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Analytic sets and extension of holomorphic maps of positive codimension

MARYAM AL-TOWAILB and NABIL OURIMI*

Department of Mathematics, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
*Corresponding author.
E-mail: mtowaileb@ksu.edu.sa; ourimi@ksu.edu.sa

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Abstract. Let D, D' be arbitrary domains in $\mathbb{C}^n$ and $\mathbb{C}^N$ respectively, $1 < n \leq N$, both possibly unbounded and $M \subseteq \partial D$, $M' \subseteq \partial D'$ be open pieces of the boundaries. Suppose that ∂D is smooth real-analytic and minimal in an open neighborhood of $\bar{M}$ and $\partial D'$ is smooth real-algebraic and minimal in an open neighborhood of $\bar{M}'$. Let $f : D \to D'$ be a holomorphic mapping such that the cluster set $\mathrm{cl}_f(M)$ does not intersect D'. It is proved that if the cluster set $\mathrm{cl}_f(p)$ of some point $p \in M$ contains some point $q \in M'$ and the graph of f extends as an analytic set to a neighborhood of $(p, q) \in \mathbb{C}^n \times \mathbb{C}^N$, then f extends as a holomorphic map to a dense subset of some neighborhood of p. If in addition, $M = \partial D$, $M' = \partial D'$ and M' is compact, then f extends holomorphically across an open dense subset of ∂D.

Keywords. Analytic sets; Holomorphic correspondences; Segre varieties.

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1. Introduction and main results

In [7], Diederich and Pinchuk proved that a proper holomorphic mapping $f : \Omega \to \Omega'$ between bounded domains in $\mathbb{C}^n$, $n > 1$, with smooth real-analytic boundaries extends holomorphically to a neighborhood of any point $p \in \partial\Omega$, if the graph of f extends as an analytic set near (p, q) for some $q \in \mathrm{cl}_f(p)$. The purpose of this article is to study this result when the boundary of the target domain is real-algebraic but of higher dimension.

Theorem 1. *Let D, D' be arbitrary domains in $\mathbb{C}^n$ and $\mathbb{C}^N$ respectively*, $1 < n \leq N$, *both possibly unbounded and $M \subseteq \partial D$, $M' \subseteq \partial D'$ be open pieces of the boundaries. Suppose that ∂D is smooth real-analytic and minimal in an open neighborhood of $\bar{M}$ and $\partial D'$ is smooth real-algebraic and minimal in an open neighborhood of $\bar{M}'$. Let $f : D \to D'$ be a holomorphic mapping such that the cluster set $\mathrm{cl}_f(M)$ does not intersect D'. If the cluster set $\mathrm{cl}_f(p)$ of a point $p \in M$ contains some point $q \in M'$ and the graph of f extends as an analytic set to a neighborhood of $(p, q) \in \mathbb{C}^n \times \mathbb{C}^N$, then f extends as a holomorphic map to a dense subset of some neighborhood of p. If p is a generic point with a generic* *fibre dimension* (*c.f. Definition* 1), *then* f *extends as a holomorphic correspondence in a neighborhood of* p.

It is an open question whether the set of points with non-generic fibre dimensions (denoted by Σ^* later in this paper) is always empty or not. It is empty if the target hypersurface in $\mathbb{C}^N$ is a sphere, according to an old result of Pinchuk [16] stating that a meromorphic map from a real-analytic strongly pseudoconvex hypersurface M with the image of $M \setminus \Sigma^* \subset \mathbb{C}^n$ contained in the sphere $S^{2N-1} \subset \mathbb{C}^N$ for $N \geq n$ extends holomorphically everywhere on M. But the answer is unknown for any other algebraic hypersurface, even if it is strongly pseudoconvex. Following [13], a brief summary concerning the generic dimension of the fibres is given at the end of section 2.

Here, *a priori* $\mathrm{cl}_f(p)$ may contain the point at infinity or boundary points which do not lie in M'. In particular, this is the main reason why our result cannot be derived from [21], even in the case where M' is strongly pseudoconvex. Contrary to the equidimensional case (see [7]), the holomorphic extension of f as a correspondence does not imply, in general, the extension as a map, as the following example shows.

Example 1. Let $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Re}(z_2) + |z_2|^2 + |z_1|^2 < 0\}$ and $\Omega' = \{(z_1', z_2', z_3') \in \mathbb{C}^3 : \mathrm{Re}(z_3') + |z_2'|^4 + |z_1'|^2 < 0\}$. Then $f : (z_1, z_2) \to (z_1, \sqrt{z_2}, z_2)$ maps properly Ω onto Ω'. The map f extends as a holomorphic correspondence at 0, but it does not extend holomorphically at 0. Note that Ω is strongly pseudoconvex, Ω' is weakly pseudoconvex and the boundaries of Ω and Ω' are real-algebraic. Note also that $f(0) = 0$ is Levi-degenerate.

For more details related to this subject, see [19]. A global version of Theorem 1 may be formulated as follows.

Theorem 2. *Let* D, D' *be bounded domains in* $\mathbb{C}^n$ *and* $\mathbb{C}^N$ *respectively*, $1 < n \leq N$, ∂D *is smooth real-analytic and* $\partial D'$ *is smooth real-algebraic and* $f : D \to D'$ *be a proper holomorphic mapping. If the graph of* f *extends as an analytic set to a neighborhood of* $(p, q) \in \mathbb{C}^n \times \mathbb{C}^N$, *for some* $p \in \partial D$ *and* $q \in \mathrm{cl}_f(p)$, *then* f *extends holomorphically across an open dense subset of* ∂D.

Theorem 2 generalizes [7] when the boundary of the target domain is real-algebraic but of higher dimension. The algebraicity of $\partial D'$ allows to show that the extension is across an open dense subset of ∂D and not only on an open dense subset of some neighborhood of p. Forstnerič [11] proved that a proper holomorphic map $f : \Omega \to \Omega'$ between strongly pseudoconvex domains $\Omega \subset \mathbb{C}^n$, $\Omega' \subset \mathbb{C}^N$, $1 < n \leq N$, with real-analytic boundaries which extends smoothly to $\partial\Omega$, necessarily extends holomorphically on a dense open subset of $\partial\Omega$. This was improved in [17] by showing that the extension is holomorphic everywhere provided that $1 < n \leq N < 2n$. The condition $N < 2n$ in [17] is due to the use of Rothstein's theorem on the analytic extension across pseudoconcave hypersurfaces (see [4]). The continuity of f up the boundary does not suffice to prove a global holomorphic extension, since it was proved in [10] and [12] that there exist proper holomorphic maps between balls of different dimensions that are continuous up to the boundary, but are not of class $\mathcal{C}^2$.

Remark 1.1. Actually, there is no argument given why the set X^* (defined in subsection 3.2) cannot have points with non-generic fibre dimensions under the projection π and this is the reason why our result is only on a dense subset of the boundary. In this paper, we use several times the ideas developed by Shafikov and Verma in [21]. Note that all holomorphic extension results in [21] are correct, but outside a complex analytic set of complex codimension at least 2 (the set of points with non-generic fibre dimensions). The problem occurs in the claim of the following implication : $\pi^{-1}(Q_z \cap U_b) \subset \pi'^{-1}(Q'_{z'})$ implies that $\pi^{-1}(Q_z \cap U_t) \subset \pi'^{-1}(Q'_{z'})$. However, in order to show that, one has to know that all components of the set $\pi^{-1}(Q_z \cap V)$ project surjectively onto $Q_z \cap V$. While the whole set $\pi^{-1}(Q_z \cap V)$ has a surjective projection under π onto its image, it is conceivable that there are some components that do not project surjectively. If this happens, one cannot conclude the above implication. This issue does not arise if the Segre varieties in the target and the preimage have the same dimension, or if the constructed sets X and X^* in subsection 3.2 (denoted by A and A^* in [21]) do not contain points with non-generic fibre dimensions. Note that the notations used here are the notations used in subsection 3.2.

The structure of this article is as follows: In section 2, we give basic definitions of holomorphic correspondences and Segre varieties. In section 3, the proof of Theorem 1 is given in several steps. In the Step 1, we show that f extends holomorphically across an open set $\Gamma \subset M$ with $p \in \bar{\Gamma}$. In Step 2, we study the holomorphic extension of f across generic submanifolds (Proposition 1). In Step 3, we conclude the proof of Theorem 1. In section 4, we prove Theorem 2. This paper develops and corrects the results announced in [1].

2. Preliminaries

Recall that a hypersurface is called minimal if it does not contain any germs of complex hypersurfaces. Let M be a smooth real-analytic hypersurface that contains the origin. By $\rho(z, \bar{z})$, we denote a real-analytic defining function of M near 0. In a small neighborhood U of the origin, the complexification $\rho(z, \bar{w})$ of ρ is well-defined by means of a convergent power series in $U \times U$. For $w \in U$, the associated Segre variety is defined as

$$Q_w = \{z \in U : \rho(z, \bar{w}) = 0\}.$$

We will write $z \in \mathbb{C}^n$ as $z = (z_1, {}'z) \in \mathbb{C} \times \mathbb{C}^{n-1}$, $(n > 1)$. By the implicit function theorem, it is possible to choose neighborhoods $U_1 \subset\subset U_2$ of the origin such that for any $w \in U_1$, Q_w is a closed, complex hypersurface in U_2 and

$$Q_w = \{(z_1, {}'z) \in U_2 : z_1 = h({}'z, \bar{w})\}, \tag{2.1}$$

where $h({}'z, \bar{w})$ is holomorphic in ${}'z$ and antiholomorphic in w. Following the terminology of [8], U_1 and U_2 are usually called a standard pair of neighborhoods of 0. It can be shown that Q_w is independent of the choice of the defining function. We denote by $S = S(U)$ the set of Segre varieties $\{Q_w,\ w \in U\}$ and λ, the so-called Segre map defined by

$$\begin{aligned} \lambda : U &\to S \\ w &\mapsto Q_w. \end{aligned}$$

Let $I_w := \{z \in U : Q_w = Q_z\}$ be the fibre of λ over Q_w. For any $w \in M$, the set I_w is a complex variety of M. If M is minimal, then for all $w \in M$ there exists a neighborhood U_w of w such that $I_w \cap U_w = \{w\}$. The set S admits the structure of a complex-analytic variety of finite dimension such that the map λ is a finite antiholomorphic branched covering. The set I_w contains at most as many points as the sheet number of λ. We next list some important properties of Q_w and I_w (see [9]):

(a) $z \in Q_w \Longleftrightarrow w \in Q_z$,
(b) $z \in Q_z \Longleftrightarrow z \in M$,
(c) $I_w = \bigcap \{Q_z : z \in Q_w\}$,
(d) The Segre map $\lambda : w \mapsto Q_w$ is locally one-to one near Levi-nondegenerate points of M.

Moreover, if $z \in M$ is strongly pseudoconvex, then there exists a neighborhood V of z such that $Q_w \cap M = \{w\}$ for all $w \in M \cap V$ [11]. For more details on properties of Segre varieties, we refer the readers to [6,8,9] and [3].

Let $\Omega \subset \mathbb{C}^n$, $\Omega' \subset \mathbb{C}^N$ be domains. By a holomorphic correspondence, we mean a complex analytic subset $\mathcal{A} \subset \Omega \times \Omega'$ such that the dimension of $\mathcal{A}$ is n and the projection $\pi : \mathcal{A} \to \Omega$ is proper. It is natural to define a multiple-valued map $f = \pi' \circ \pi^{-1}$ associated with $\mathcal{A}$, where $\pi' : \mathcal{A} \to \Omega'$ is the coordinate projection to the second component. Since π is proper, in particular, it is a branched analytic covering. Then there exist an $(n-1)$-dimensional complex-analytic subset V_f of $\mathcal{A}$ and an integer m such that π is an m-sheeted covering map from the set $\mathcal{A} \backslash \pi^{-1}(\pi(V_f))$ onto $\Omega \backslash \pi(V_f)$. Hence, $f(z) = \{f^1(z), \ldots, f^m(z)\}$ for all $z \in \Omega \backslash \pi(V_f)$ and f^j's are distinct holomorphic functions in a neighborhood of $z \in \Omega \backslash \pi(V_f)$. The integer m is called the multiplicity of f and $\pi(V_f)$ is its branch locus. One says that f is irreducible if $\mathcal{A}$ is irreducible as an analytic set and proper if both π and π' are proper. We say that the graph Γ_f of a holomorphic mapping $f : \Omega \to \Omega'$ extends as an analytic set to a neighborhood of $(p, q) \in \partial\Omega \times \partial\Omega'$, if there exist neighborhoods $U \ni p$, $U' \ni q$, an irreducible analytic subset $\mathcal{F} \subset U \times U'$ of pure dimension n and a sequence $\{a_\nu\} \subset U \cap \Omega$ with $a_\nu \to p$ and $f(a_\nu) \to q$ such that $\mathcal{F}$ contains an open piece of the graph Γ_f near $(a_\nu, f(a_\nu))$ for each ν. Note that, *a priori*, $\mathcal{F}$ does not necessarily contain the whole graph of f over $U \cap \Omega$ even for an arbitrary small U. If, in addition, the natural projection $\pi : \mathcal{F} \to U$ is proper, we say that f extends as a holomorphic correspondence near p. The extension of f as a holomorphic correspondence, at first, usually provided only a multi-valued extension which may be best interpreted as an analytic set extending the graph of f.

Let $f : \Omega \to \Omega'$ be a holomorphic mapping (or correspondence) which extends as a holomorphic correspondence F to a neighborhood of a point $p \in \partial\Omega$. Assume that $p = 0$, $0' \in f(0)$ and choose standard neighborhoods $U_2 \supset\supset U_1 \ni 0$ and $U_2' \supset\supset U_1' \ni 0'$. Then, we have the following invariance property for the Segre variety under F (see [8,22]):

$$\text{For all } (w, w') \in \text{graph}(F) \cap (U_1 \times U_1'), \quad F(Q_w) \subset Q'_{w'}. \tag{2.2}$$

This means that any branch of F maps any point from Q_w to $Q'_{w'}$ for any point $w' \in F(w)$.

Recall that if $f : \Omega \to \mathbb{C}^N$ is a holomorphic mapping and $M \subset \partial\Omega$, then the cluster set of f at $p \in M$ is the set of accumulation points of $(f(z_j))_j$, where $(z_j)_j$ is a sequence in Ω, $z_j \to p$. The cluster set $\text{cl}_f(M)$ is defined as

$$\begin{aligned} \text{cl}_f(M) = \{w \in \mathbb{C}^N \cup \{\infty\} : \; & \lim_{j\to\infty} \text{dist}(z_j, M) = 0 \text{ and} \\ & \lim_{j\to\infty} \text{dist}(f(z_j), w) = 0 \text{ for some sequence } \{z_j\} \text{ in } \Omega\}. \end{aligned}$$

Following [13], the generic dimension of the fibres of a holomorphic mapping $f : A \to B$ of locally analytic sets A, B in $\mathbb{C}^n$ and $\mathbb{C}^N$, respectively is defined by

$$\lambda(f) = \min\{\dim f^{-1}(f(z)) : z \in A\}.$$

The rank of f is defined by the formula: $\operatorname{rank} f = \max\{\operatorname{rank}_z f,\ z \in \operatorname{Reg}(A)\}$. Clearly, $\dim f^{-1}(w) \geq \lambda(f)$ for all $w \in f(A)$. When $\dim f^{-1}(w) = \lambda(f)$, we say that w has a generic fibre dimension. According to [13], if A is irreducible $\lambda(f) = \dim A - \dim f(A)$ and $\dim f^{-1}(f(z)) = \lambda(f)$ if $z \in A\backslash C(f)$, where

$$C(f) = \{z \in \operatorname{Reg}(A) : \operatorname{rank}_z f < \operatorname{rank} f\} \cup \operatorname{Sing}(A).$$

3. Proof of Theorem 1

3.1 *Extension across an open set*

We may assume that $p = 0$, $q = 0'$ and 0 is not in the envelope of holomorphy of D. Let U, U' be small neighborhoods of 0 and $0'$ respectively. We denote by $U^- = D \cap U$ and $U^+ = U\backslash(\overline{D \cap U})$. Let $\mathcal{A}$ be the irreducible, closed complex-analytic set in $U \times U'$ which extends the graph Γ_f of f and $\pi : \mathcal{A} \to U$ be the coordinate projection to the first component. Set $E = \{z \in U : \dim \pi^{-1}(z) \geq 1\}$. First, note that $U\backslash E \neq \emptyset$. Otherwise, $\pi^{-1}(z)$ is at least one-dimensional for all $z \in \pi(\mathcal{A})$ and hence

$$\dim \mathcal{A} \geq 1 + \dim \pi(\mathcal{A})$$

and so $\dim \pi(\mathcal{A}) \leq n-1$. This is a contradiction; since $\pi(\mathcal{A})$ contains Γ_f over U^-. Denote by $F : U\backslash E \to \mathbb{C}^N$ the multi-valued map corresponding to $\mathcal{A}$ and by S_F its branch locus (i.e., for $z \in U\backslash E$, $z \in S_F$ if the coordinate projection π is not locally biholomorphic near $\pi^{-1}(z)$).

From the continuity principle and the strong disc theorem, we may prove the following lemma, stating that $\mathcal{A}$ has points lying over U^+ (see [18] for details).

Lemma 1. $\mathcal{A} \cap (U^+ \times U') \neq \emptyset$.

As a consequence of Lemma 1, one has the following:

Lemma 2. *There exists an open set* $\Gamma \subset M \cap U$ *such that* f *extends holomorphically to a neighborhood of* $U^- \cup \Gamma$, *and the graph of* f *near any point* $(z, f(z))$, $z \in \Gamma$, *is contained in* $\mathcal{A}$. *Moreover*, $0 \in \bar{\Gamma}$ *and* $\lim_{\substack{z \to 0 \\ z \in \Gamma}} f(z) = 0'$.

Proof. Let $V_F := \{(z, z') \in \mathcal{A} : z \in S_F\}$. Since $\mathcal{A}$ is irreducible and $\mathcal{A}\backslash V_F$ is a non-empty open set of $\mathcal{A}$, the complex dimension of V_F is at most $n - 1$. Therefore $\mathcal{A}\backslash V_F$ is path connected. Let $(a, b) \in \Gamma_f \cap (\mathcal{A}\backslash V_F) \cap (U^- \times U')$ and choose $(a', b') \in (\mathcal{A}\backslash V_F) \cap (U^+ \times U')$, and connect them by a path $\gamma \subset \mathcal{A}\backslash V_F$. It follows that $\pi(\gamma) \cap M \neq \emptyset$. Let z_0 be the point where $\pi(\gamma)$ first intersects M. Then f extends analytically along $\pi(\gamma)$ from a to z_0 and the graph of f over this part of $\pi(\gamma)$ is connected in $(\mathcal{A}\backslash V_F)$. It follows that z_0 is a point of holomorphic extendibility for f. The second part immediately follows from the fact that U and U' can be chosen arbitrarily small. □

3.2 *Extension across generic submanifolds*

In view of Lemma 2, we can choose U and U' small enough so that $f(\Gamma) \subset M'$ and without loss of generality, we can assume that Γ is connected. In this subsection, consider the restriction $f : \Gamma \to M'$ (still denoted by f). This restriction is a real analytic CR-map. It is holomorphic in a neighborhood of Γ and its graph extends as an analytic set $\mathcal{A} \subset U \times U'$ to a neighborhood of $(0, 0')$. Our aim is to prove the following.

PROPOSITION 1

If $\partial\Gamma \cap M$ is a smooth generic submanifold, then $f : \Gamma \to M'$ extends holomorphically to an open dense subset of some neighborhood of 0. *If additionally*, 0 *has a generic fibre dimension* (*c.f. Definition* 1), *then f extends as a holomorphic correspondence near* 0.

Recall that a real submanifold $M \subset \mathbb{C}^n$ of real dimension $k \geq n$ is called generic if, for any $z \in M$; $\dim_{\mathbb{C}} T_z^c M = k - n$ ($T_z^c M$ is the complex tangent space to M at the point z). As usual, a smooth CR-function on a hypersurface $M \subset \mathbb{C}^n$ is defined to be a function that satisfies the tangential Cauchy–Riemann equations on M. A smooth map $f : M \to M'$ between hypersurfaces $M \subset \mathbb{C}^n$, $M' \subset \mathbb{C}^N$ is called a CR-map if all its components are CR-functions. Concerning the holomorphic extension of smooth CR-mappings between real-analytic and real-algebraic CR-manifolds, we refer the readers (for examples) to [5] and [15].

Proof. In view of Proposition 4.1 in [20], there exists an open subset ω of Q_0 such that for all $b \in \omega$, $Q_b \cap \Gamma$ is nonempty. Furthermore, there exists a non-constant curve $\gamma \subset \Gamma \cap Q_b$ with the end point at 0. Thus, we may choose t and b such that $b \in Q_0$ and $t \in \gamma \subset \Gamma \cap Q_b$. For simplicity, we will also denote by $f : U_t \to \mathbb{C}^N$ a germ of a holomorphic mapping defined from the extension of f in some neighborhood U_t of t. Let V be a neighborhood of Q_t and define

$$X = \{(w, w') \in V \times \mathbb{C}^N : f(Q_w \cap U_t) \subset Q'_{w'}\}.$$

Since $w \in Q_t$ implies that $t \in Q_w$, we may choose V such that $Q_w \cap U_t$ is nonempty for all $w \in V$. The set X allows us to extend the graph of f as an analytic set along Q_t, $t \in \Gamma$. In contrast with the equidimensional case, the dimension of X may be bigger than the dimension of the graph of f and this leads us to construct another analytic set X^* from X extending the graph of f with dimension equal to n (the dimension of the graph of f). For this construction, we will follow the ideas in [21]. The analytic set X^* allows us to prove that f extends as a holomorphic correspondence to a neighborhood of 0, if 0 has a generic fibre dimension (c.f. Definition 1). This extension is guaranteed to be single-valued if $M' \cap U'$ is strongly pseudoconvex or $M' \cap U'$ is Levi-nondegenerate with additional conditions on the eigenvalues of the Levi-form of M' at z' (c.f. [19]).

According to [21], X is a complex analytic subset of $V \times \mathbb{C}^N$. By the invariance property of Segre varieties, X contains the germ at t of the graph of f. From the algebraicity of M', the set X extends to an analytic subset of $V \times \mathbb{P}^N$. Since $\mathbb{P}^N$ is compact and X is closed in $V \times \mathbb{P}^N$, the projection $\pi : X \to V$ is proper. By the Remmert proper mapping theorem, $\pi(X)$ is a complex analytic subset of V. As V is connected, it follows that $\pi(X) = V$. Since X contains the germ at t of the graph of f, we may consider only the irreducible component of the least dimension which contains the graph of f.

So, we may assume that $\dim(X) \equiv m \geq n$. For $\xi \in X$, let $I_\xi\pi \subset X$ be the germ of the fibre $\pi^{-1}(\pi(\xi))$ at ξ. For a generic point $\xi \in X$, $\dim(I_\xi\pi) = m - n$ which is the smallest possible dimension of the fibre. By Cartan–Remmert's theorem (see [13]), the set $\Sigma := \{\xi \in X : \dim(I_\xi\pi) > m - n\}$ is complex-analytic and by Remmert's proper mapping theorem, $\pi(\Sigma)$ is a complex-analytic set in V. Furthermore, $\dim \pi(\Sigma) < n - 1$. By the above considerations, we deduce that $\pi(\Sigma)$ does not contain $Q_0 \cap V$. We may rearrange points $b \in Q_0$ and $t \in \Gamma \cap Q_b$ so that b and t are not in $\pi(\Sigma)$. Denote the corresponding multiple-valued map to X by $\hat{F}$. That is, $\hat{F} := \pi' \circ \pi^{-1} : V \to \mathbb{P}^N$, where $\pi' : X \to \mathbb{P}^N$ denotes the other coordinate projection. We choose suitable neighborhoods, U_γ of γ (including its endpoints), U_b of b and U_t of t such that U_b and U_t do not intersect $\pi(\Sigma)$, and $Q_w \cap U_b$ is nonempty and connected for any $w \in U_\gamma$. Consider the set

$$X^* = \left\{(w, w') \in U_\gamma \times \mathbb{P}^N : \hat{F}(Q_w \cap U_b) \subset Q'_{w'}\right\}.$$

According to [21] (Lemma 3.1), X^* is a complex-analytic subset of $U_\gamma \times \mathbb{P}^N$. The same arguments used for π show that the projection $\pi^* : X^* \to U_\gamma$ is surjective and proper. Now, define $\pi'^* : X^* \to \mathbb{P}^N$ and consider the multiple-valued mapping $\widehat{F^*} := \pi'^* \circ \pi^{*-1} : U_\gamma \to \mathbb{P}^N$. We will denote by w^s the symmetric point of $w \in U$, which is the unique point in the intersection $Q_w \cap \{z \in U : {}'z = {}'w\}$. Define further

$$\Sigma^* = \{z \in U_\gamma : \dim \pi^{*-1}(z) > \dim X^* - n\}.$$

The set Σ^* is a complex analytic set of dimension at most $n-2$. In our situation the generic fibre dimension of a point $z \in U_\gamma$ may be defined as follows. □

DEFINITION 1

A point $z \in U_\gamma$ has a generic fibre dimension if $\dim \pi^{*-1}(z) = \dim X^* - n$ (i.e. $z \notin \Sigma^*$).

As it was mentioned at the end of section 2, a generic fibre dimension is the smallest possible dimension of the fibre.

Lemma 3. *X^* is a non-empty closed set in $U_\gamma \times \mathbb{P}^N$. Furthermore*,

$$X^* \cap ((U_t \cap V \cap U_\gamma) \times \mathbb{P}^N) \subset X.$$

Proof. First, we prove that $X^* \neq \emptyset$. Let $z \in V \cap U_\gamma$ be sufficiently close to t. We show that $(z, f(z)) \in X^*$, that is,

$$\hat{F}(Q_z \cap U_b) \subset Q'_{f(z)}. \tag{3.1}$$

Let $w \in Q_z \cap U_b$ be an arbitrary point, and let $w' \in \pi^{-1}(w)$, i.e, $(w, w') \in X$. Then $f(Q_w \cap U_t) \subset Q'_{w'}$. In particular, since $z \in Q_w \cap U_t$, we have $f(z) \in Q'_{w'}$ which implies that $w' \in Q'_{f(z)}$ and so $(w, w') \in \{w\} \times Q'_{f(z)}$. Since w' is an arbitrary point in $\pi^{-1}(w)$, we conclude that $\pi^{-1}(w) \subset \pi'^{-1}(Q'_{f(z)})$. Thus, (3.1) holds.

On the other hand, note that if $w \in U_t \cap V \cap U_\gamma$ and $(w, w') \in X^*$, then $\hat{F}(Q_w \cap U_t) \subset Q'_{w'}$; since Q_w is connected in V. Note that $X \cap ((U_t \cap V \cap U_\gamma) \times \mathbb{P}^N)$ does not contain points with non-generic fibre dimensions. But X contains the germ of the graph of f near t. Then it follows that $f(Q_w \cap U_t) \subset Q'_{w'}$. This by definition means that $(w, w') \in X$ and proves that $X^* \cap ((U_t \cap V \cap U_\gamma) \times \mathbb{P}^N)$ is contained in X.

Finally, suppose that $(w_i, w'_i) \to (w_0, w'_0)$ as $i \to \infty$, where $(w_i, w'_i) \in X^*$ and $(w_0, w'_0) \in U_\gamma \times \mathbb{P}^N$. This means that $\pi^{-1}(Q_{w_i} \cap U_t) \subset \pi'^{-1}(Q'_{w'_i})$. Since $Q_{w_i} \to Q_{w_0}$ and $Q'_{w'_i} \to Q'_{w'_0}$, by analyticity, $\pi^{-1}(Q_{w_0} \cap U_t) \subset \pi'^{-1}(Q'_{w'_0})$ which implies that $(w_0, w'_0) \in X^*$. Thus, X^* is a closed set and this completes the proof. □

Lemma 4. *For any point* $w \in U_t \cap V \cap U_\gamma$,

$$\widehat{F^*}(Q^s_w) \subset Q'_{w'}, \quad \textit{for all } w' \in \widehat{F^*}(w) \tag{3.2}$$

Furthermore, the set $U_t \cap V \cap U_\gamma$ *does not intersect* Σ^*.

Proof. Let $w \in U_t \cap V \cap U_\gamma$ be an arbitrary point and $w' \in \widehat{F^*}(w)$. The set $Q_w \cap U_t$ coincides with $Q^s_w \cap U_t$; because $z^s = z$ for any $z \in M$. Let $z \in Q_w \cap U_t$. Then $z' \in \widehat{F^*}(z)$ means $\hat{F}(Q_z \cap U_b) \subset Q'_{z'}$. For z and w sufficiently close to t, Q_z is connected in V, and therefore, $\hat{F}(Q_z \cap U_t) \subset Q'_{z'}$. Since $w \in Q_z \cap U_t$, we get $\hat{F}(w) \subset Q'_{z'}$ and this implies that $w' \in Q'_{z'}$ or $z' \in Q'_{w'}$ for any $w' \in \hat{F}(w)$. By Lemma 3, $X^* \cap ((U_t \cap V \cap U_\gamma) \times \mathbb{P}^N) \subset X$. Then it follows that $z' \in Q'_{w'}$ for all $w' \in \widehat{F^*}(w)$. This completes the proof of the first part of the lemma. □

The set $\pi^{*^{-1}}(z)$ is non-empty for any $z \in M \cap U_\gamma$, since π^* is surjective. We claim that for a given $z_0 \in U_t \cap V \cap U_\gamma$, the set $\pi^{*^{-1}}(z_0)$ is discrete near $(z_0, f(z_0)) \in X^*$. From the fact that $(z, z') \in X^* \cap ((U_t \cap V \cap U_\gamma) \times \mathbb{P}^N)$ implies that $\widehat{F^*}(Q^s_z) \subset Q'_{z'}$. It follows that $\widehat{F^*}(z) \subset Q'_{z'}$ and hence $z' \in Q'_{z'}$. Then we have

$$(z, z') \in X^* \cap ((U_t \cap V \cap U_\gamma) \times \mathbb{P}^N) \Rightarrow z' \in M' \tag{3.3}$$

for any $z \in M$ close to z_0, and z' close to $f(z_0)$. Since M' is minimal and $\widehat{F^*}(z)$ is a locally countable union of complex analytic sets, it follows that $\pi^{*^{-1}}(z)$ is discrete near $(z_0, f(z_0))$. This completes the proof of the lemma. □

Let Ω^* be a small connected neighborhood of the path $\gamma \backslash \Sigma^*$, where γ is the curve which connects t and 0, such that for any $w \in \Omega^*$, the symmetric point w^s belongs to U_γ, and let Q^s_w denote the connected component of $Q_w \cap U_\gamma$ which contains w^s. We may choose Ω^* so that $\Omega^* \cap \Sigma^* = \emptyset$ and we may shrink U_t so that $U_t \subset \Omega^*$.

Lemma 5. *For any point* $w \in \Omega^*$,

$$\widehat{F^*}(Q^s_w) \subset Q'_{w'}, \quad \textit{for all } w' \in \widehat{F^*}(w). \tag{3.4}$$

Proof. Let S be the set of points in Ω^* for which (3.4) holds and $S_0 \subset S$ be the largest connected open set which contains t. By Lemma 4, $U_t \cap V \cap U_\gamma \subset S_0$. We will show that $\partial S_0 \cap \Omega^* \subset S_0$. Let $w \in \partial S_0 \cap \Omega^*$ and $(w, w') \in X^*$ for some w'. Since $\dim \Sigma^* < \dim Q^s_w = n - 1$, we may find a point $\alpha \in (Q^s_w \backslash \Sigma^*)$. So, by repeating the argument used in the proof of Lemma 3.1 in [21], we can choose neighborhoods U_w and U_α of w and α respectively, so that the set

$$X_w = \{(x, x') \in U_w \times \mathbb{P}^N : \widehat{F^*}(Q_x^s \cap U_\alpha) \subset Q'_{x'}\},$$

is a complex analytic set. Note that $(x, x') \in X_w$, for every point $x \in U_w \cap S_0$, and every x' such that $(x, x') \in X^*$. This implies that

$$X^* \cap ((S_0 \cap U_w) \times \mathbb{P}^N) \subset X_w,$$

and hence the set of X_w is non-empty. By the uniqueness theorem, it follows that $X^* \cap (U_w \times \mathbb{P}^N) \subset X_w$, and therefore, the projection $\pi_w : X_w \to U_w$ is surjective. Thus, for any $x \in U_w$, there exists some point $x' \in \mathbb{P}^N$ such that

$$\widehat{F^*}(Q_x^s \cap U_\alpha) \subset Q'_{x'},$$

and the set $Q_x \cap U_\alpha$ will also be mapped by X^* into $Q'_{x'}$, whenever $(x, x') \in X^*$. So $x \in S_0$ and hence $U_w \subset S_0$. Since Ω^* is connected, it now follows that $S = \Omega^*$, and hence the lemma is proved. □

Lemma 6. $\dim X^* = n$.

Proof. By (3.3), for any $z \in M$ close to $z_0 \in U_t \cap V \cap U_\gamma$ and z' close to $f(z_0)$,

$$(z, z') \in X^* \cap ((U_t \cap V \cap U_\gamma) \times \mathbb{P}^N) \Rightarrow z' \in M'.$$

Hence, the same arguments used in Lemma 4 show that ${\pi^*}^{-1}(z)$ is discrete near $(z_0, f(z_0))$. So, $\dim X^* = n$ near $(z_0, f(z_0))$. We may consider only the irreducible component of X^* which has the smallest dimension containing the germ of the graph of f at t. This component is still denoted by X^*. Then, we may assume that $\dim X^* = n$. □

Lemma 7. $X^* \cap [(U \cap U_\gamma) \times \mathbb{P}^N] = \mathcal{A} \cap [(U \cap U_\gamma) \times \mathbb{P}^N]$.

Proof. The map f extends holomorphically across t ($t \in \gamma$), and the graph of f near $(t, f(t))$ is contained in $\mathcal{A}$. Since X^* contains the graph of f near t, then by considering dimensions of X^* and $\mathcal{A}$, for a small neighborhood U_t of t,

$$X^*|_{U_t \times \mathbb{P}^N} = \mathcal{A}\,|_{U_t \times \mathbb{P}^N}.$$

Now, the result follows by the uniqueness theorem for analytic sets. □

To complete the proof of the proposition, suppose first that 0 has a generic fibre dimension ($0 \notin \Sigma^*$). We will prove that f extends as a holomorphic correspondence to a neighborhood of 0. In view of Lemma 7, $(0, 0') \in X^*$. From (3.4), $(z, z') \in X^* \cap (\Omega^* \times \mathbb{P}^N)$ implies that $\widehat{F^*}(Q_z^s) \subset Q'_{z'}$. In particular, $\widehat{F^*}(z) \subset Q'_{z'}$ for any $z \in M \cap \Omega^*$. Hence, $z' \in Q_{z'}$ and so $z' \in M'$. Then, for any $z \in M$ close to 0 and any z' close to $0'$, the inclusion $(z, z') \in X^*$ implies that $z' \in M'$. Since $\pi'^*({\pi^*}^{-1}(z))$ is contained in a countable union of complex analytic sets and M' is minimal, it follows that $\pi^{*^{-1}}(z)$ is discrete near $(0, 0')$. Therefore, we may choose small neighborhoods U_0 and $U'_{0'}$ of 0 and $0'$, respectively, such that $X^* \cap (U_0 \times \partial U'_{0'}) = \emptyset$. Consequently, $\pi'^*|_{X^* \cap (U_0 \times U'_{0'})} \circ \pi^{*-1}|_{U_0}$ is the desired extension of f as a holomorphic correspondence.

Since the complex dimension of Σ^* is $\leq n-2$, any small neighborhood of 0 contain points of generic fibre dimensions in $M \cap \Omega^*$. According to the above argument, it follows that f extends holomorphically to a dense subset of some neighborhood of 0. This completes the proof of Proposition 1. □

3.3 *Conclusion of the proof of Theorem 1*

Near any regular point, $\partial\Gamma \cap M$ is a smooth generic submanifold of dimension $2n-2$. Since the set of regular points is dense, the proof clearly follows from Proposition 1. □

4. Proof of Theorem 2

We denote by $M = \partial D$ and $M' = \partial D'$. We define the set $M_h \subset M$ as follows: $z \in M_h$ if there exists a neighborhood U_z of z in $\mathbb{C}^n$ such that f extends holomorphically across an open dense subset of $M \cap U_z$. The set M_h is open by construction and non-empty ($t \in M_h$). To prove the theorem, it suffices to show that M_h is closed in M. By contradiction, assume that $\overline{M_h} \neq M_h$, and let $a \in \partial M_h$. Following the ideas of construction of ellipsoids developed in [14] and used in [21] and [2], there exists a CR-curve γ passing through a and entering M_h. After shortening γ, we may assume that γ is a smoothly embedded segment. Then γ can be described as a part of an integral curve of some non-vanishing smooth CR-vector field L near a. By integrating L, we obtain a smooth coordinate system $(t, s) \in \mathbb{R} \times \mathbb{R}^{2n-2}$ on M such that for any fixed s_0 the segments (t, s_0) are contained in the trajectories of L. We may choose a point $z_0 \in \gamma \cap M_h$ sufficiently close to a. After a translation, we may assume that $z_0 = (0, 0)$. For $\epsilon > 0$ and $\tau > 0$, define the family of ellipsoids on M centered at $(0, 0)$ by $E_\tau = \{(t, s) : |t|^2/\tau + |s|^2 < \epsilon\}$, where $\epsilon > 0$ is so small that for some $\tau_0 > 0$, the ellipsoid E_{τ_0} is compactly contained in M_h. Observe that every ∂E_τ is transverse to the trajectories of L out of the set $\Lambda := \{(0, s) : |s|^2 = \epsilon\}$. So, ∂E_τ is generic at every point except the points of Λ. Note that Λ is contained in M_h. For some $\tau_1 > \tau_0$, the set E_{τ_1} touches ∂M_h at some point $b \in \partial E_{\tau_1}$, b may be different from a. Near b, ∂E_{τ_1} is a smooth generic manifold of M; since the non-generic points of ∂E_{τ_1} are contained in Λ, which is contained in M_h. Then by Proposition 1, f extends holomorphically past a dense subset of some neighborhood of b. This contradiction shows that $M_h = M$ and the proof of Theorem 2 is complete. □

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