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Uniformly locally univalent harmonic mappings

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Abstract. The primary aim of this paper is to characterize the uniformly locally univalent harmonic mappings in the unit disk. Then, we obtain sharp distortion, growth and covering theorems for one parameter family $\mathcal{B}_H(\lambda)$ of uniformly locally univalent harmonic mappings. Finally, we show that the subclass of k-quasiconformal harmonic mappings in $\mathcal{B}_H(\lambda)$ and the class $\mathcal{B}_H(\lambda)$ are contained in the Hardy space of a specific exponent depending on λ, respectively, and we also discuss the growth of coefficients for harmonic mappings in $\mathcal{B}_H(\lambda)$.

Keywords. Harmonic mapping; pre-Schwarzian derivatives; uniformly locally univalence; growth estimate; coefficient estimate; harmonic Bloch space; Hardy space.

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1. Introduction

The class of complex-valued harmonic mappings f defined on a simply connected domain D of the complex plane $\mathbb{C}$ has attracted the attention of function theorists because it generalizes the class of analytic functions with a lot of rich applications in many different fields. Every such f has the canonical decomposition $f = h + \overline{g}$, where both h and g are analytic in D and $g(z_0) = 0$ for some prescribed point $z_0 \in D$ (cf. [10,13]). For a complex-valued and continuously differentiable mapping f, let

$$\lambda_f = |f_z| - |f_{\overline{z}}| \quad \text{and} \quad \Lambda_f = |f_z| + |f_{\overline{z}}|$$

so that the Jacobian J_f of f takes the form

$$J_f = \lambda_f \Lambda_f = |f_z|^2 - |f_{\overline{z}}|^2.$$

Moreover, a necessary and sufficient condition for harmonic mappings $f = h + \overline{g}$ to be locally univalent and sense preserving in D is that $J_f = |h'|^2 - |g'|^2 > 0$, or equivalently, its dilatation $\omega_f(z) = g'(z)/h'(z)$ satisfies the inequality $|\omega_f(z)| < 1$ for $z \in D$ (see [17] and [10,13,22]). Let $k \in [0, 1)$ be a constant. Then, we say that a sense-preserving harmonic mapping $f = h + \overline{g}$ in D is a k-quasiconformal mapping if $|\omega_f(z)| \le k$ holds in D.

Throughout this paper, we consider harmonic mappings defined on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{H}$ the class of harmonic mappings $f = h + \overline{g}$ in $\mathbb{D}$ such that $h(0) = g(0) = h'(0) - 1 = 0$ and consider the family

$$\mathcal{S}_H = \{f \in \mathcal{H} : f \text{ is sense-preserving and univalent in } \mathbb{D}\}.$$

Often it is convenient to work with

$$\mathcal{S}_H^0 = \{f \in \mathcal{S}_H : f_{\overline{z}}(0) = 0\}.$$

Although both the families $\mathcal{S}_H$ and $\mathcal{S}_H^0$ are known to be normal, only $\mathcal{S}_H^0$ is compact (see [10]). We also denote the class of analytic functions f in $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$ by $\mathcal{A}$ so that $\mathcal{H}$ reduces to $\mathcal{A}$ when the co-analytic part g of $f = h + \overline{g} \in \mathcal{H}$ vanishes identically in $\mathbb{D}$. Then the set $\mathcal{S} := \mathcal{A} \cap \mathcal{S}_H^0$ of all normalized univalent analytic functions in $\mathbb{D}$ is the central object in the study of geometric function theory so that $\mathcal{S} \subset \mathcal{S}_H^0 \subset \mathcal{S}_H$.

We denote $d_h(z, w)$ as the hyperbolic distance of $z, w \in \mathbb{D}$, that is,

$$d_h(z, w) = \frac{1}{2} \log \left(\frac{1 + \left| \frac{z-w}{1-\overline{z}w} \right|}{1 - \left| \frac{z-w}{1-\overline{z}w} \right|} \right).$$

We say that a harmonic mapping $f = h + \overline{g}$ in $\mathbb{D}$ is uniformly locally univalent if f is univalent in each hyperbolic disk

$$D_h(a, \rho) = \{z \in \mathbb{D} : d_h(z, a) < \rho\}$$

with center $a \in \mathbb{D}$ and hyperbolic radius ρ (independent of the center), $0 < \rho \le \infty$. The subscript h in d_h and D_h should not be confused with the analytic part h of the harmonic mapping f.

If f is analytic in the above definition, then it reduces to the uniformly locally univalent (analytic) functions. We know that a holomorphic universal covering map of a plane domain D is uniformly locally univalent if and only if the boundary of D is uniformly perfect (cf. [20,23]). Also, in [24], Yamashita showed that an analytic function f in $\mathbb{D}$ is uniformly locally univalent in $\mathbb{D}$ if and only if the pre-Schwarzian derivative $T_f = f''/f'$ of f is hyperbolically bounded, i.e., the norm

$$\|T_f\| = \sup_{z \in \mathbb{D}} (1 - |z|^2)|T_f(z)|$$

is finite and this means that $\log f'$ belongs to the Bloch space $\mathscr{B}$ (cf. [3,12]).

In section 2 (see Theorem 2.1), we characterize the uniformly locally univalent harmonic mappings $f = h + \overline{g}$ in terms of the pre-Schwarzian derivative of $h + e^{i\theta} g$ for each $\theta \in [0, 2\pi]$. This result and the corresponding results in [15] helps to obtain sharp distortion, growth and covering theorems (see section 3) for the class $\mathcal{B}_H(\lambda)$, where λ is a positive real number, and

$$\mathcal{B}_H(\lambda) = \{f = h + \overline{g} \in \mathcal{H} : \|T_f\| \le 2\lambda\}$$

with

$$\|T_f\| := \sup_{z \in \mathbb{D},\ \theta \in [0, 2\pi]} (1 - |z|^2) \left| \frac{h''(z) + e^{i\theta} g''(z)}{h'(z) + e^{i\theta} g'(z)} \right|. \tag{1.1}$$

Henceforth, $\|T_f\|$ is defined by (1.1) in the case of harmonic mappings $f = h + \overline{g}$ in $\mathbb{D}$.

It is known that for $\lambda > 1$, the class $\mathcal{B}(\lambda)$ and the subclass $\mathcal{B}(\lambda) \cap \mathcal{S}$ are contained in the Hardy space H^p with $0 < p < 1/(\lambda^2 - 1)$ and $0 < p < 1/(\lambda - 1)$, respectively (cf. [14,16]).

In section 4, we consider relationships between the space $\mathcal{B}_H(\lambda)$ and the harmonic Hardy space. We also prove that a k-quasiconformal harmonic mapping $f \in \mathcal{B}_H(\lambda)$ $(\lambda > 1)$ is contained in the harmonic Hardy space h^p with $0 < p < 1/(\lambda - 1)$, and also obtain that $\mathcal{B}_H(\lambda) \subset h^p$ with $0 < p < 1/(\lambda^2 - 1)$. Finally, in the last section, as applications of distortion estimate obtained in Section 3, we discuss the growth of coefficients for harmonic mappings in $\mathcal{B}_H(\lambda)$ $(\lambda > 1)$.

In [15], the authors discussed the set $\mathcal{B}(\lambda) := \mathcal{A} \cap \mathcal{B}_H(\lambda)$ and obtained distortion estimates for analytic functions in $\mathcal{B}(\lambda)$ in terms of λ, and characterization for functions in $\mathcal{B}(\lambda)$ (cf. [15, Proposition 1.1]). As a consequence of Theorem 2.1 in section 2 and [15, Proposition 1.1], we can easily obtain the following corollary which characterizes harmonic mappings in $\mathcal{B}_H(\lambda)$. We omit its proof and this particular case is indeed a generalization of earlier known result (see [15, Proposition 1.1]) to the case of harmonic mappings.

COROLLARY 1.1

A locally univalent harmonic mapping $f = h + \overline{g} \in \mathcal{H}$ belongs to $\mathcal{B}_H(\lambda)$ if and only if, for each pair of points z_1, z_2 in $\mathbb{D}$ and $\theta \in [0, 2\pi]$,

$$|u_\theta(z_1) - u_\theta(z_2)| \le 2\lambda d_h(z_1, z_2),$$

where $u_\theta(z) = \log(h'(z) + e^{i\theta} g'(z))$.

2. Characterizations of uniformly locally univalent harmonic mappings

We now state our first result which is indeed a generalization of [24, Theorem 1] to the case of harmonic mappings.

Theorem 2.1. *A harmonic mapping $f = h + \overline{g}$ is uniformly locally univalent in $\mathbb{D}$ if and only if $\|T_f\| < \infty$.*

For the proof of the sufficiency of Theorem 2.1, we need the following classical result due to Noshiro [19].

Lemma A. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be analytic for $|z| < R$ and $|f'(z)| < M$ for $|z| < R$. Then the disk $|z| < R/M$ is mapped on a star-like domain with respect to the origin by f and also by all its polynomial sections $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$ $(n = 2, 3, \ldots)$.

Proof of Theorem 2.1. Let $f = h + \overline{g}$ be harmonic in $\mathbb{D}$ and assume that $\|T_f\| < \infty$. Define $F(\xi) = (f \circ T)(\xi)$ for $\xi \in \mathbb{D}$, where

$$w = T(\xi) = \frac{R\xi + a}{1 + \overline{a}R\xi}$$

with $R = \tanh\rho$, the constants $a \in \mathbb{D}$ and ρ $(0 < \rho \le \infty)$. Then $F = H + \overline{G}$ is harmonic in $\mathbb{D}$. Elementary computations yield

$$\frac{H''(\xi) + e^{i\theta}G''(\xi)}{H'(\xi) + e^{i\theta}G'(\xi)} = \frac{h''(w) + e^{i\theta}g''(w)}{h'(w) + e^{i\theta}g'(w)}T'(\xi) + \frac{T''(\xi)}{T'(\xi)},$$

where

$$T'(\xi) = \frac{R(1-|a|^2)}{(1+\overline{a}R\xi)^2} \quad \text{and} \quad T''(\xi) = -\frac{2\overline{a}(1-|a|^2)R^2}{(1+\overline{a}R\xi)^3}$$

so that

$$\frac{T''(\xi)}{T'(\xi)} = -\frac{2\overline{a}R}{1+\overline{a}R\xi}.$$

Since

$$1 - |w|^2 = \frac{(1-|a|^2)(1-|\xi|^2R^2)}{|1+\overline{a}R\xi|^2},$$

we easily have $|T'(\xi)|(1-|\xi|^2) \le 1-|w|^2$ and therefore, it follows that

$$(1-|\xi|^2)\left|\frac{H''(\xi) + e^{i\theta}G''(\xi)}{H'(\xi) + e^{i\theta}G'(\xi)}\right| \le (1-|w|^2)\left|\frac{h''(w) + e^{i\theta}g''(w)}{h'(w) + e^{i\theta}g'(w)}\right| + (1-|\xi|^2)\left|\frac{2\overline{a}R}{1+\overline{a}R\xi}\right|.$$

This inequality implies that

$$\sup_{\xi\in\mathbb{D}}(1-|\xi|^2)\left|\frac{H''(\xi) + e^{i\theta}G''(\xi)}{H'(\xi) + e^{i\theta}G'(\xi)}\right| \le k_0 < \infty, \tag{2.1}$$

where

$$k_0 = \sup_{w\in\mathbb{D}}(1-|w|^2)\left|\frac{h''(w) + e^{i\theta}g''(w)}{h'(w) + e^{i\theta}g'(w)}\right| + \frac{2R}{1-R}.$$

Let φ be an analytic branch of $\log(H'(\xi) + e^{i\theta}G'(\xi))$ in $\mathbb{D}$. Then

$$\varphi'(\xi) = \frac{H''(\xi) + e^{i\theta}G''(\xi)}{H'(\xi) + e^{i\theta}G'(\xi)}.$$

This choice is clearly possible, because $H'(\xi) + e^{i\theta}G'(\xi) \neq 0$ for $\xi \in \mathbb{D}$, by (2.1). It then follows from (2.1) that

$$\left|\log\left|\frac{H'(\xi) + e^{i\theta}G'(\xi)}{H'(0) + e^{i\theta}G'(0)}\right|\right| \le |\varphi(\xi) - \varphi(0)| \le \frac{k_0}{2}\log\left(\frac{1+|\xi|}{1-|\xi|}\right). \tag{2.2}$$

Now, we introduce $H_\theta(\xi)$ by

$$H_\theta(\xi) = \frac{H(\xi) + e^{i\theta}G(\xi)}{H'(0) + e^{i\theta}G'(0)}.$$

We see that H_θ is analytic in $\mathbb{D}$ and is normalized so that $H_\theta'(0) - 1 = 0$. We infer from (2.2) that

$$\log\left|H_\theta'(\xi)\right| \le \frac{k_0}{2}\log 3 \quad \text{for } |\xi| < \frac{1}{2},$$

whence

$$\left|H'_\theta(\xi)\right| < 3^{k_0/2} \ \text{ for } |\xi| < \frac{1}{2}.$$

Therefore, by Lemma A of [19], $H_\theta(\xi) - H_\theta(0)$ is univalent in the disk $|\xi| < \frac{3^{-k_0/2}}{2}$ for each θ. The radius of convexity for univalent functions is known to be $2-\sqrt{3}$ (cf. [13, Theorem 2.13]). Thus, $H_\theta(\xi) - H_\theta(0)$ and $H(\xi) + \mathrm{e}^{i\theta}G(\xi)$ are convex in $|\xi| < (2-\sqrt{3})\frac{3^{-k_0/2}}{2} = \rho_0$. This implies that F is harmonic (convex) univalent in $|\xi| < (2-\sqrt{3})\frac{3^{-k_0/2}}{2}$ (cf. [10]).

Consequently, f is univalent in the hyperbolic disk $D_h(a, \rho_0)$ with $\tanh \rho_0 = (2-\sqrt{3})\frac{3^{-k_0/2}}{2} \tanh \rho$ if $(2-\sqrt{3})\frac{3^{-k_0/2}}{2} \le 1$, and $\rho_0 = \rho$ if $(2-\sqrt{3})\frac{3^{-k_0/2}}{2} > 1$. Hence, f is uniformly locally univalent.

To prove the necessity, we assume that f is uniformly locally univalent in $\mathbb{D}$, that is, f is univalent in each hyperbolic disk $D_h(a, \rho)$, where $a \in \mathbb{D}$ and $0 < \rho \le \infty$.

Again, as above, we consider $w = T(\xi)$ and let

$$F(\xi) = (f \circ T)(\xi) = f(w) \ \text{ for } \xi \in \mathbb{D}.$$

Then $F = H + \overline{G}$ is univalent in $\mathbb{D}$. By Lewy's theorem (cf. [13]), the Jacobian J_F is different from 0 for all $z \in \mathbb{D}$, and then, without loss of generality, we assume that F is sense-preserving. Let

$$F_0(\xi) = \frac{H(\xi) - H(0)}{H'(0)} + \overline{\frac{G(\xi) - G(0)}{H'(0)}} = H_0(\xi) + \overline{G_0(\xi)}.$$

Obviously, $F_0 \in \mathcal{S}_H$. For $\xi \in \mathbb{D}$, set

$$F_1(z) = \frac{F_0\left(\frac{z+\xi}{1+\overline{\xi}z}\right) - F_0(\xi)}{(1-|\xi|^2)H_0'(\xi)} = H_1(z) + \overline{G_1(z)},$$

which again belongs to $\mathcal{S}_H$. The analytic function $H_1(z)$ has the form

$$H_1(z) = z + A_2(\xi)z^2 + A_3(\xi)z^3 + \cdots$$

and a direct computation shows that

$$A_2(\xi) = \frac{1}{2}\left\{(1-|\xi|^2)\frac{H_0''(\xi)}{H_0'(\xi)} - 2\overline{\xi}\right\} = \frac{1}{2}\left\{(1-|\xi|^2)\frac{H''(\xi)}{H'(\xi)} - 2\overline{\xi}\right\}.$$

Let $\alpha = \sup\{|a_2| : f(z) = \sum_{k=1}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b}_k \overline{z}^k \in \mathcal{S}_H\}$. For $f \in \mathcal{S}_H$, we have $\frac{f(z)-\overline{b_1 f(z)}}{1-|b_1|^2} \in \mathcal{S}_H^0$. It is known that for $f^*(z) = \sum_{k=1}^{\infty} a_k^* z^k + \sum_{k=1}^{\infty} \overline{b}_k^* \overline{z}^k \in \mathcal{S}_H^0$, the coefficient $|a_2^*| < 49$ and $|b_2^*| < \frac{1}{2}$ (cf. [13]). Using this estimate, by computations, it is possible to get $|a_2| < 98$. It has been recently shown by Abu-Muhanna *et al.* [2] that $|a_2^*| \le 20.5$ which indeed is the best known upper bound for $|a_2^*|$. Since $F_1 \in \mathcal{S}_H$, we must have $|A_2(\xi)| \le \alpha$ and therefore,

$$(1-|\xi|^2)\left|\frac{H''(\xi)}{H'(\xi)}\right| < 2(\alpha+1), \quad \xi \in \mathbb{D}.$$

For each $c \in \mathbb{D}$, the composition of sense preserving affine mapping $\phi(w) = w + c\overline{w}$ with F, namely, the function $F + c\overline{F}$, is univalent and sense-preserving in $\mathbb{D}$. Then by what we have just proved, we obtain

$$(1-|\xi|^2)\left|\frac{H''(\xi) + cG''(\xi)}{H'(\xi) + cG'(\xi)}\right| < 2(\alpha+1), \quad \xi \in \mathbb{D},$$

which, in particular, implies that for each $\theta \in [0, 2\pi]$,

$$(1-|\xi|^2)\left|\frac{H''(\xi)+e^{i\theta}G''(\xi)}{H'(\xi)+e^{i\theta}G'(\xi)}\right| < 2(\alpha+1), \quad \xi \in \mathbb{D}.$$

Thus, for $f = h + \overline{g}$, we have

$$A(\theta) := \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{h''(z)+e^{i\theta}g''(z)}{h'(z)+e^{i\theta}g'(z)}\right| < \infty.$$

Since $A(\theta)$ is a continuous function of θ in $[0, 2\pi]$, it follows from $A(\theta) < \infty$ that

$$\sup_{z\in\mathbb{D},\ \theta\in[0,2\pi]}(1-|z|^2)\left|\frac{h''(z)+e^{i\theta}g''(z)}{h'(z)+e^{i\theta}g'(z)}\right| < \infty.$$

The proof of the theorem is complete. □

3. Growth estimate for the class $\mathcal{B}_H(\lambda)$

For a nonnegative real number λ, we consider

$$H_\lambda(z) = \int_0^z \left(\frac{1+t}{1-t}\right)^\lambda dt.$$

It is easy to verify that $\|T_{H_\lambda}\| = 2\lambda$, and thus $H_\lambda \in \mathcal{B}_H(\lambda)$. If $\lambda \geq 0$, then it is known that H_λ is univalent in $\mathbb{D}$ if and only if $0 \leq \lambda \leq 1$ (see [15, Lemma 2.1]). We will see later that H_λ is extremal in the class $\mathcal{B}_H(\lambda)$. It follows from Theorem 2.1 that if $f = h+\overline{g} \in \mathcal{B}_H(\lambda)$, then $\frac{h+e^{i\theta}g}{1+e^{i\theta}b_1} \in \mathcal{B}(\lambda)$. This fact and [15, Theroem 2.3] give the following result.

Theorem 3.1 (Distortion theorem). *Let λ be a nonnegative real number and $f(z) = h(z)+\overline{g(z)} = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n \in \mathcal{B}_H(\lambda)$. Then for $z \in \mathbb{D}$, we have*

$$|\lambda_f(z)| = |\,|h'(z)| - |g'(z)|\,| \geq |1-|b_1||\left(\frac{1-|z|}{1+|z|}\right)^\lambda = |1-|b_1||\,|H'_\lambda(-|z|),$$

$$|\Lambda_f(z)| = |h'(z)| + |g'(z)| \leq (1+|b_1|)\left(\frac{1+|z|}{1-|z|}\right)^\lambda = (1+|b_1|)H'_\lambda(|z|)$$

and $|f(z)| \leq (1+|b_1|)H_\lambda(|z|)$. Furthermore, if $f \in \mathcal{S}_H^0 \cap \mathcal{B}_H(\lambda)$, then

$$-H_\lambda(-|z|) \leq |f(z)| \leq H_\lambda(|z|).$$

Equality occurs in each case when $f(z) = \overline{\mu}H_\lambda(\mu z)$ for a unimodular constant μ.

COROLLARY 3.2

For $\lambda > 1$, each $f(z) = h(z)+\overline{g(z)} = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n \in \mathcal{B}_H(\lambda)$ satisfies the growth condition

$$f(z) = O((1-|z|)^{1-\lambda})$$

as $|z| \to 1$. On the other hand, for $\lambda < 1$, each mapping $f \in \mathcal{B}_H(\lambda)$ is bounded with the bound $(1+|b_1|)H_\lambda(1)$. Moreover, if $\lambda > 0$ and $f \in \mathcal{S}_H^0 \cap \mathcal{B}_H(\lambda)$ in $\mathbb{D}$, then the image $f(\mathbb{D})$ contains the disk $\{w : ,|w| < -H_\lambda(-1)\}$.

By [4,5], for $\lambda \le 1/2$, $\mathcal{B}(\lambda) \subset \mathcal{S}$ and so, by Theorem 2.1, for $\lambda \le 1/2$, $f \in \mathcal{B}_H(\lambda)$ must be univalent in $\mathbb{D}$. We also note that for $0 \le \lambda \le 1$, we have

$$-H_\lambda(-1) \ge -H_1(-1) = 2\log 2 - 1 = 0.38629\ldots,$$

and therefore the result is an improvement of the covering theorem for harmonic mappings in $\mathcal{S}_H^0$.

In Corollary 3.2, the case $\lambda = 1$ is critical. By Theorem 3.1, we have that, for $f \in \mathcal{B}_H(1)$,

$$|f(z)| \le (1 + |b_1|)H_1(|z|) = (1 + |b_1|)(-2\log(1 - |z|) - |z|),$$

which shows that functions in $\mathcal{B}_H(1)$ need not be bounded. The next theorem, which follows from Theorem 2.1 and [15, Proposition 2.5] gives a boundedness criterion for mappings in $\mathcal{B}_H(1)$.

Theorem 3.3. *If a harmonic mapping $f = h + \overline{g}$ in $\mathbb{D}$ satisfies the condition*

$$\overline{\lim_{|z|\to 1^-}} \left\{(1 - |z|^2)\left|\frac{h''(z) + \mathrm{e}^{i\theta} g''(z)}{h'(z) + \mathrm{e}^{i\theta} g'(z)}\right| - 2\right\} \log \frac{1}{1 - |z|^2} < -2$$

for each $\theta \in [0, 2\pi]$, then f is bounded. Here the constant -2 on the right-hand side is sharp.

We conclude this section with the Hölder continuity of mappings in $\mathcal{B}_H(\lambda)$.

Theorem 3.4. *Let $0 \le \lambda < 1$. Then each mapping $f \in \mathcal{B}_H(\lambda)$ is Hölder continuous of exponent $1 - \lambda$ in $\mathbb{D}$.*

The proof follows from Theorem 2.1 and [15, Theorem 2.6] and so, we omit its detail.

4. The space $\mathcal{B}_H(\lambda)$ and the Hardy space

We begin this section with the following concepts.

DEFINITION 4.1

For $0 < p < \infty$, the *Hardy space* H^p is the set of all functions f analytic in $\mathbb{D}$ for which

$$M_p(r, f) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(r\mathrm{e}^{i\theta})|^p \,\mathrm{d}\theta\right\}^{1/p}$$

is bounded on $0 < r < 1$.

The space h^p consists of all harmonic mappings f in $\mathbb{D}$ for which $M_p(r, f)$ $(0 < r < 1)$ are bounded (cf. [13]).

For a harmonic mapping $f = h + \overline{g}$ in $\mathbb{D}$, the Bloch seminorm is given by (cf. [11])

$$\|f\|_{\mathscr{B}_H} = \sup_{z\in\mathbb{D}} (1 - |z|^2)\big(|h'(z)| + |g'(z)|\big),$$

and f is called a Bloch mapping when $\|f\|_{\mathscr{B}_H} < \infty$. In the recent years, the class of harmonic Bloch mappings has been studied extensively together with its higher dimensional analog (see for example, [7–9,11] and the references therein).

By Theorem 3.1, we have, for $f \in \mathcal{B}_H(\lambda)$,

$$|f(z)| \leq (1+|b_1|)\int_0^{|z|}\left(\frac{1+t}{1-t}\right)^{\lambda} \mathrm{d}t,$$

which shows that

- f is bounded when $\lambda < 1$,
- $f(z) = O(-\log(1-|z|))$ $(|z| \to 1)$ when $\lambda = 1$, and
- $f(z) = O((1-|z|)^{1-\lambda})$ $(|z| \to 1)$ when $\lambda > 1$.

Let BMOA (resp. BMOH) denote the class of analytic functions (resp. harmonic mappings) that have bounded mean oscillation on the unit disk $\mathbb{D}$ (cf. [1]). In [14], Kim proved the following result for analytic functions.

Theorem B.

(1) *If* $\lambda < 1$, $\mathcal{B}(\lambda) \cap \mathcal{S} \subset H^{\infty}$,
(2) *If* $\lambda = 1$, $\mathcal{B}(\lambda) \cap \mathcal{S} \subset \mathrm{BMOA}$,
(3) *If* $\lambda > 1$, $\mathcal{B}(\lambda) \cap \mathcal{S} \subset H^p$ *for every* $0 < p < 1/(\lambda - 1)$.

In order to state a generalization of this result for harmonic mappings, we introduce

$$\mathcal{S}_{H_k} = \{f = h + \overline{g} \in \mathcal{S}_H : f \text{ is } k\text{-quasiconformal}\}$$

for $0 \leq k < 1$. We now state the analog of Theorem B.

Theorem 4.2.

(1) *If* $\lambda < 1$, *then* $\mathcal{B}_H(\lambda) \cap \mathcal{S}_H \subset h^{\infty}$.
(2) *If* $\lambda = 1$, *then* $\mathcal{B}_H(\lambda) \cap \mathcal{S}_H \subset \mathrm{BMOH}$.
(3) *If* $\lambda > 1$, *then* $\mathcal{B}_H(\lambda) \cap \mathcal{S}_{H_k} \subset h^p$ *for every* $0 < p < 1/(\lambda - 1)$.

For the proof of Theorem 4.2, we need some preparation.

Lemma 4.3. *If* $f = h + \overline{g} \in \mathcal{B}_H(1)$, *then* $\|f\|_{\mathscr{B}_H} \leq 4(1+|b_1|)$.

Proof. For $f = h + \overline{g} \in \mathcal{B}_H(1)$, by Theorem 2.1, we have $h + \mathrm{e}^{i\theta} g \in \mathcal{B}(1)$ for each $\theta \in [0, 2\pi]$. It follows from [14, Theorem 2.1] that $\|h + \mathrm{e}^{i\theta} g\|_{\mathscr{B}} \leq 4(1+|b_1|)$, which implies that $\|f\|_{\mathscr{B}_H} \leq 4(1+|b_1|)$. □

In the next lemma, we shall consider the problem of how the integral means of k-quasiconformal harmonic univalent mappings f behaves. Here the integral means of f is defined by

$$I_p(r) = I_p(r, f) = \frac{1}{2\pi}\int_0^{2\pi} |f(r\mathrm{e}^{i\theta})|^p \,\mathrm{d}\theta. \tag{4.1}$$

The following lemma is regarded as a generalization of [21, Proposition 8.1] to the case of harmonic mappings.

Lemma 4.4. *Let $f \in \mathcal{S}_{H_k}$ and $p > 0$. Then*

$$I_p(r) \le \frac{2(1+k^2)(|p-2|+1)}{1-k^2} \int_0^r M(\rho)^p \rho^{-1}\, \mathrm{d}\rho \quad (0 \le r < 1),$$

where

$$M(r) := M(r, f) = \max_{0 \le \theta \le 2\pi} |f(r\mathrm{e}^{i\theta})|.$$

Proof. Let $f = h + \overline{g} \in \mathcal{S}_{H_k}$ and write $z = r\mathrm{e}^{i\theta}$, where $0 \le r < 1$. Writing

$$|f(z)|^p = [(h(z) + \overline{g(z)})(\overline{h(z)} + g(z))]^{p/2},$$

elementary computations give

$$r\frac{\partial}{\partial r}(|f(z)|^p) = p|f(z)|^{p-2}\mathrm{Re}\,\{(zh'(z) + \overline{zg'(z)}\,)\overline{f(z)}\,\}$$

and

$$\frac{\partial}{\partial \theta}(|f(z)|^p) = p|f(z)|^{p-2}\mathrm{Re}\,\{(izh'(z) + \overline{izg'(z)}\,)\overline{f(z)}\,\}.$$

Further computations yield

$$\begin{aligned}\left(r\frac{\partial}{\partial r}\right)^2(|f(z)|^p) &= p(p-2)|f(z)|^{p-4}(\mathrm{Re}\,\{(zh'(z) + \overline{zg'(z)}\,)\overline{f(z)}\})^2\\ &\quad + p|f(z)|^{p-2}\mathrm{Re}\,\{(z^2h''(z) + \overline{z^2g''(z)} + zh'(z) + \overline{zg'(z)}\,)\overline{f(z)}\,\}\\ &\quad + p|f(z)|^{p-2}|zh'(z) + \overline{zg'(z)}|^2\end{aligned}$$

and

$$\begin{aligned}\left(\frac{\partial}{\partial \theta}\right)^2(|f(z)|^p) &= p(p-2)|f(z)|^{p-4}(\mathrm{Re}\,\{(izh'(z) + \overline{izg'(z)}\,)\overline{f(z)}\,\})^2\\ &\quad + p|f(z)|^{p-2}\mathrm{Re}\,\{(-z^2h''(z) - \overline{z^2g''(z)} - zh'(z)\\ &\quad - \overline{zg'(z)})\overline{f(z)}\,\} +\ p|f(z)|^{p-2}|izh'(z) + \overline{izg'(z)}|^2.\end{aligned}$$

Adding the last two expressions shows that

$$\begin{aligned}&\left(r\frac{\partial}{\partial r}\right)^2 (|f(z)|^p) + \left(\frac{\partial}{\partial \theta}\right)^2(|f(z)|^p)\\ &\quad = p(p-2)|f(z)|^{p-4}[(\mathrm{Re}\,\{(zh'(z) + \overline{zg'(z)}\,)\overline{f(z)}\,\})^2\\ &\qquad + (\mathrm{Re}\,\{(izh'(z) + \overline{izg'(z)}\,)\overline{f(z)}\,\})^2]\\ &\qquad + p|f(z)|^{p-2}[|zh'(z) + \overline{zg'(z)}|^2 + |izh'(z) + \overline{izg'(z)}|^2]\\ &\quad \le 2p(1+k^2)(|p-2|+1)r^2|f(z)|^{p-2}|h'(z)|^2.\end{aligned}$$

It follows that

$$\frac{1}{2\pi}\int_0^{2\pi}\left[\left(r\frac{\partial}{\partial r}\right)^2(|f(re^{i\theta})|^p)+\left(\frac{\partial}{\partial\theta}\right)^2(|f(re^{i\theta})|^p)\right]\mathrm{d}\theta$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\left(r\frac{\partial}{\partial r}\right)^2(|f(re^{i\theta})|^p)\,\mathrm{d}\theta$$
$$=r\frac{\mathrm{d}}{\mathrm{d}r}(rI_p'(r))$$
$$\le p(1+k^2)(|p-2|+1)r^2\frac{1}{\pi}\int_0^{2\pi}|f(re^{i\theta})|^{p-2}|h'(re^{i\theta})|^2\,\mathrm{d}\theta,$$

where we have used the fact that the integral corresponding to the second term above vanishes because of the periodicity of the function involved in the integrand. As $J_f(z)=|h'(z)|^2-|g'(z)|^2$ and

$$|h'(z)|^2=\frac{|h'(z)|^2}{|h'(z)|^2-|g'(z)|^2}J_f(z)\le\frac{1}{1-k^2}J_f(z),$$

we may integrate the last expression on both sides and obtain the inequality

$$rI_p'(r)\le p(1+k^2)(|p-2|+1)\iint_{|z|\le r}|f(z)|^{p-2}|h'(z)|^2\,\mathrm{d}\sigma(z)$$
$$\le\frac{p(1+k^2)(|p-2|+1)}{1-k^2}\iint_{|z|\le r}|f(z)|^{p-2}J_f(z)\,\mathrm{d}\sigma(z),$$

where $\mathrm{d}\sigma(z)=(1/\pi)\mathrm{d}x\,\mathrm{d}y$ denotes the normalized area element. Now, we substitute $w=f(z)$. Since f is univalent in $\mathbb{D}$ and $M(r)=\max_{0\le\theta\le 2\pi}|f(re^{i\theta})|$, the last inequality gives

$$rI_p'(r)\le\frac{p(1+k^2)(|p-2|+1)}{1-k^2}\iint_{|w|\le M(r)}|w|^{p-2}\,\mathrm{d}\sigma(w)$$
$$=\frac{2p(1+k^2)(|p-2|+1)}{1-k^2}\int_0^{M(r)}t^{p-1}\,\mathrm{d}t$$
$$=\frac{2(1+k^2)(|p-2|+1)}{1-k^2}M(r)^p,$$

which upon integration on both sides shows that

$$I_p(r)\le\frac{2(1+k^2)(|p-2|+1)}{1-k^2}\int_0^r M(\rho)^p\rho^{-1}\,\mathrm{d}\rho.$$

The desired conclusion follows. □

4.1 *The proof of Theorem* 4.2

Let $f\in\mathcal{B}_H(\lambda)$ for some $\lambda<1$. Then, by Corollary 3.2, f is bounded.

Next we assume that $f = h + \overline{g} \in \mathcal{B}_H(1) \cap \mathcal{S}_H$. Then, by Lemma 4.3, it follows that f is Bloch and thus, h is Bloch, since, for $f = h + \overline{g} \in \mathcal{S}_H$, h is Bloch if and only if h is BMOA if and only if f is BMOH (cf. [1]). Consequently, $f \in$ BMOH.

Finally, we assume that $f \in \mathcal{B}_H(\lambda) \cap \mathcal{S}_{H_k}$ for some $\lambda > 1$. Then, by Theorem 3.1 and Corollary 3.2, we deduce that $f(z) = O((1-|z|)^{1-\lambda})$ as $|z| \to 1$. Furthermore, using Lemma 4.4, we find that

$$I_p(r) \le \frac{2(1+k^2)(|p-2|+1)}{1-k^2} \int_0^r M(\rho)^p \rho^{-1}\, \mathrm{d}\rho.$$

Hence

$$I_p(r) = O\left(\frac{1}{(1-|z|)^{(\lambda-1)p-1}}\right) \text{ as } |z| \to 1.$$

Hence, $f \in h^p$ if $0 < p < 1/(\lambda - 1)$. □

Obviously, the assertion (1) in Theorem B remains valid if we replace $\mathcal{B}(\lambda)$ by $\mathcal{B}_H(\lambda)$. By [16, Theorem 1], we see that the assertion (3) does not hold for $\mathcal{B}_H(\lambda)$. On the other hand, we will show that $\mathcal{B}_H(\lambda)$ is contained in some Hardy space.

Theorem 4.5. *Let $\lambda \ge 1$. Then $\mathcal{B}_H(\lambda) \subset h^p$ with $0 < p < \frac{1}{\lambda^2-1}$.*

In the above, the expression $\frac{1}{\lambda^2-1}$ is interpreted as ∞ when $\lambda = 1$.

Proof. Assume that $f = h + \overline{g} \in \mathcal{B}_H(\lambda)$. By Theorem 2.1 and [16, Theorem 2], for each θ, $h + e^{i\theta} g \in H^p$ with $0 < p < \frac{1}{\lambda^2-1}$. It follows that $h - g \in H^p$ and $h + g \in H^p$ which implies that $f \in h^p$.

COROLLARY 4.6

A uniformly locally univalent harmonic mapping f in $\mathbb{D}$ is contained in the Hardy space h^p for some $p = p(f) > 0$.

In [14], Kim also conjectured that the assertion (3) in Theorem B holds for $\mathcal{B}(\lambda)$.

5. Coefficient estimates for the class $\mathcal{B}_H(\lambda)$

Let $f(z) = h(z) + \overline{g(z)} = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n$ with $a_1 = 1$ and $b_1 = 0$. If $f \in \mathcal{B}_H(\lambda)$, then by Theorem 2.1, for each $\theta \in [0, 2\pi]$,

$$\left|\frac{h''(0) + \mathrm{e}^{i\theta} g''(0)}{h'(0) + \mathrm{e}^{i\theta} g'(0)}\right| \le 2\lambda,$$

which shows that $\big||a_2| - |b_2|\big| \le |a_2| + |b_2| \le \lambda$. Of course, this estimate is sharp because equality holds for H_λ.

In order to estimate the coefficients of harmonic mappings f in $\mathbb{D}$, we consider the integral mean $I_p(r, f)$ of f defined by (4.1), where p is a real number. For $f(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n \in \mathcal{B}_H(\lambda)$ with $a_1 = 1$ and $b_1 = 0$, by Theorem 2.1, [15, Theorem 2.3] and similar arguments as in [15, p. 190], we have $|a_n + \mathrm{e}^{i\theta} b_n| = O(n^{\lambda-1})$ uniformly for $\theta \in [0, 2\pi]$ as $n \to \infty$, and then $|a_n| + |b_n| = O(n^{\lambda-1})$ as $n \to \infty$.

Moreover, if $\lambda < 1$ and $f = h + \overline{g}$ is univalent, then, by Theorem 2.1 and [15, Corollary 2.4], $H_\theta = h + e^{i\theta} g$ is uniformly bounded for $\theta \in [0, 2\pi]$. So

$$\text{Area}\,(H_\theta(\mathbb{D})) = \pi\left(1 + \sum_{n=2}^{\infty} n|a_n + e^{i\theta} b_n|^2\right) < \infty,$$

which implies that $|a_n| + |b_n| = o(n^{-1/2})$ as $n \to \infty$.

In the following theorem, we improve the exponents in these order estimates.

Theorem 5.1. *Let* $f(z) = h(z) + \overline{g(z)} = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n \in \mathcal{B}_H(\lambda)$ *with* $a_1 = 1$ *and* $b_1 = 0$. *Then, for each* $\varepsilon > 0$, *a real number* p *and uniformly for* $\theta \in [0, 2\pi]$, *we have*

$$I_p(r, h' + e^{i\theta} g') = O((1-r)^{\alpha(|p|\lambda) - \varepsilon}),$$

and thus,

$$I_p(r, f) = O((1-r)^{-\alpha(|p|\lambda) - \varepsilon}), \quad |a_n| + |b_n| = O(n^{\alpha(\lambda) - 1 + \varepsilon}),$$

where $\alpha(\lambda) = \frac{\sqrt{1+4\lambda^2} - 1}{2}$.

We can prove this theorem by using Theorem 2.1 and [15, Theroem 3.1]. Here we omit its detail.

Given a harmonic mapping $f(z) = h(z) + \overline{g(z)} = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n$ with $a_1 = 1$ and $b_1 = 0$ in $\mathbb{D}$, let $\gamma(f)$ denote the infimum of exponents γ such that $|a_n| + |b_n| = O\big(n^{\gamma - 1}\big)$ as $n \to \infty$, that is,

$$\gamma(f) = \overline{\lim_{n\to\infty}} \frac{\log n(|a_n| + |b_n|)}{\log n}.$$

For the subset X of $\mathcal{H}$, we let $\gamma(X) = \sup_{f\in X} \gamma(f)$. As for the class $\mathcal{S}_b$ of all normalized bounded univalent functions in $\mathbb{D}$, it is proved that $0.24 < \gamma(\mathcal{S}_b) < 0.4886$ (cf. [6,18]), and conjectured by Carleson and Jones [6] that $\gamma(\mathcal{S}_b) = 0.25$. For a bounded and univalent function f, we note that the growth of coefficients seems to involve the irregularity of boundary of image under f (cf. [21, Chapter 10]), and Makarov and Pommerenke [18] observed a remarkable phenomenon of phase transition of the functional $\gamma(f)$ with respect to the Minkowski dimension of the boundary curve. Recently, in [15], authors established the boundedness of $\gamma(\mathcal{B}(\lambda))$ in terms of λ. As a generalization, we consider the class $\mathcal{B}_H(\lambda)$ and prove that $\gamma(\mathcal{B}_H(\lambda))$ have the same bound with $\gamma(\mathcal{B}(\lambda))$.

For the class $\mathcal{B}_H(\lambda)$, Theorem 5.1 implies that $\gamma(\mathcal{B}_H(\lambda)) \le \alpha(\lambda)$. The extremal function H_λ satisfies the relation $\gamma(H_\lambda) = \lambda - 1$. By [15, Example 3.1], it follows that $\gamma(\mathcal{B}_H(\lambda)) \ge 0$ for $\lambda > 0$. Hence, we have

Theorem 5.2. *For each* $\lambda \in (0, \infty)$, *we have*

$$\max\{\lambda - 1, 0\} \le \gamma(\mathcal{B}_H(\lambda)) \le \alpha(\lambda),$$

where $\alpha(\lambda) = \frac{\sqrt{1+4\lambda^2} - 1}{2}$. *In particular,* $\gamma(\mathcal{B}_H(\lambda)) = O(\lambda^2)$ *as* $\lambda \to 0$.

Now we mention a connection with integral means for univalent analytic functions. For a univalent harmonic mapping $f \in \mathcal{S}_H$ and a real number p, we let

$$\beta_{f,\theta}(p) = \overline{\lim_{r\to 1^-}} \frac{\log I_p(r, h' + e^{i\theta} g')}{\log \frac{1}{1-r}}.$$

Clearly, for an univalent analytic function $f \in \mathcal{S}$,

$$\beta_f(p) = \overline{\lim_{r\to 1^-}} \frac{\log I_p(r, f')}{\log \frac{1}{1-r}}.$$

Brennan conjectured that $\beta_f(-2) \leq 1$ for univalent analytic functions f (cf. [21, Chapter 8]).

As a corollary of Theorem 5.1, we have

Theorem 5.3. *For $f \in \mathcal{B}_H(\lambda)$ and a real number p,*

$$\beta_{f,\theta}(p) \leq \alpha(|p|\lambda) = \frac{\sqrt{1 + 4p^2\lambda^2} - 1}{2}$$

holds for each $\theta \in [0, 2\pi]$. In particular, the Brennan conjecture is true for univalent functions f with $\|T_f\| \leq \sqrt{2}$.

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