_b - U(b, 0)x_0 - \int_0^b U(b, s) g(s)\mathrm{d}s.$$

Thus we have

$$\|p\big(x_\lambda(\cdot)\big) - \zeta\| \le \left\| \int_0^b U(b, s)\big[f(s, x_\lambda(s)) - g(s)\big]\mathrm{d}s \right\|. \tag{3.5}$$

We observe that as in Theorem 3.2 using compactness of $U(t, s)$, we can prove that

$$x(t) \mapsto \int_0^t U(t, s) x(s)\mathrm{d}s$$

from $L^2([0, b], X)$ to $C([0, b], X)$ is compact, that is the Cauchy operator $G : L^2([0, b], X) \to C([0, b], X)$ (defined by (3.1)) is compact. Since $f(\cdot, x_\lambda(\cdot)) \rightharpoonup g(\cdot)$ weakly in $L^2([0, b], X)$, therefore we have

$$\int_0^b U(b, s)[f(s, x_\lambda(s)) - g(s)]\mathrm{d}s \longrightarrow 0 \quad \text{as } \lambda \to 0^+.$$

Thus in view of (3.5), we get that

$$\|p(x_\lambda) - \zeta\| \to 0, \quad \text{as } \lambda \to 0^+. \tag{3.6}$$

Hence it follows by (3.4), (3.6) and the condition (H3) that we have

$$\begin{aligned}\|x_\lambda(b) - x_b\| &\leq \|\lambda R(\lambda, \Lambda_b)p(x_\lambda)\| \\ &\leq \|\lambda R(\lambda, \Lambda_b).\|p(x_\lambda) - \zeta\| + \|\lambda R(\lambda, \Lambda_b)\zeta\| \\ &\to 0, \quad \text{as } \lambda \to 0^+,\end{aligned}$$

which implies that the non-autonomous control system (1.1) is approximately controllable on $[0, b]$. □

4. Application

As an application to Theorem 3.3, we study a control system governed by the partial differential equation of the form

$$\begin{cases} \dfrac{\partial z(t, \xi)}{\partial t} = a(t, \xi)\dfrac{\partial^2}{\partial \xi^2}z(t, \xi) + w\mu(t, \xi) + \mathcal{F}(t, z(t, \xi)), \\ \quad t \in [0, T] = J, \;\; \xi \in \Omega, \\ z(t, \xi) = 0, \qquad \xi \in \partial\Omega, \;\; t \in J, \\ z(0, \xi) = \phi(\xi), \quad \xi \in \Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded domain in $\mathbb{R}$ with sufficiently smooth boundary $\partial\Omega$.

Let $X = L^2(\Omega)$ be the space of square integrable functions. It is well-known that the linear operator $A : D(A) \subset X \to X$ defined by $Ax = x''$, with the domain given by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) = \{x \in X : x'' \in X, \; x(0) = x(\pi) = 0\},$$

has a discrete spectrum whose eigenvalues are $-n^2$, $n \in \mathbb{N}$ and the corresponding normalized eigenfunctions are given by $w_n(\xi) = \sqrt{\frac{2}{\pi}} \sin(n\xi)$. Moreover, A generates an analytic and compact semigroup $\{T(t)\}_{t\geq 0}$, which is given by

$$T(t)x = \sum_{n=1}^{\infty} \mathrm{e}^{-n^2 t}\langle x, w_n\rangle w_n, \text{ where } w_n(\xi) = \sqrt{\frac{2}{\pi}}\sin(n\xi).$$

Further, for each $t \in J$, we have $T^*(t)x = T(t)x$, where $x \in X$. We require the following assumptions for the system (4.1).

(i) $a(t, \xi)$ is uniformly Hölder continuous with respect to first variable $t \in \mathbb{R}$.
(ii) $a(t, \xi) \leq -\delta_0$, $(\delta_0 > 0)$ for all $\xi \in \Omega$.
(iii) $w > 0$ and $\mu : J \times \Omega \to \Omega$ is continuous in t.

Let us take for each $t \in [0, T]$, $D(A(t)) = D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Then we define the operator $A(t) : D(A(t)) \subset X \to X$ as follows:

$$-A(t)x(\xi) = a(t, \xi)Ax(\xi), \quad \text{for } x \in D(A(t)),\ t \in J \text{ and } \xi \in \Omega.$$

Under the assumptions (i) and (ii), it is easy to see that $-A(t)$ generates an unique evolution system $(U(t, s))_{t \geq s}$ on X, which can be explicitly given by

$$U(t, s)y = T\left(-a(t, \xi)(t - s)\right) y.$$

We define $x(t)(\xi) := z(t, \xi)$, for $t \in J$, $\xi \in \Omega$. Let the function $f : J \times X \to X$, be given by

$$f(t, x(t))(\xi) = \mathcal{F}(t, z(t, \xi)),$$

and let the operator $B : \mathcal{U} = L^2(J, \Omega) \to X$, be a linear continuous map defined as follows:

$$Bu(t)(\xi) = u(t)(\xi) = w\mu(t, \xi),$$

where $\xi \in \Omega$ and $t \in J$. Then we can write (4.1) in an abstract form

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} + A(t)x(t) = Bu(t) + f(t, x(t)), \\ x(0) = \phi. \end{cases} \tag{4.2}$$

Note that the boundary conditions are absorbed into the definition of domain of the operator $A(t)$ and into the requirement that $x(t) \in D(A)$ for all $t \geq 0$.

For $t, s \in J$, it can be shown that

$$\left\| B^* U^*(t, s)y \right\| = 0 \Rightarrow \|U(t, s)y\| = 0 \Rightarrow y = 0.$$

Therefore, by [6, Theorem 4.1.7], the linear system corresponding to (4.2) is approximately controllable. Now, [16, Theorem 7] implies that $\lambda R(\lambda, \Lambda_T) \to 0$ as $\lambda \to 0$ in the strong operator topology. Hence, the hypothesis (H3) holds. Further, let $f(t, x(t)) = \sin(x(t))$. It is easy to see that the maps f and B satisfy the assumptions (H1) and (H2) with $K = 1$ and $N = 1$ respectively. Therefore, by Theorem 3.3, the nonlinear system (4.2) (and hence the system (4.1)) is approximately controllable.

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